ANNALES
POLONICI MATHEMATICI
LXVI (1997)

On strongly monotone flows

by WOLFGANG WALTER (Karlsruhe)

Włodzimierz Mlak in memoriam

Abstract. M. Hirsch's famous theorem on strongly monotone flows generated by autonomous systems $u'(t) = f(u(t))$ is generalized to the case where f depends also on t, satisfies Carathéodory hypotheses and is only locally Lipschitz continuous in u. The main result is a corresponding Comparison Theorem, where $f(t, u)$ is quasimonotone increasing in u; it describes precisely for which components equality or strict inequality holds.

1. Introduction. One of M. Hirsch's theorems on monotone flows [1] states that the flow generated by a C^1-function f is strongly monotone if the Jacobian $f'(x) = (\partial f_i/\partial x_j)$ is essentially positive (i.e., $\partial f_i/\partial x_j \geq 0$ for $i \neq j$) and irreducible. In the language of differential equations the theorem says that the conditions

$$v' = f(v), \quad w' = f(w) \quad \text{in } J = [a, b], \quad v(a) \leq w(a), \quad v(a) \neq w(a)$$

imply the strict inequality $v(t) < w(t)$ in $J_0 = (a, b]$. Here, $f : \mathbb{R}^n \supset D \to \mathbb{R}^n$ and $v, w : J \to \mathbb{R}^n$, while $\leq$ and $<$ refer to the componentwise ordering in $\mathbb{R}^n$.

Hirsch's original proof was subject to criticism, and other proofs have been given by several authors. Our objective is to present a simple proof for a more general theorem. We allow that f depends explicitly on t and satisfies only Carathéodory hypotheses. Furthermore, we consider the case where v, w are not necessarily solutions but satisfy differential inequalities, and finally we assume that $f(t, x)$ is only locally Lipschitzian in x; let us remark that there are important applications, e.g., in nonsymmetric mechanical systems, where f is Lipschitzian but not of class C^1. Consequently, the handsome irreducibility assumption has to be replaced by an assumption that avoids derivatives.

1991 *Mathematics Subject Classification*: 34A40, 34C11, 34C35.

Key words and phrases: system of ordinary differential equations, initial value problem, comparison theorem, monotone flow, quasimonotonicity.

Theorem 1 deals with a system of linear differential inequalities. The main point in this theorem is the assertion that each component of $u(t)$ is either strictly positive in $(a,b]$ or vanishes in an interval $[a,a+\delta_i]$ and is positive thereafter. Theorem 2, which covers the nonlinear case, is reduced to Theorem 1 by simple, well-known means. From these results, rather weak additional assumptions which imply strict inequalities in J_0 for all components are easily obtained.

2. The linear case. For $x,y\in\mathbb{R}^n$ we define

$$x\le y \Leftrightarrow x_i\le y_i \quad \text{for } i\in N \quad\text{and}\quad x<y\Leftrightarrow x_i<y_i \quad \text{for } i\in N,$$

where $N=\{1,\dots,n\}$. The spaces $AC(J)$ and $L(J)$ contain all functions $x(t)$ that are absolutely continuous or integrable in J, resp. Here and below, $J=[a,b]$, $J_0=(a,b]$ and $e_n=(1,\dots,1)\in\mathbb{R}^n$.

THEOREM 1. *Let $C(t)=(c_{ij}(t))\in L(J)$ be an essentially positive $n\times n$ matrix, i.e., $c_{ij}\ge 0$ a.e. in J for $i\ne j$. Then $u\in AC(J)$,*

$$u(a)\ge 0,\quad u'\ge Cu \quad \text{a.e. in } J \tag{1}$$

imply $u(t)\ge 0$ in J. Moreover, the index set N can be split up in two disjoint sets α, β ($\alpha\cup\beta=N$, $\alpha\cap\beta=\emptyset$) such that

- *for $i\in\alpha$, $u_i(t)>0$ in J_0,*
- *for $j\in\beta$, $u_j(t)=0$ in $[a,a+\delta_j]$ and $u_j(t)>0$ in $(a+\delta_j,b]$,*

where $\delta_j>0$.

Proof. Let $|c_{ij}(t)|\le m(t)\in L(J)$ for $i,j\in N$ and $M(t)=\int_a^t m(s)\,ds$. The function $w(t)=e^{M(t)}u(t)$ satisfies $w(a)\ge 0$ and

$$w'\ge D(t)w \quad \text{a.e. in } J, \quad \text{where } D(t)=C(t)+m(t)I\ge 0,$$

i.e., $d_{ij}(t)\ge 0$ for all $i,j\in N$. The function

$$h(t)=(\varrho,\dots,\varrho) \quad \text{with } \varrho(t)=e^{(n+1)M(t)}$$

satisfies $h(a)=e_n=(1,\dots,1)$ and $h'\ge Dh$. Hence $w_\varepsilon=w+\varepsilon h$ ($\varepsilon>0$) has the properties

$$w_\varepsilon'\ge Dw_\varepsilon \quad \text{a.e. in } J \quad\text{and}\quad w(a)\ge \varepsilon e_n>0.$$

As long as $w_\varepsilon\ge 0$, we have $w_\varepsilon'\ge 0$. It follows easily that $w_\varepsilon(t)\ge\varepsilon e_n$ for all $t\in J$. Since $\varepsilon>0$ is arbitrary, $w(t)\ge 0$ in J and also $w'(t)\ge 0$ a.e. in J.

This shows that $u(t)\ge 0$ in J. Assume now that a component u_i is positive at $t_0\in J$. Then $w_i(t_0)>0$ and therefore $w_i(t)>0$ for $t>t_0$, which in turn implies $u_i(t)>0$ for $t>t_0$. We let α be the set of all indices i such that $u_i>0$ in $J_0=(a,b]$. Then each u_j with $j\notin\alpha$ vanishes at some point $t_i\in J_0$ and therefore in $[a,t_i]$. ∎

3. The nonlinear, quasimonotone case. We consider the nonlinear equation

$$(2) \qquad u'(t) = f(t, u(t)) \quad \text{a.e. in } J$$

and assume for simplicity that $f(t,x)$ is defined in the strip $S = J \times \mathbb{R}^n$ and satisfies the following conditions: $f(t,\cdot)$ is continuous in $\mathbb{R}^n$ for almost all (fixed) $t \in J$, $f(\cdot,x)$ is measurable in J for all (fixed) $x \in \mathbb{R}^n$, $f(t,0) \in L(\mathbb{R})$, and for each constant $A > 0$ there is a function $m(t) \in L(J)$ such that

$$(3) \qquad |f(t,x) - f(t,y)| \le m(t)|x-y| \quad \text{for } t \in J \text{ and } |x|, |y| \in A.$$

The *defect* P of a function $v \in AC(J)$ with respect to equation (2) is defined by

$$(Pv)(t) = v'(t) - f(t, v(t)).$$

The function f is said to be *quasimonotone increasing* in x if $f_i(t,x)$ is (weakly) increasing in x_j for all $j \neq i$, or equivalently, if

$$x \le y,\ x_i = y_i \Rightarrow f_i(t,x) \le f_i(t,y) \quad \text{a.e. in } J \quad (i = 1, \ldots, n).$$

THEOREM 2. *Assume that the function $f(t,x)$ satisfies the conditions given above and is quasimonotone increasing in x, and let $v, w \in AC(J)$ satisfy*

$$(4) \qquad v(a) \le w(a) \quad \textit{and} \quad Pv \le Pw \quad \textit{a.e. in } J.$$

Then $v \le w$ in J, and there exist disjoint index sets α, β with $\alpha \cup \beta = N$ and positive numbers δ_j such that

$$v_i < w_i \quad \textit{in } J_0 \quad \textit{for } i \in \alpha,$$

$$v_j = w_j \quad \textit{in } [a, a+\delta_j] \quad \textit{and} \quad v_j < w_j \quad \textit{in } (a+\delta_j, b] \quad \textit{for } j \in \beta.$$

Proof. Let $|v(t)|, |w(t)| \le A$ in J and assume that (3) holds. Let $u(t) = w(t) - v(t)$. In the scalar case ($n = 1$) one can write

$$\Delta f := f(t, w(t)) - f(t, v(t)) = c(t)u(t) \quad \text{with } |c(t)| \le m(t)$$

and $c(t) \ge 0$ in case f is increasing in x (take $c(t) = \Delta f/u$ if $u \neq 0$ and $c(t) = 0$ otherwise). In the general case $n > 1$, the same is accomplished by writing Δf_i as a sum of differences $\Delta_1, \ldots, \Delta_n$, where

$$\Delta_1 = f_i(t, w_1, v_2, \ldots, v_n) - f_i(t, v_1, v_2, \ldots, v_n), \ \ldots$$

In this way one obtains

$$f(t,w) - f(t,v) = C(t)u \quad \text{with } |c_{ij}(t)| \le m(t)$$

and $c_{ij}(t) \ge 0$ for $i \neq j$ because f is quasimonotone increasing in x. Now the theorem follows from Theorem 1. ∎

4. Strong monotonicity. We are looking for conditions such that in Theorems 1 and 2 the set β is empty, which means that in the conclusions strict inequality holds in J_0. The following notation is used. A measurable set $M \subset J$ is said to be *dense* at a if the set $M \cap [a, a+\varepsilon]$ has positive measure for every $\varepsilon > 0$. For measurable real-valued functions φ, ψ we write $\varphi < \psi$ at $a+$ if the set $\{t \in J : \varphi(t) < \psi(t)\}$ is dense at a.

THEOREM 3. *Suppose $u(t)$ and $C(t)$ satisfy the assumptions of Theorem 1, and there exists a nonempty index set α_1 such that for $i \in \alpha_1$ either $u_i(a) > 0$ or $u_i' > (Cu)_i$ at $a+$. If for every index set $\alpha_0 \supset \alpha_1$ with $\beta_0 = N \setminus \alpha_0 \neq \emptyset$ there exist numbers $k \in \alpha_0$, $j \in \beta_0$ such that $c_{jk} > 0$ at $a+$, then $u > 0$ in J_0.*

In particular, the assertion $u > 0$ in J_0 holds under each of the following conditions:

(i) $u(a) > 0$;
(ii) $u' > Cu$ *at* $a+$;
(iii) $u(a) \neq 0$ *and the matrix* $C(t)$ *is irreducible at* $a+$.

Irreducibility at $a+$ is defined as follows: For every nonempty index set α with $\beta = N \setminus \alpha \neq \emptyset$ there exist indices $k \in \alpha$, $j \in \beta$ such that $c_{jk} > 0$ at $a+$.

Proof of Theorem 3. According to Theorem 1, $u_\alpha > 0$ in J_0 and $u_\beta = 0$ in an interval $J_\delta = [a, a+\delta]$, $\delta > 0$. Assume $\beta \neq \emptyset$. Our assumptions imply that $\alpha \supset \alpha_1$. Putting $\alpha = \alpha_0$, we find indices $k \in \alpha$, $j \in \beta$ such that $c_{ij} > 0$ at $a+$, which implies

$$u_j' = 0 \geq \sum_l c_{jl} u_l \geq c_{jk} u_k > 0 \quad \text{at } a+ .$$

This contradiction shows that $\beta = \emptyset$. ∎

Now assume that the assumptions of Theorem 2 hold and that the set β in the conclusion is not empty. We write $x = (x_\alpha, x_\beta)$, $v(t) = (v_\alpha(t), v_\beta(t)), \ldots$ with an obvious meaning. Let $\delta = \min\{\delta_j : j \in \beta\}$. Then $v_\beta = w_\beta$ in $J_\delta = [a, a+\delta]$, and $Pv \leq Pw$ implies

$$f_\beta(t, v_\alpha, v_\beta) \geq f_\beta(t, w_\alpha, w_\beta).$$

But from quasimonotonicity and $v \leq w$ we get $f_\beta(t, v_\alpha, v_\beta) \leq f_\beta(t, w_\alpha, v_\beta)$ and hence

$$(5) \qquad f_\beta(t, v_\alpha, v_\beta) = f_\beta(t, w_\alpha, v_\beta) \quad \text{and} \quad v_\alpha < w_\alpha \quad \text{in } J_\delta = (a, a+\delta],$$

which implies, by the way, that $P_\beta v = P_\beta w$ in J_δ. So, in order to obtain $\beta = \emptyset$, we must add an assumption which is incompatible with (5).

THEOREM 4. *Suppose $v(t)$, $w(t)$ and f satisfy the assumptions of Theorem 2. Each of the following conditions is sufficient for the strong inequality $v < w$ in J_0:*

(i) $v(a) < w(a)$;

(ii) $Pv < Pw$ *at* $a+$;

(iii) *For every pair* (α, β) *of nonempty, disjoint index sets with* $\alpha \cup \beta = N$ *there are* $j \in \beta$, $k \in \alpha$ *such that* $f_j(t, x)$ *is strictly increasing in* x_k *for* $t \in M$, $x \in U$, *where* $M \subset J$ *is dense at* a *and* U *is a neighborhood of* $v(a)$.

Proof. It is obvious that β is empty in cases (i), (ii). In case (iii) we use the notation $x = (x_k, \widetilde{x}_k)$ with $\widetilde{x}_k \in \mathbb{R}^{n-1}$. There is $\delta > 0$ such that for $a < t \le \delta$ and $0 < s < \delta$,

$$v(t) \in U, \quad v^s(t) := (v_k(t) + s, \widetilde{v}_k(t)) \in U$$

and $v_k(t) + s \le w_k(t)$, which implies $v^s(t) \le w(t)$. It follows from quasimonotonicity of f and the strict monotonicity of f_j that

$$f_j(t, v(t)) < f_j(t, v^s(t)) \le f_j(t, w(t)) \quad \text{for } t \in M.$$

This is a contradiction to (5). ■

Remarks. 1. If $f(t, x)$ is of class C^1 with respect to x, then (iii) follows from

(iii′) The Jacobian $\partial f(t, x)/\partial x$ is irreducible for $t \in M$, $x \in U(v(a))$.

2. In Theorem 3 it was assumed that f is defined in $J \times \mathbb{R}^n$. If f is only defined in a set $G = J \times D$, where $D \subset \mathbb{R}^n$ is open, then it is naturally assumed that $\operatorname{graph} v, \operatorname{graph} w \subset G$. But in the representation of $\Delta f = f(t, w(t)) - f(t, v(t))$ as a sum of differences (cf. the proof of Theorem 2), the auxiliary points must also belong to G. This is the case if D is an open n-dimensional interval. Yet convexity of D suffices. The proof runs as follows. Let $x, y \in D$, $h = (y - x)/p$ and $x^k = x + kh$ $(k = 0, \ldots, p)$, in particular $x^0 = x$, $x^p = y$. Then $f(t, y) - f(t, x) = \sum_{k=1}^{p} [f(t, x^k) - f(t, x^{k-1})]$, and each of those differences can be treated as in the proof of Theorem 2 (the line segment $\overline{xy}$ has positive distance to the boundary of D, and for large p all auxiliary points are close to $\overline{xy}$).

3. The two definitions of quasimonotonicity in Section 3 are not always equivalent if f is only defined in $G = J \times D$. This fact was first observed by Ważewski [4] in 1950. He introduced two conditions (H) and (K): Condition (H) is the one given above by means of inequalities, Condition (K) requires that $f_i(t, x)$ is (weakly) increasing in x_j for $i \ne j$. Obviously (H) implies (K), but (K) implies (H) only if the set D has a certain property P which is described in [4]. For example, convex sets have this property. What is really needed in inequalities is always the form (H) first introduced by Ważewski. Let us remark that the general quasimonotonicity condition for differential equations in ordered Banach spaces (or topological vector spaces), which was given by Volkmann [2], is also of the (H) type.

4. We denote the solution of $y' = f(t,y)$, $y(a) = \eta$ by $y(t,\eta)$. Under the assumptions of Remark 1 the Jacobian $Z(t,\eta) = \partial y(t,\eta)/\partial\eta$ satisfies $Z_{ij}(t,\eta) > 0$ for $t \geq a$ and all $i,j \in N$. This follows from Theorems 1 and 3, since $Z(t,\eta)$ is a solution to the linear system

$$Z' = C(t)Z, \quad Z(0) = I, \quad \text{where } C(t) = \frac{\partial f}{\partial x}(t, y(t,\eta));$$

cf. [3], Theorem 13.X. The matrix $C(t)$ is essentially positive and irreducible. Note that the columns $z^i = \partial y(t,\eta)/\partial\eta_i$ satisfy $z_i' = C(t)z_i$, $z_i(0) \neq 0$ and hence are positive in J_0. This is a simple proof of Theorem 1.1 in [1].

References

[1] M. Hirsch, *Systems of differential equations that are competitive or cooperative, II: convergence almost everywhere*, SIAM J. Math. Anal. 16 (1985), 423–439.

[2] P. Volkmann, *Gewöhnliche Differentialgleichungen mit quasimonoton wachsenden Funktionen in topologischen Vektorräumen*, Math. Z. 127 (1972), 157–164.

[3] W. Walter, *Gewöhnliche Differentialgleichungen*, 5th ed., Springer, 1993.

[4] T. Ważewski, *Systèmes des équations et des inégalités différentielles ordinaires aux deuxièmes membres monotones et leurs applications*, Ann. Soc. Polon. Math. 23 (1950), 112–166.

Mathematisches Institut I
Universität Karlsruhe
D-76128 Karlsruhe, Germany
E-mail: walter@ma1ws1.mathematik.uni-karlsruhe.de

Reçu par la Rédaction le 21.8.1995