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Holomorphic bijections of algebraic sets

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Abstract. We prove that every holomorphic bijection of a quasi-projective algebraic set onto itself is a biholomorphism. This solves the problem posed in [CR].

1. Introduction. It is well known that every injective endomorphism of an algebraic space over an algebraically closed field is an automorphism (see [CR] for the case of affine varieties and [N] for the general case). On the other hand, the authors proved in [CR] that there exists an analytic curve in $\mathbb{C}^6$ and its holomorphic bijection which is not biholomorphic.

In this context the question formulated in [CR] whether each holomorphic bijective self-transformation of an algebraic set has a holomorphic inverse seems to be interesting.

The aim of this paper is to answer this question. This is given by

Theorem 1. *Let X be a quasi-projective complex algebraic set and let $f : X \to X$ be a holomorphic bijection. Then the mapping $f^{-1} : X \to X$ is holomorphic.*

Our proof is essentially based on the recent result of the first author on singularities of weakly holomorphic (w-holomorphic) functions [C, Thm. 5.1]. We summarize here all the necessary information on those functions.

Let Y be a complex space. A complex-valued function g is said to be *w-holomorphic* on Y if there exists a nowhere dense analytic subset Z of Y such that g is defined and holomorphic on $Y \setminus Z$ and locally bounded near the set Z (for details see [W, Sect. 4.3]).

The set $S_g := \{x \in Y : g \text{ is not holomorphic at } x\}$ is called the *singular set* of g.

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If Y_1, Y_2 are complex spaces then by a *w-holomorphic mapping* from Y_1 into Y_2 we mean a holomorphic mapping $f : Y_1 \setminus Z \to Y_2$, where Z is a nowhere dense analytic subset, such that the closure $\bar{f}$ in $Y_1 \times Y_2$ of the graph of f is analytic and the projection of $\bar{f}$ onto Y_1 is a finite map (i.e. a proper map with finite fibers). It is easy to see that if Y_2 is an analytic subset of $\mathbb{C}^n$ then a mapping $f : Y_1 \setminus Z \to Y_2$ is *w-holomorphic* iff the components of f are w-holomorphic functions on Y_1. The notion of the *singular set* of a w-holomorphic mapping is defined in the same way as in the case of w-holomorphic functions.

Besides standard facts in analytic and algebraic geometry, the main points in the proof of Theorem 1 are Proposition 2.1 and the above-announced result from [C], which may be formulated as follows:

THEOREM 2. *Let X be a quasi-projective complex algebraic set. Then there exist algebraic subsets $V_1, \ldots, V_p$ of X such that for every w-holomorphic function g on X we have $S_g = V_i$.*

2. Local ring of an analytic subvariety. Let X be a complex space and let $V \subset X$ be an irreducible nowhere dense analytic subset of X. We shall denote by $\mathcal{O}_{X,V}$ the ring of all functions which are holomorphic on $X \setminus W$, where W is a nowhere dense analytic subset of X not containing V. We shall call $\mathcal{O}_{X,V}$ the *local ring* of X along V.

The ring $\mathcal{O}_{X,V}$ is local with the maximal ideal $\{f \in \mathcal{O}_{X,V} : f|V = 0\}$.

Let us point out that for $x_0 \in X$ the ring $\mathcal{O}_{X,\{x_0\}}$ does not coincide with the ring $\mathcal{O}_{X,x_0}$ of holomorphic germs at x_0.

PROPOSITION 2.1. *If X is locally a Stein space in the* (*analytic*) *Zariski topology then the ring $\mathcal{O}_{X,V}$ is noetherian.*

Proof. Suppose that the ring $\mathcal{O}_{X,V}$ is not noetherian and choose a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{O}_{X,V}$ such that $f_{i+1} \notin (f_1, \ldots, f_i), i = 1, 2, \ldots$ Let W_i be an analytic subset of X not containing V and such that the function f_i is holomorphic on $X \setminus W_i$.

Observe that $V \not\subset \bigcup_{i=1}^{\infty} W_i$ and fix a point $x_0 \in V \setminus \bigcup_{i=1}^{\infty} W_i$. The ring $\mathcal{O}_{X,x_0}$ is noetherian (see [Ł]), therefore there exists $k \in \mathbb{N}$ such that the germ $(f_{k+1})_{x_0}$ belongs to the ideal $((f_1)_{x_0}, \ldots, (f_k)_{x_0})$ of the ring $\mathcal{O}_{X,x_0}$.

Denote by $\mathcal{J}_k$ (resp. $\mathcal{J}_{k+1}$) the sheaf of ideals on $X \setminus \bigcup_{i=1}^{k+1} W_i$ defined by $(f_1, \ldots, f_k)$ (resp. $(f_1, \ldots, f_{k+1})$). Let $Z := \mathrm{Supp}(\mathcal{J}_{k+1}/\mathcal{J}_k)$. Then Z is an analytic subset of $X \setminus \bigcup_{i=1}^{k+1} W_i$ and $x_0 \notin Z$.

Take a Zariski open neighborhood X_0 of x_0 in $X \setminus (\bigcup_{i=1}^{k+1} W_i \cup Z)$ which is a Stein space. Then $(f_{k+1})_x \in ((f_1)_x, \ldots, (f_k)_x)$ for every $x \in X_0$ and by Cartan's Theorem B there exist functions $g_1, \ldots, g_k$, holomorphic on X_0, such that $f_{k+1} = f_1 g_1 + \ldots + f_k g_k$. Since $g_1, \ldots, g_k \in \mathcal{O}_{X,V}$ we have $f_{k+1} \in (f_1, \ldots, f_k)$, contrary to our assumption. This proves the proposition. ■

3. Proof of the main result. Before the proof of Theorem 1 we state two lemmas.

LEMMA 3.1. *Let V be an irreducible quasi-projective algebraic set and let $f : V \to V$ be a holomorphic injection such that for every $q \in \mathbb{N}$ the set $V \setminus f^q(V)$ is contained in a proper algebraic subset of V (f^q denotes the q-th iterate of f). Then the mapping f^{-1} is w-holomorphic on V.*

Proof. Let $\widehat{f} : \widehat{V} \to \widehat{V}$ be the lifting of f to an algebraic normalization $\widehat{V}$ of V. It is easy to see that it suffices to show that the mapping $\widehat{f}$ is biholomorphic.

Let W_q denote the closure of $\widehat{V} \setminus \widehat{f}^q(\widehat{V})$ in the (algebraic) Zariski topology on $\widehat{V}$. Then the set W_q is either empty or of pure dimension $d-1$, where $d := \dim V$, and the mapping $\widehat{f}^{-q} : \widehat{V} \setminus W_q \to \widehat{V}$ is holomorphic. Let $W_q' := \operatorname{Reg} W_q \setminus \operatorname{Sing} V$. Using the Laurent expansion of any component of $\widehat{f}^{-q}$ near points of W_q' (in suitable local coordinates on $\widehat{V}$) and the identity principle we see that the set of points of W_q' such that the mapping $\widehat{f}^{-q}$ can be holomorphically continued through them is open and closed in W_q'. Therefore $\widehat{V} \setminus \widehat{f}^q(\widehat{V}) = W_q$.

Let $H^i_{\mathrm{c}}(T)$ denote the ith Alexander–Spanier cohomology group with compact supports of the locally compact topological space T, with complex coefficients, and let $H^*_{\mathrm{c}}(T)$ be the direct sum of all $H^i_{\mathrm{c}}(T)$. In our case $H^*_{\mathrm{c}}(\widehat{V})$ is a finite-dimensional vector space and by [B, Lemma 1.2] the sequence $(\dim H^*_{\mathrm{c}}(W_q))_{q\in\mathbb{N}}$ is uniformly bounded.

By [B, Lemma 2.2] the sequence $(W_q)_{q\in\mathbb{N}}$ is stationary. But $W_{q+1} = W_q \cup \widehat{f}^q(W_1)$, and $W_q \cap \widehat{f}^q(W_1) = \emptyset$, so $W_1 = \emptyset$, which concludes the proof. ■

LEMMA 3.2. *Let V be a quasi-projective algebraic set and let $f : V \to V$ be a holomorphic bijection. Then the mapping f^{-1} is w-holomorphic on V.*

Proof. Let $V = V_1 \cup \ldots \cup V_r$ be the decomposition of V into irreducible components. Since any irreducible component of V is mapped by f into an irreducible component and f is surjective, the mapping f permutes the components of V.

Fix a component V_k of V. Replacing, if necessary, the mapping f by its suitable iterate, we may assume that $f(V_k) \subset V_k$. Since f is bijective and $f(V_i) \subset \bigcup_{j\neq k} V_j$ for $i \neq k$, we see that $V_k \setminus f(V_k) \subset \bigcup_{j\neq k} V_j$ and the same holds for any iterate of f.

By Lemma 3.1 the function $f^{-1}|V_k$ is w-holomorphic on V_k. Therefore the mapping f^{-1} is w-holomorphic on V. ■

Proof of Theorem 1. Assume that X is a quasi-projective algebraic set and $f : X \to X$ is a holomorphic bijection. Then, by Lemma 3.2, the mapping f^{-1} is w-holomorphic on X.

Denote by S_ν the singular set of the w-holomorphic mapping $f^{-\nu}$, for $\nu = 1, 2, \ldots$ Of course $S_\nu \subset S_{\nu+1}$ and $f(S_\nu) \subset S_{\nu+1}$. By Theorem 2, the sequence $(S_\nu)_{\nu\in\mathbb{N}}$ is stationary, i.e. $S_\nu = S$ for $\nu \geq \nu_0$ with S algebraic and $f(S) \subset S$.

Let V be an irreducible component of S. Without loss of generality we may assume that $f(V) \subset V$. Let D be a universal denominator on X (see [Ł]) and let, for any positive integer ν,

$$I_\nu := \{D \cdot (h \circ f^{-\nu}) \in \mathcal{O}_{X,V} : h \in \mathcal{O}_{X,V}\}.$$

These are ideals in the ring $\mathcal{O}_{X,V}$ and $I_\nu \subset I_{\nu+1}$. By Proposition 2.1 there exists $n_0 \geq \nu_0$ such that $I_n = I_{n_0}$ for $n \geq n_0$.

Choose an affine part $\widetilde{X} \subset \mathbb{C}^d$ of the set X in the manner that $\widetilde{X}$ contains a dense subset of V. Then, if $z_1, \ldots, z_d$ are the coordinate functions on $\widetilde{X}$, we have

$$D \cdot (z_i \circ f^{-2n_0}) = D \cdot (h_i \circ f^{-n_0})$$

for some $h_1, \ldots, h_d \in \mathcal{O}_{X,V}$.

Hence we get $z_i \circ f^{-n_0} = h_i$. This means that the mapping f^{-n_0} is holomorphic at the generic point of V. Therefore the set V is empty and $f^{-1} : X \to X$ is a holomorphic mapping. ∎

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