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# ESTIMATES FOR SIMPLE RANDOM WALKS ON FUNDAMENTAL GROUPS OF SURFACES

BY

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Numerical estimates are given for the spectral radius of simple random walks on Cayley graphs. Emphasis is on the case of the fundamental group of a closed surface, for the usual system of generators.

**Introduction.** Let $X$ be a connected graph, with vertex set $X^0$. We denote by $k_x$ the number of neighbours of a vertex $x \in X^0$. The *Markov operator* $M_X$ of $X$ is defined on functions on $X^0$ by

$$(M_X f)(x) = \frac{1}{k_x} \sum_{y \sim x} f(y), \qquad f : X^0 \to \mathbb{C},\ x \in X^0,$$

where the summation is taken over all neighbours $y$ of $x$ (we assume that $1 \leq k_x < \infty$ for all $x \in X^0$).

If $X$ is a *regular graph,* i.e. if $k_x = k$ is independent of $x \in X^0$, this operator induces a bounded self-adjoint operator on the Hilbert space $\ell^2(X^0)$, again denoted by $M_X$. The *spectral radius* $\mu(X)$ of the graph $X$ is the norm of this bounded operator. It is also a measure of the asymptotic probability for a path of length $n$ in $X$ to be closed, and has several other interesting interpretations (see e.g. [Woe]). This carries over to the case of a not necessarily regular graph, but the definition of the appropriate Hilbert space is slightly more complicated (see again [Woe], Section 4.B).

Let $\Gamma$ be a group generated by a finite set $S$ which is symmetric ($s \in S \Leftrightarrow s^{-1} \in S$) and which does not contain the unit element $1 \in \Gamma$. Denote by $\mathrm{Cay}(\Gamma, S)$ the *Cayley graph* with vertex set $X^0 = \Gamma$ and, for $x, y \in \Gamma$, with $\{x, y\}$ an edge if $x^{-1}y \in S$. We denote by $\mu(\Gamma, S)$ the spectral radius of the graph $\mathrm{Cay}(\Gamma, S)$.

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Let us recall two important results due to Kesten [Ke1], [Ke2]. The first one is the relation

$$\frac{2\sqrt{k-1}}{k} \le \mu(\Gamma, S) \le 1$$

with equality on the right if and only if $\Gamma$ is amenable ($k$ is the number of generators in $S$). For the second one let us assume (for simplicity) that $\Gamma$ does not have any element of order 2, so that $k = 2h$ for some integer $h \ge 1$; assume also (again for simplicity) that $h \ge 2$. Then one has the equality

$$\frac{\sqrt{2h-1}}{h} = \frac{2\sqrt{k-1}}{k} = \mu(\Gamma, S)$$

if and only if $\Gamma$ is a free group on a set $S_+ = \{s_1, \ldots, s_h\}$ such that $S = S_+ \amalg S_+^{-1}$ (where $\amalg$ indicates a disjoint union).

There are few examples of exact *computations* of $\mu(\Gamma, S)$ for non-amenable groups. Most of those we are aware of are for groups which contain free subgroups of finite index, even if there are a few known cases beyond these "almost free" groups (see e.g. [Car, Theorem 2] and [CaM]). One direction for further progress is to find good *estimates* for new classes of examples.

As a test case, we consider here the fundamental group of an orientable closed surface of genus $g \ge 2$, namely the group $\Gamma_g$ given by the presentation

$$\Gamma_g = \left\langle a_1, b_1, \ldots, a_g, b_g \;\middle|\; \prod_{j=1}^{g} a_j b_j a_j^{-1} b_j^{-1} = 1 \right\rangle$$

and the generating set

$$S_g = \{a_1, a_1^{-1}, b_1, b_1^{-1}, \ldots, a_g, a_g^{-1}, b_g, b_g^{-1}\}$$

with $k = 4g$ elements; the resulting Cayley graph is denoted by $X_g$.

Setting $\mu_g = \mu(X_g) = \mu(\Gamma_g, S_g)$, one has

$$\frac{\sqrt{4g-1}}{2g} < \mu_g < 1$$

by Kesten's estimates recalled above. In particular,

$$0.6614 \approx \frac{\sqrt{7}}{4} < \mu_2 < 1$$

when $g = 2$. As $\Gamma_g$ has $2g$ generators and as $X_g$ has cycles of length $4g$, the previous estimate may be improved to

$$\frac{\sqrt{4g-1}}{2g} + \frac{4 - 2\sqrt{3}}{(4g+2)(4g)^{4g+2}} \le \mu_g < 1$$

(see Formula (4.15) in [Kes]), which gives for $g = 2$ an improvement of order $5 \times 10^{-11}$. There is a better result due to Paschke, for which the improvement is about $1.75 \times 10^{-4}$ [Pas].

In Section 1 below, we present a very simple method based on an observation of O. Gabber to show that

$$\mu_g \leq \frac{\sqrt{2g-1}}{g} \quad \text{and in particular} \quad \mu_2 \leq \frac{\sqrt{3}}{2} \approx 0.8660.$$

Section 2 records a computation with Poisson kernels; though it is in our view the most interesting part of the present work, its numerical outcome so far is limited to the inequality

$$\mu_2 \leq 0.7675$$

and to similar inequalities for other small values of $g$. Section 3 uses embedding of trees in graphs to improve the results of Section 1; more precisely, one has

$$\mu_g \leq \frac{\sqrt{4g-2}}{2g} + \frac{1}{4g} \quad \text{and in particular} \quad \mu_2 \leq \frac{\sqrt{6}}{4} + \frac{1}{8} \approx 0.7373.$$

(One can extend much of Sections 1 and 3 to $C'(1/6)$ small cancellation groups and to one relator groups.) It follows from Section 3 and from Kesten's result that

$$\mu_g = g^{-1/2} + O(g^{-1})$$

for large $g$.

Our numerical results for $g \leq 10$ are summarized in the following table.

| genus | Kesten | Section 1 | Section 2 | Section 2 | Section 3 |
|---|---|---|---|---|---|
| $g$ | $\frac{\sqrt{4g-1}}{2g}$ | $\frac{\sqrt{2g-1}}{g}$ | $\nu$ | $1-\alpha$ | $\frac{\sqrt{4g-2}}{2g} + \frac{1}{4g}$ |
| 2 | .6614 | .8660 | .2990 | .7675 | .7373 |
| 3 | .5529 | .7453 | .2944 | .6588 | .6104 |
| 4 | .4841 | .6615 | .2932 | .5872 | .5303 |
| 5 | .4359 | .6000 | .2926 | .5352 | .4742 |
| 6 | .3997 | .5529 | .2920 | .4953 | .4325 |
| 7 | .3712 | .5153 | .2916 | .4633 | .3999 |
| 8 | .3480 | .4841 | .2912 | .4369 | .3736 |
| 9 | .3287 | .4581 | .2908 | .4147 | .3518 |
| 10 | .3123 | .4359 | .2905 | .3956 | .3332 |

For example, for $g = 3$, one has the lower bound $\mu_3 \geq 0.5529$ (Kesten) and the upper bounds

$$\mu_3 \leq \frac{\sqrt{5}}{3} \approx 0.7453 \qquad \text{(method of Section 1)},$$

$$\mu_3 \leq 0.6588 \qquad \text{(method of Section 2 with } \nu = 0.2944\text{)},$$

$$\mu_3 \leq \frac{\sqrt{10}}{6} + \frac{1}{12} \approx 0.6104 \qquad \text{(method of Section 3)}.$$

After completion of this work, the method of Section 1 has been improved by A. Żuk [Żuk], who has shown in particular that

$$\mu_g < 1/\sqrt{g}$$

for all $g \geq 2$, and again by T. Nagnibeda [Nag], who has shown in particular that

$$\mu_2 \leq 0.6629.$$

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**1. Upper bounds from discrete 1-forms.** Let $X$ be a graph with vertex set $X^0$ and with edge set $X^1$. Denote by $\mathbb{X}^1$ the set of *oriented* edges of $X$ (if $X$ is finite, then $|\mathbb{X}^1| = 2|X^1|$). For each $e \in \mathbb{X}^1$ we denote by $\overline{e}$ the oriented edge obtained from $e$ by reversing the orientation. A 1-*form* on $X$ with values in some group $G$ is a map $\omega : \mathbb{X}^1 \to G$ such that $\omega(\overline{e}) = \omega(e)^{-1}$ for all $e \in \mathbb{X}^1$. We denote by $\mathbb{R}_+^*$ the multiplicative group $]0, \infty[$.

The following proposition is due to O. Gabber. It can be found in [CdV] (with the proof below) and its corollary in [ChV] (with a different proof).

Proposition 1. *Let $X$ be a regular graph of degree $k$. Suppose there exists a 1-form $\omega : \mathbb{X}^1 \to \mathbb{R}_+^*$ and a constant $c > 0$ such that*

$$\frac{1}{k} \sum_{e \in \mathbb{X}^1, e_+ = x} \omega(e) \leq c$$

*for all $x \in X^0$. Then*

$$\mu(X) \leq c.$$

(The summation is over all oriented edges $e$ heading to the vertex $x$.)

Corollary 1. *One has*

$$\mu_g \leq \frac{\sqrt{2g-1}}{g}$$

*for all $g \geq 2$. In particular*,

$$\mu_2 \leq \frac{\sqrt{3}}{2} \approx 0.8660.$$

Proof of Corollary 1. As the only relation in the chosen presentation of $\Gamma_g$ has even length, any edge $e$ in the Cayley graph $X_g$ of $(\Gamma_g, S_g)$ joins two vertices $e_+, e_-$ at different distances from the vertex 1. Let $d(x, y)$ denote the combinatorial distance in a graph between two vertices $x, y$, and write $\ell(x)$ for $d(1, x)$. For a number $b \geq 1$ (to be made precise below), one may thus define a 1-form on $X_g$ by

$$\omega(e) = \begin{cases} b^{-1} & \text{if } \ell(e_+) < \ell(e_-), \\ b & \text{if } \ell(e_+) > \ell(e_-). \end{cases}$$

Say that a vertex $x$ in $X_g$ is *of type $t$* if the set

$$\{y \in X_g \mid d(y,x) = 1 \text{ and } \ell(y) = \ell(x) - 1\}$$

is of cardinality $t$. For example $x$ is of type 1 if $0 < \ell(x) < 2g$, and $x$ is of type 2 if $x$ is at distance $2g$ from 1 on a $4g$-gon containing 1. It follows from the definition that 1 is the only vertex of type 0.

It is a fact that any other vertex is either of type 1 or of type 2. This is well known and goes back to M. Dehn (or H. Poincaré?); it is for example a straightforward consequence of Lemma 2.2 in [Ser]. Compare with [Can] and [Wag]; note, however, that a vertex is type 1 [respectively type 2] in our sense if and only if its Cannon type is in $\{1, \ldots, 2g-1\}$ [resp. is $2g$]. For convenience of the reader, we give a proof of the fact we use in Appendix A below.

One has

$$\sum_{e \in \mathbb{X}^1, e_+ = x} \omega(e) = \begin{cases} 4gb^{-1} & \text{if } x = 1 \text{ (type 0)}, \\ (4g-1)b^{-1} + b & \text{if } x \text{ is of type 1}, \\ (4g-2)b^{-1} + 2b & \text{if } x \text{ is of type 2}, \end{cases}$$

and Proposition 1 applies with

$$c = \frac{(4g-2)b^{-1} + 2b}{k}.$$

To minimize $c$, one sets $b = \sqrt{2g-1}$, so that

$$c = \frac{4\sqrt{2g-1}}{4g}. \quad \blacksquare$$

P r o o f o f P r o p o s i t i o n 1. Let $f \in \ell^2(X^0)$. Choose $e \in \mathbb{X}^1$; set $x = e_+$ and $y = e_-$. From

$$\left( \sqrt{\omega(e)}|f(x)| - \frac{1}{\sqrt{\omega(e)}}|f(y)| \right)^2 \geq 0$$

one has

$$2|f(x)| \cdot |f(y)| \ \leq \ \omega(e)|f(x)|^2 + \omega(\overline{e})|f(y)|^2.$$

Summing over $e \in \mathbb{X}^1$ one obtains

$$\begin{aligned} 2 \sum_{x \in X^0} |f(x)| \sum_{e \in \mathbb{X}^1, e_+ = x} |f(e_-)| & \\ \leq \sum_{x \in X^0} |f(x)|^2 \sum_{e \in \mathbb{X}^1, e_+ = x} \omega(e) &+ \sum_{y \in X^0} |f(y)|^2 \sum_{e \in \mathbb{X}^1, \overline{e}_+ = y} \omega(\overline{e}) \end{aligned}$$

and

$$2k|\langle f \mid M_X f \rangle| = 2k \Big| \sum_{x \in X^0} \overline{f(x)} (M_X f)(x) \Big| \leq 2kc \, \|f\|^2 .$$

As this holds for all $f \in \ell^2(X^0)$, and as the operator $M_X$ on $\ell^2(X^0)$ is self-adjoint, one has $\|M_X\| \leq c$ and the conclusion follows. ∎

GENERALIZATION. *Let $\Gamma = \langle S_+ \mid R \rangle$ be a group presentation satisfying a small cancellation hypothesis $C'(1/6)$. If $h \doteq |S_+| \geq 2$ and if $S = S_+ \cup (S_+)^{-1}$, one has*

$$\mu(\Gamma, S) \leq \frac{2\sqrt{h-1}}{h}.$$

Proof. One has $|S| = 2h$ because small cancellation groups cannot have elements of order 2 (see e.g. Section V.4 in [LyS]). Types being defined as in the proof of Corollary 1, it is known that any vertex distinct from the identity in the Cayley graph of $(\Gamma, S)$ is either of type 1 or of type 2 (lemme 4.19 in [Cha]). Defining a 1-form $\omega$ on this Cayley graph by

$$\omega(e) = \begin{cases} b^{-1} & \text{if } \ell(e_+) < \ell(e_-), \\ 1 & \text{if } \ell(e_+) = \ell(e_-), \\ b & \text{if } \ell(e_+) > \ell(e_-), \end{cases}$$

one may apply verbatim the argument of Corollary 1. ∎

**2. Upper bounds from Poisson kernels.** Let again $X = \mathrm{Cay}(\Gamma, S)$ be as in the introduction and let $M_X$ be the corresponding Markov operator. The *combinatorial Laplacian* of $X$ is defined to be

$$\Delta_X = 1 - M_X.$$

Let $\alpha \in \mathbb{R}$; a function $f : \Gamma \to [0, \infty[$ is said to be *$\alpha$-superharmonic* if $f \neq 0$ and if $\Delta_X f \geq \alpha f$. (If there exists such a function $f$, one has $f \geq \Delta_X f \geq \alpha f$ and consequently $\alpha \leq 1$. One may also show that $f(\gamma) > 0$ for all $\gamma \in \Gamma$.) The function is said to be *$\alpha$-harmonic* if moreover $\Delta_X f = \alpha f$.

PROPOSITION 2. *Let $\alpha \in \mathbb{R}$. The following are equivalent.*

(i) $\alpha \leq 1 - \mu(X) = \inf\{$*spectrum of $\Delta_X$ on the Hilbert space $\ell^2(\Gamma)$*$\}$.
(ii) *There exists a function $f : \Gamma \to [0, \infty[$ which is $\alpha$-superharmonic.*
(iii) *There exists a function $f : \Gamma \to [0, \infty[$ which is $\alpha$-harmonic.*

There is one proof in terms of graphs in [DoK, Proposition 1.5]. But there are earlier proofs in the literature on irreducible stationary discrete Markov chains; the equivalence of (i) and (ii) is standard; the equivalence with (iii) is more delicate (see [Har] and [Pru]).

COROLLARY 2. *One has $\mu_2 \leq 0.784$. More generally, upper estimates for $\mu_g$ and small $g$'s are given by the table in the introduction.*

We begin the proof of Corollary 2 with the following lemma.

Lemma 1. *Let $g$ be an integer*, $g \geq 2$. *Set*

$$D_g = 2 \operatorname{arccosh}\left(\cot \frac{\pi}{4g}\right) \tag{1}$$

*For* $\phi \in [0, 2\pi[$, *set*

$$b(\varrho, \phi) = \frac{1}{\cosh \varrho - \sinh \varrho \cos \phi} \tag{2}$$

*for all* $\varrho > 0$ *and*

$$F_g(\nu, \phi) = \frac{1}{4g} \sum_{j=0}^{4g-1} \left\{ b\left(D_g, \phi + j\frac{2\pi}{4g}\right)\right\}^{\nu} \tag{3}$$

*for all* $\nu \in \mathbb{R}$. *Then*

$$\mu_g \leq \max_{0 \leq \phi < 2\pi} F_g(\nu, \phi)$$

*for all* $\nu \in \mathbb{R}$.

Proof. First step: *definition of a function* $f_\nu$. Let $H^2$ be the hyperbolic plane.

There is a free discrete isometric action of $\Gamma_g$ on $H^2$ and a point $z_0 \in H^2$ such that the Dirichlet cells of the orbit $\Gamma_g z_0$ constitute a tesselation of $H^2$ by regular $4g$-gons with all inner angles equal to $\pi/(2g)$. There is consequently an embedding of the graph $X_g = \mathrm{Cay}(\Gamma_g, S_g)$ in $H^2$, with vertices of the graph corresponding to points of the orbit $\Gamma_g z_0$ and edges corresponding to pairs of adjacent Dirichlet cells. Trigonometric computations for a hyperbolic triangle with angles $\pi/2$, $\pi/(4g)$, $\pi/(4g)$ show that $D_g$ in (1) is the distance between the centres of two adjacent Dirichlet cells.

Let $\omega_0 \in \partial H^2$ be a point at infinity. Let $P : H^2 \to ]0, \infty[$ be the function given by the value at $\omega_0$ of the Poisson kernel. For computations we choose

$$H^2 = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\} \quad \text{and} \quad \omega_0 = \infty i \quad \text{so that} \quad P(x + iy) = y. \tag{4}$$

Let $\Delta_H$ be the hyperbolic Laplacian on $H^2$. One has

$$\Delta_H P^\nu = -\nu(\nu - 1)P^\nu$$

for all $\nu \in \mathbb{R}$. (We have chosen a *positive* Laplacian $\Delta_H$. This implies that the spectrum of the corresponding self-adjoint operator on the Hilbert space $L^2(H^2, y^{-2}dxdy)$ is $[1/4, \infty[$. The equality $\Delta_H P^\nu = -\nu(\nu-1)P^\nu$ shows that there exist $\alpha$-harmonic functions for $\Delta_H$ for all $\alpha \leq 1/4$, in accordance with an analogue for $\Delta_H$ of the previous proposition. Much more on this in [Sul].)

We define

$$f_\nu : \Gamma_g \to ]0, \infty[$$

by $f_\nu(\gamma) = P^\nu(\gamma z_0)$. For $\gamma \in \Gamma$, let $z_{\gamma,j}$ $(0 \le j \le 4g-1)$ denote the centres of the Dirichlet cells adjacent to the Dirichlet cell centred at $\gamma z_0$. One has

$$(\Delta_X f_\nu)(\gamma) = P^\nu(\gamma z_0) - \frac{1}{4g}\sum_{j=0}^{4g-1} P^\nu(z_{\gamma,j})$$

for each $\gamma \in \Gamma$. The strategy of the proof is to find some $\alpha \in \mathbb{R}$ such that $\Delta_X f_\nu \ge \alpha f_\nu$, and to deduce from the previous proposition that $\mu_g \le 1-\alpha$.

Second step: *lower estimate for $\Delta_X f_\nu$.* For $z \in H^2, \varrho > 0$ and $\phi \in [0, 2\pi[$, let $z(\varrho,\phi) \in H^2$ be the point at hyperbolic distance $\varrho$ from $z$ for which the oriented angle between the geodesic ray $\overrightarrow{z_0,\omega_0}$ and the geodesic segment $\overrightarrow{z_0, z(\varrho,\phi)}$ is $\phi$. Set

$$c_g(\nu,\varrho,\phi,z) = \frac{P^\nu(z) - \frac{1}{4g}\sum_{j=0}^{4g-1} P^\nu\Big(z\Big(\varrho,\phi + j\frac{2\pi}{4g}\Big)\Big)}{P^\nu(z)}. \tag{5}$$

Observe that there is one well-defined value $\phi_\gamma \in [0, 2\pi/(4g)[$ such that

$$(\Delta_X f_\nu)(\gamma) = c_g(\nu, D_g, \phi_\gamma, \gamma z_0) f_\nu(\gamma)$$

for each $\gamma \in \Gamma$. But computing the angles $\phi_\gamma$ is a difficult task, and we rather look for an estimate of the right-hand side in the inequality

$$\Delta_X f_\nu \ge \big(\min_{\substack{0\le\phi<2\pi\\ z\in H^2}} c_g(\nu, D_g, \phi, z)\big) f_\nu.$$

Now (5) shows that $c_g(\nu,\varrho,\phi,z)$ depends neither on the real part of $z$, because $P(x+iy) = y$ for all $x \in \mathbb{R}$, nor on the imaginary part of $z$, because $P^\nu(\lambda z) = \lambda^\nu P^\nu(z)$ for all $\lambda > 0$. Thus one has

$$\Delta_X f_\nu \ge \big(\min_{0\le\phi<2\pi} c_g(\nu, D_g, \phi, z_0)\big) f_\nu.$$

Choosing moreover $z_0 = i$, one has

$$P(z_0) = 1$$

and

$$c_g(\nu, D_g, \phi, z_0) = 1 - \frac{1}{4g}\sum_{j=0}^{4g-1}\Big\{\Im\Big(z_0\Big(D_g, \phi + j\frac{2\pi}{4g}\Big)\Big)\Big\}^\nu$$

by (5).

Third step: *computation of $\Im(z_0(\varrho,\phi))$.* Let $\mathcal{C}$ be a hyperbolic circle of hyperbolic radius $\varrho$ centred at the point $z_0 = i$ of the Poincaré half-plane. The Cartesian coordinates $(a,b)$ of a point on $\mathcal{C}$ satisfy

$$a^2 + (b - \cosh\varrho)^2 = (\sinh\varrho)^2. \tag{6}$$

For each $\phi \in ]-\pi, \pi[$, let $\mathcal{C}_\phi$ be the hyperbolic geodesic through $z_0$ defining at this point an angle $\phi$ with the vertical axis. The Cartesian coordinates of a point on $\mathcal{C}_\phi$ satisfy

$$\left(a - \frac{1}{\tan\phi}\right)^2 + b^2 = 1 + \frac{1}{\tan^2\phi}. \tag{7}$$

Let us compute the second coordinates of the two points of $\mathcal{C} \cap \mathcal{C}_\phi$ (see Figure 1). Subtracting (7) from (6), one finds

$$\frac{a}{\tan\phi} - b\cosh\varrho = -1$$

and inserting this in (7) one obtains

$$(\cosh^2\varrho\tan^2\phi + 1)b^2 - 2(\cosh\varrho\ (\tan^2\phi + 1))b + 1 + \tan^2\phi = 0.$$

Straightforward manipulations show that

$$(\cosh\varrho\ (\tan^2\phi + 1))^2 - (\cosh^2\varrho\ \tan^2\phi + 1)(1 + \tan^2\phi) = \left(\frac{\sinh\varrho}{\cos\phi}\right)^2$$

and consequently that

$$b = \frac{\cosh\varrho(\tan^2\phi + 1) \pm \frac{\sinh\varrho}{cos\phi}}{\cosh^2\varrho\tan^2\phi + 1} = \frac{\cosh\varrho \pm \sinh\varrho\cos\phi}{\cosh^2\varrho\sin^2\phi + \cos^2\phi} = \frac{1}{\cosh\varrho \mp \sinh\varrho\cos\phi}. \tag{8}$$

Thus one has

$$\Im(z_0(\varrho,\phi)) = \frac{1}{\cosh\varrho - \sinh\varrho\cos\phi} = b(\varrho,\phi)$$

where the last equality is (2). (The other sign in (8) would give $b(\varrho, \phi + \pi)$.)

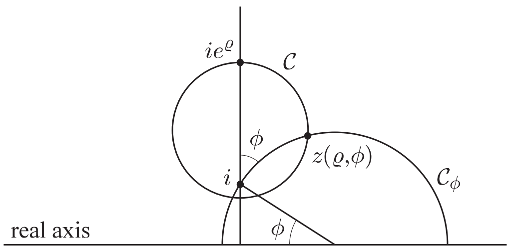


Fig. 1

F o u r t h s t e p: *coda*. The previous computations show that one has

$$\Delta_X f_\nu \geq \alpha f_\nu$$

for

$$\alpha = \min_{0\leq\phi<2\pi}\{1 - F_g(\nu,\phi)\}$$

where $F_g$ is defined in (3). As $\mu_g \le 1-\alpha$ by Proposition 2, this ends the proof. ∎

At this point, the problem is to compute $\inf_\nu \max_\phi F_g(\nu,\phi)$. One could use just here a computer system such as MAPLE and obtain a table of numerical results. However, we rather adopt the following program.

A first step consists of a lemma of calculus showing that, for any $\nu \in [0,1]$, the function $\phi \mapsto F_g(\nu,\phi)$ reaches its maximum at $\phi = 0$. (This at least for $g \le 27$; we have not found a reasonably short proof working for all $g$.) This is stated below, and proved in Appendix B at the end of our paper.

Only in a second step do we use a computer, first to find an efficient value of $\nu$ (which turns out to be near 0.3 for all $g$) and then to compute $F_g(\nu,0)$ for this $\nu$, so that one has a numerical estimate

$$\mu_g \le F_g(\nu,0)$$

for the spectral radius of $\mu_g = \mu(\mathrm{Cay}(\Gamma_g, S_g))$.

For $g$ and $\nu$ fixed, the function $\phi \mapsto 4gF_g(\nu,\phi)$ is a sum of a function

$$\beta : \phi \mapsto (\cosh(D_g) - \sinh(D_g)\cos\phi)^{-\nu}$$

and of $4g-1$ translates of $\beta$. It is straightforward to check that $\dot\beta(0) = 0$ and $\ddot\beta(0) < 0$, so that $\beta$ has a local maximum at the origin. The purpose of Lemma 2 (which is proved in Appendix B) is to show that this local maximum is strong enough for $\phi \mapsto F_g(\nu,\phi)$ to have an absolute maximum at the origin.

LEMMA 2. *For* $2 \le g \le 27$ *and* $0 \le \nu \le 1$ *one has*

$$\max_{0\le\phi\le 2\pi} F_g(\nu,\phi) = F_g(\nu,0).$$

*Thus, for these g's,*

$$\mu_g \le F_g(\nu,0)$$

*for all* $\nu \in [0,1]$, *by Lemma* 1.

End of proof of Corollary 2. Thanks to the previous lemma, we may consider the function

$$\nu \mapsto F_g(\nu,0) = \frac{1}{4g}\sum_{j=0}^{4g-1} \beta\left(j\frac{2\pi}{4g}\right),$$

and compute its minimum over $0 \le \nu \le 1$, yielding an upper bound for $\mu_g$. The computer algebra program MAPLE was used here, giving for $g \le 10$ the values of the table in the introduction. ∎

**3. Upper bounds from regular subtrees.** Let $X$ be a regular graph of degree $k$, as in Section 1. Assume that there is a subgraph $Y$ of $X$ which is spanning (that is, which contains all vertices of $X$) and which is regular of degree $l$ for some $l \in \{2, \ldots, k-1\}$ (we assume $k \geq 3$). The Markov operators $M_X$ and $M_Y$ act on the same space $\ell^2(X^0) = \ell^2(Y^0)$. One has

$$\begin{aligned}(M_X f)(x) &= \frac{1}{k}\Big\{ \sum_{e \in \mathbb{Y}^1, e_+ = x} f(e_-) + \sum_{e \in \mathbb{X}^1 \setminus \mathbb{Y}^1, e_+ = x} f(e_-) \Big\} \\ &= \frac{l}{k}(M_Y f)(x) + \frac{1}{k} \sum_{e \in \mathbb{X}^1 \setminus \mathbb{Y}^1, e_+ = x} f(e_-)\end{aligned}$$

so that

$$\|M_X\| \leq \frac{l}{k}\,\|M_Y\| + \frac{k-l}{k}.$$

In case $Y$ is a disjoint union of regular trees, $\|M_Y\|$ is explicitly known from Kesten's computations and one has the following.

PROPOSITION 3. *Let $X$ be a regular graph of degree $k \geq 3$ and let $Y$ be a spanning subgraph of $X$ which is a disjoint union of regular trees of degree $l$, for some $l \in \{2, \ldots, k-1\}$. Then*

$$\frac{2\sqrt{k-1}}{k} \leq \|M_X\| \leq \frac{2\sqrt{l-1}}{k} + \frac{k-l}{k}.$$

LEMMA 3. *The graph $X_g$ contains a spanning subgraph $Y_g$ which is a disjoint union of regular trees of degree $4g-1$.*

Proof. Recall from Section 1 that $\ell(x)$ denotes the combinatorial distance in $X_g$ between a vertex $x$ and the base point 1, and from Appendix A that vertices in $X_g$ are shared amongst three *types* numbered $0, 1$ and 2. Recall also that

(a) two vertices of type 2 are at distance at least 3 from each other,
(b) any vertex $x$ of type 1 has a *convenient* neighbour $y \in X_g^0$ such that
- $\ell(y) = \ell(x) + 1$,
- $y$ is of type 1,
- all neighbours of $y$ in $X_g$ are of type 1

[indeed $x$ has at least $4g-2$ such neighbours].

The construction goes in two steps.

First step. Let $Z_g$ be the spanning subgraph of $X_g$ obtained from $X_g$ by erasing, for each vertex $x$ of type 2, one edge connecting $x$ to a neighbour $y$ of $x$ such that $\ell(y) = \ell(x) - 1$. (This edge is chosen arbitrarily from 2 candidates.) By (a) above, any vertex of type 1 has degree $4g-1$ or $4g$ in $Z_g$ and any vertex of type 2 has degree $4g-1$ in $Z_g$.

Second step. For each $k \geq -1$, define inductively a graph $Y_g^{(k)}$ as follows. First, set $Y_g^{(-1)} = Z_g$. Then, if $k \geq 0$, let $Y_g^{(k)}$ be a spanning subgraph of $X_g$ obtained from $Y_g^{(k-1)}$ by erasing, for each vertex $x$ with

$|x| = k$ which is of degree $4g$ in $Y_g^{(k-1)}$, one edge connecting $x$ to one of its convenient neighbours. (This edge is chosen arbitrarily from at least $4g-2$ candidates.) By (b) above, any vertex with $|x| \leq k$ in $Y_g^{(k)}$ is of degree $4g-1$.

Observe that, for all $l \geq k$, the graphs $Y_g^{(k)}$ and $Y_g^{(l)}$ coincide "in the ball defined by $|x| \leq k$". Thus one may set $Y_g = Y_g^{(\infty)}$; any vertex in $Y_g$ is of degree $4g-1$.

Let us check that $Y_g$ does not contain any circuit. For this, we will show that $Z_g$ has no circuit.

Observe that two neighbours in $Z_g$ are never at the same distance from 1 (because this is already so in $X_g$, a consequence of the relation defining the group $\Gamma_g$ being of *even* length). If there were a circuit in $Z_g$, it would contain a vertex $x$ at maximum distance, say $n$, from 1, and this $x$ would have two neighbours at distance $n-1$; in particular, $x$ would be of type 2; this is ruled out by the first step above.

Thus $Y_g$ is indeed a spanning forest of degree $4g-1$ in $X_g$.

Though this fact is not needed for what follows, let us observe that $Y_g$ has infinitely many connected components. Indeed, choose a vertex $x$ of type 1 and a convenient neighbour $y$ of $x$ such that the edge connecting $x$ to $y$ has been erased in the second step above; then any neighbour $z$ of $y$ in $Y_g$ is such that $\ell(z) = \ell(y) + 1$. Choose similarly a vertex $x' \neq x$ and a convenient neighbour $y'$, with the same properties as $x$ and $y$. Then $y$ and $y'$ are not in the same component of $Y_g$, because any path from $y$ to $y'$ in $Y_g$ should have a maximum strictly between $y$ and $y'$, and this is ruled out by the first step above.

There are infinitely many such $x$'s, because from (a) there are infinitely many vertices of type 1 and degree $4g$ in $Z_g$. ■

Remark. In another terminology, Lemma 3 shows that the set of edges of $X_g$ which are not edges of $Y_g$ constitute a *perfect matching* of $X_g$, also called a 1-*factor*.

COROLLARY 3. *One has*

$$\mu_g \leq \frac{\sqrt{4g-2}}{2g} + \frac{1}{4g}$$

*for all* $g \geq 2$. *In particular*,

$$\mu_2 \leq \frac{\sqrt{6}}{4} + \frac{1}{8} \approx 0.7373.$$

Proof. Immediate from Proposition 3 and Lemma 3. ■

*Comparison with Corollary* 1. Computations in this section are more efficient than computations of Section 1 (with discrete 1-forms), because

$$\frac{\sqrt{4g-2}}{2g} + \frac{1}{4g} < \frac{\sqrt{2g-1}}{g}$$

for all $g \geq 2$. But computations of Section 1 can be improved to beat the present ones [Nag]!

COROLLARY 4. *Let $\Gamma = \langle S_+ \mid R \rangle$ be a one-relator group, with $S_+ \subset \Gamma \setminus \{1\}$ of order $h \geq 2$. Then*

$$\frac{\sqrt{2h-1}}{h} < \mu(\Gamma, S) \leq \frac{\sqrt{2h-3}+1}{h}$$

*for $S = S_+ \cup (S_+)^{-1}$.*

Proof. Let $T_+$ be a subset obtained from $S_+$ by erasing one letter appearing in $R$ (we assume $R$ to be cyclically reduced). Then $T_+$ is free by the Dehn–Magnus' Freiheitssatz (see e.g. [ChM, Chapter II.5]). Set $T = T_+ \cup (T_+)^{-1}$. Let $Y$ be the spanning subgraph of the Cayley graph $\mathrm{Cay}(\Gamma, S)$ for which two vertices $x, y$ are connected by an edge whenever $xy^{-1} \in T$. As $T_+$ is free in $\Gamma$, the graph $Y$ is a disjoint union of regular trees of degree $2h-2$. The corollary follows from Proposition 3. ∎

**Appendix A: on planar graphs.** Let $X$ be a connected graph embedded in the plane, edges of $X$ being piecewise smooth curves which are pairwise disjoint (but for common vertices). If $X$ is infinite, we assume that the following *strong planarity* condition holds: for any simple closed curve in $X$, the corresponding bounded region of the plane (via the Jordan curve theorem) contains only *finitely many* vertices of $X$. A *face* of $X$ is the closure of a connected component of the complement of $X$ in the plane.

Let $d(x, y)$ denote the combinatorial distance between two vertices $x, y \in X^0$; let $x_0 \in X^0$ be a base point and set $\ell(x) = d(x_0, x)$. If $X$ is bipartite, two neighbouring vertices $x, y \in X^0$ are necessarily such that $|\ell(x) - \ell(y)| = 1$. Recall that the *type* $t(x)$ of a vertex $x \in X^0$ is *here* the number of neighbours $y$ of $x$ such that $\ell(y) < \ell(x)$. Observe that, for $x \in X^0$, one has $t(x) = 0$ if and only if $x = x_0$.

GEOMETRIC PROPOSITION. *Let $X$ be a strongly planar graph with base point $x_0 \in X^0$. Assume that $X$ is connected, bipartite, and satisfies the following conditions*:

(i) (large degree) *each vertex $x \in X^0$ has $k_x \geq 4$ neighbours in $X$*;
(ii) (large faces) *each face $F$ of $X$ contains $k_F \geq 4$ vertices of $X$*;
(iii) (no-sink-vertex) *each vertex $x \in X^0$ has at least one neighbour $y \in X^0$ such that $\ell(y) = \ell(x) + 1$.*

*Then $t(x) \leq 2$ for all $x \in X^0$.*

*Assume moreover that each face $F$ of $X$ contains $k_F \geq 8$ vertices of $X$. Then*

(a) *for two vertices $x, y$ of type $t(x) = t(y) = 2$, one has $d(x, y) \geq 3$,*

(b) *any vertex $x$ of type 1 has a neighbour $y \in X^0$ such that $d(x_0, y) = d(x_0, x) + 1$ and such that all neighbours of $y$ are also of type 1.*

Proof. We will make use of the following *maximum principle*: if $C$ is a simple closed curve in $X$ enclosing a bounded open region $R$ of the plane, then

$$\max_{x \in R \cap X^0} d(x_0, x) < \max_{y \in C \cap X^0} d(x_0, y).$$

To show this, consider a point $x' \in R$ and a geodesic segment from $x_0$ to $x'$. By (iii), this can be extended to an arbitrarily long geodesic segment starting at $x_0$. By strong planarity, such an extension has to escape $R$ and does so crossing $C$ in some vertex $y'$. One has clearly $d(x_0, x') < d(x_0, y')$, and this proves the inequality above.

We will also make use of another standard fact: for two distinct faces $F$ and $G$, the intersection $F \cap G$ is either empty, or a vertex of the graph, or one edge of the graph. (To rule out the case of several edges, one may evaluate the Euler characteristics of the closure of a bounded component of the complement of $F \cup G$.)

Claim A. *For each face $F$ of $X$, the function*

$$f_F : F \cap X^0 \to \mathbb{N}, \qquad x \mapsto \ell(x),$$

*has a unique local minimum (say $m_F$) and a unique local maximum (say $M_F$). In other words, the function $f_F$ is unimodal.*

To prove the claim, it is enough to show that, for any $n \in \mathbb{N}$, the cardinal of the fibre $f_F^{-1}(n)$ is at most 2.

Suppose *ab absurdo* that this is not the case. Let $x, y, z \in F \cap X^0$ be three distinct vertices such that $f_F(x) = f_F(y) = f_F(z)$. Denote by $[x, y]$, $[y, z]$, $[z, x]$ the three sides of a triangle with vertices $x, y, z$ contained in the boundary of $F$. Choose geodesic segments $L_x, L_y, L_z$ from $x_0$ to $x, y, z$ respectively. Then appropriate subsegments of $[x, y], L_x, L_y$ constitute a simple closed curve $C_{x,y}$ defining a bounded open region $R_{x,y}$ of the plane; one has similarly curves $C_{y,z}$, $C_{z,x}$ and regions $R_{y,z}$, $R_{z,x}$. Let $R$ be the interior of $\overline{R}_{x,y} \cup \overline{R}_{y,z} \cup \overline{R}_{z,x}$. There is exactly one of the three points $x, y, z$ which is inside $R$; upon changing notations for $x, y, z$, one may assume that $y \in R$ (as in Figure 2).

The geodesic segment $L_y$ can be extended infinitely, by (iii). Such an extension of $L_y$ has to escape $R$ through its boundary, and this is impossible; thus Claim A is proved.

It follows that the two geodesic segments in $F \cap X$ from $m_F$ to $M_F$ have the same number $\ell(M_F) - \ell(m_F) - 1$ of interior vertices—this number being strictly positive by (ii).

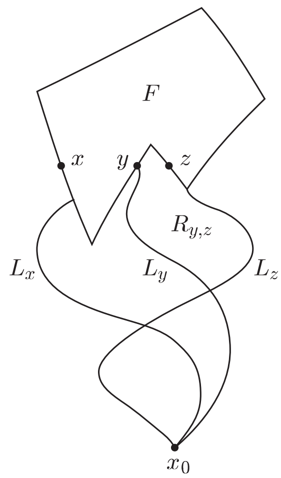


Fig. 2

CLAIM B. *There is no vertex $x \in X^0$ with type $t(x) \geq 3$.*

Indeed, suppose *ab absurdo* that $X$ has vertices of type at least 3 and let $m$ be one of these for which the distance to $x_0$ is minimum. Let $v_1, \ldots, v_r, w_1, \ldots, w_s$ be the neighbours of $m$, listed in such a way that

$$\begin{aligned} \ell(v_i) &= \ell(m) - 1, \quad 1 \leq i \leq r \ \ (r \geq 3), \\ \ell(w_k) &= \ell(m) + 1, \quad 1 \leq k \leq s \ (s \geq 1). \end{aligned}$$

For $i \in \{1, \ldots, r\}$, choose a geodesic segment $L_i$ from $x_0$ to $v_i$.

For $i, j \in \{1, \ldots, r\}$ with $i \neq j$, the segment $[v_i, m, v_j]$ and appropriate subsegments of $L_i, L_j$ constitute a simple closed curve $C_{i,j}$ defining a bounded open region $R_{i,j}$ of the plane. By the maximum principle, $w_k \notin R_{i,j}$ for all $k \in \{1, \ldots, s\}$. Thus, upon renumbering the $v_i$'s and the $w_k$'s, one may assume that $v_1, \ldots, v_r, w_1, \ldots, w_s$ are arranged in cyclic order around the vertex $m$. It follows that there is a face $F_1$ containing $v_1, m, v_2$, a face $F_2$ containing $v_2, m, v_3$, and that $F_1, F_2$ are adjacent along $[v_2, m]$ (see Figure 3).

For $h \in \{1, 2\}$, let $u_h$ denote the vertex of $F_h$ such that $d(u_h, v_2) = 1$ and $\ell(u_h) = \ell(v_2) - 1$; let also $m_h$ denote the vertex of $F_h$ nearest to $x_0$ and choose a geodesic segment $\widetilde{L}_h$ from $x_0$ to $m_h$. (We have used Claim A here.) By (i), the vertex $v_2$ has a neighbour $u_0 \in X^0 \setminus \{m, u_1, u_2\}$. Using again the maximum principle for a region enclosed by appropriate subsegments of $\widetilde{L}_1 \cup [m_1, v_2]$ and $\widetilde{L}_2 \cup [m_2, v_2]$, one checks that $\ell(u_0) = \ell(v_2) - 1$. It

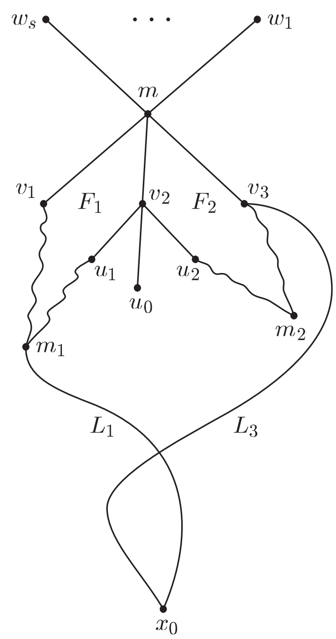


Fig. 3

follows that $v_2$ is of type at least 3 (because it has neighbours $u_0, u_1, u_2$), in contradiction with the choice of $m$ (because $\ell(v_2) < \ell(m)$); thus Claim B is proved.

Proof of (a). Let $x, y \in X^0$ be such that $x \neq y$ and $t(x) = t(y) = 2$. There is a face $F$ such that $x$ is the vertex of $F$ maximizing the distance to the origin on $F \cap X^0$, and a face $G$ similarly associated with $y$. The equality $d(x, y) = 1$ would contradict Claim B, as indicated in Figure 4 (this uses only $k_H \geq 6$ for all faces $H$ of $X$).

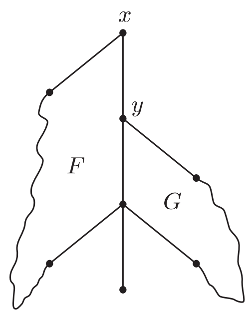


Fig. 4

The equality $d(x, y) = 2$ gives rise to two types of configuration, each in contradiction with Claim B, as indicated in Figure 5.

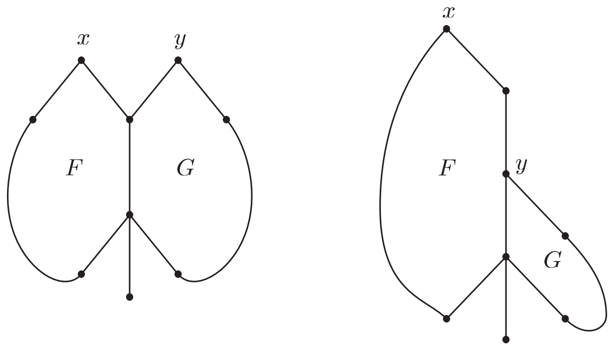


Fig. 5

Proof of (b). Let $x \in X^0$ be a vertex of type 1. Let $v, w_1, \ldots, w_s$ be the neighbours of $x$, listed in cyclic order around the vertex $x$, with

$$\ell(v) = \ell(x) - 1,$$
$$\ell(w_k) = \ell(x) + 1, \quad 1 \le k \le s \ (s \ge 3).$$

We leave it to the reader to check the following facts:

- the vertices $w_1$ and $w_s$ are of types 1 or 2 (not both of type 2 by Claim B),
- the intermediate vertices $w_2, \ldots, w_{k-1}$ are all of type 1,
- any of these has all its neighbours of type 1.

This ends the proof of the proposition. ■

## Appendix B: proof of Lemma 2

LEMMA 4. *For $g \ge 2$, set*

$$C_g = \cosh(D_g), \quad \delta_g = \arccos\left(\frac{S_g}{C_g}\right) = \arccos(\tanh(D_g)),$$
$$S_g = \sinh(D_g), \quad \varepsilon_g = \arccos\left(\frac{S_g}{C_g} - \frac{1}{S_g C_g}\right).$$

*Then*

$$0 < \delta_g < \varepsilon_g < \frac{\pi}{4g}, \tag{9}$$

*and*

$$\frac{d}{d\phi}\frac{1}{(C_g - S_g\cos\phi)^\nu} \le 0 \quad \textit{for all } \phi \in [0,\pi],$$
$$\frac{d^2}{d\phi^2}\frac{1}{(C_g - S_g\cos\phi)^\nu} \ge 0 \quad \textit{for all } \phi \in [\delta_g,\pi],$$
$$\frac{d^3}{d\phi^3}\frac{1}{(C_g - S_g\cos\phi)^\nu} \le 0 \quad \textit{for all } \phi \in [\varepsilon_g,\pi].$$

P r o o f. F i r s t s t e p: *inequalities of* (9) *in Lemma* 4. Obviously $0 < \delta_g$, as $C_g$ and $S_g$ are both positive. Better, $S_g > 1$ because $D_g > 1$; indeed, $D_g$ is an increasing function of $g$ (being the composite of two decreasing functions and an increasing one), and $D_2 \approx 3.057 > 1$. This allows us to write $S_g > S_g - 1/S_g > 0$; dividing by $C_g$ and taking arccosines yields $\delta_g < \varepsilon_g$.

Next $\varepsilon_g < \pi/(4g)$. For this, as "cos" is decreasing, we must show that

$$\frac{S_g}{C_g} - \frac{1}{S_g C_g} \overset{?}{>} \cos\left(\frac{\pi}{4g}\right) \tag{10}$$

holds without the ? sign. We set $X = \cot^2(\pi/(4g))$ and we express $C_g$, $S_g$, $\cos(\pi/(4g))$ in terms of $X$; as $C_g = \cosh(D_g) = 2(\cosh(D_g/2))^2 - 1$, one has

$$C_g = 2X - 1, \quad S_g = 2\sqrt{X(X-1)}, \quad \cos\left(\frac{\pi}{4g}\right) = \sqrt{\frac{X}{X+1}},$$

whence (10) becomes

$$\frac{2\sqrt{X(X-1)}}{2X-1} - \frac{1}{2\sqrt{X(X-1)}(2X-1)} \overset{?}{>} \sqrt{\frac{X}{X+1}}.$$

Squaring,

$$4X(X-1) - 2 + \frac{1}{4X(X-1)} \overset{?}{>} \frac{X}{X+1}(2X-1)^2$$

or, provided $X > 1$,

$$16X^4 - 44X^3 + 20X^2 + 9X + 1 \overset{?}{>} 0.$$

We rewrite this as

$$16(X-2)^4 + 84(X-2)^3 + 140(X-2)^2 + 73(X-2) + 3 \overset{?}{>} 0.$$

This inequality is true for all $X > 2$, as the left hand side is a polynomial in $X-2$ with all coefficients positive. It remains to check that $\cot^2(\pi/(4g)) > 2$ for all $g \geq 2$; but this is clear because $\cot^2(\pi/(4g))$ is an increasing function of $g$ with value $3 + 2\sqrt{2}$ at $g = 2$.

S e c o n d s t e p: *the function* $\beta$. Set

$$\beta(\phi) = b(D_g, \phi)^\nu = \frac{1}{(C_g - S_g \cos\phi)^\nu}$$

so that

$$F_g(\nu, \phi) = \frac{1}{4g} \sum_{j=0}^{4g-1} \beta\left(\phi + j\frac{2\pi}{4g}\right). \tag{11}$$

The first derivative of $\beta$ is

$$\dot\beta(\phi) = \frac{-\nu S_g \sin\phi}{(C_g - S_g\cos\phi)^{\nu+1}} \tag{12}$$

so that $\dot\beta(\phi) \le 0$ for all $\phi \in [0,\pi]$. The second derivative of $\beta$ is

$$\ddot\beta(\phi) = \nu S_g \frac{S_g - C_g\cos\phi + \nu S_g \sin^2\phi}{(C_g - S_g\cos\phi)^{\nu+2}} \ge \nu S_g \frac{S_g - C_g\cos\phi}{(C_g - S_g\cos\phi)^{\nu+2}} \tag{13}$$

so that $\ddot\beta(\phi) \ge 0$ as soon as $\cos\phi \le S_g/C_g$, namely as soon as $\phi \in [\delta_g,\pi]$. The third derivative of $\beta$ is

$$\begin{aligned}\dddot\beta(\phi) &= \nu S_g \sin\phi \frac{1-(3\nu+1)S_g(S_g - C_g\cos\phi) - \nu^2 S_g^2\sin^2\phi}{(C_g - S_g\cos\phi)^{\nu+3}}\\ &\le \nu S_g\sin\phi\frac{1-(3\nu+1)S_g(S_g-C_g\cos\phi)}{(C_g-S_g\cos\phi)^{\nu+3}}\end{aligned}$$

so that $\dddot\beta(\phi) \le 0$ for $\phi \in [\varepsilon_g,\pi]$. ■

Proof of Lemma 2. Let $g \ge 2$ and $\nu \in [0,1]$ be fixed. As the function $\phi \mapsto F_g(\nu,\phi)$ is smooth, even and periodic of period $\pi/(2g)$ it is enough to show that

$$F_g(\nu,\phi) \le F_g(\nu,0)$$

for all $\phi \in [0,\pi/(4g)]$.

In the range $[\delta_g, \pi/(2g) - \delta_g]$, the functions $\phi \mapsto b\big(D_g, \phi + j\frac{2\pi}{4g}\big)^\nu$ are convex for all $j \in \{0,1,\dots,4g-1\}$ by Lemma 4. Their convex sum $\phi \mapsto F_g(\nu,\phi)$ is thus also convex, so that

$$F_g(\nu,\phi) \le F_g(\nu,\delta_g)$$

for all $\phi \in [\delta_g,\pi/(4g)]$.

We now suppose $\phi \in [0,\delta_g]$ and we want to show that $\frac{d}{d\phi}F_g(\nu,\phi) \le 0$. One has

$$\frac{d}{d\phi}\,4gF_g(\nu,\phi) = \dot\beta(\phi) + \sum_{j=1}^{4g-1}\dot\beta\left(\phi + j\frac{\pi}{2g}\right)$$

by (11). As $\dot\beta$ is an odd function $\sum_{j=0}^{4g-1}\dot\beta\big(j\frac{\pi}{2g}\big) = 0$; as $\dot\beta(0) = \dot\beta(\pi) = 0$ one also has

$$\frac{d}{d\phi}\,4gF_g(\nu,\phi) = \dot\beta(\phi) + \sum_{j=1}^{4g-1}\left(\dot\beta\left(\phi + j\frac{\pi}{2g}\right) - \dot\beta\left(j\frac{\pi}{2g}\right)\right).$$

By the theorem of Rolle,

$$\frac{d}{d\phi}\,4gF_g(\nu,\phi) = \dot\beta(\phi) + \sum_{j=1}^{4g-1}\phi\ddot\beta\left(\psi_j + j\frac{\pi}{2g}\right)$$

for some $\psi_j \in [0,\phi]$. By the computation for $\dddot\beta$ in Lemma 4, one has $\ddot\beta\big(\psi_j + j\frac{\pi}{2g}\big) \le \ddot\beta\big(\frac{\pi}{2g}\big)$ and

$$\frac{d}{d\phi}\,4gF_g(\nu,\phi)\le\dot\beta(\phi)+(4g-1)\ \phi\ddot\beta\left(\frac{\pi}{2g}\right).$$

Using (12) and (13) one finds

$$\begin{aligned}\frac{d}{d\phi}\,4gF_g(\nu,\phi)\le{}& -\nu S_g\frac{\sin\phi}{(C_g-S_g\cos\phi)^{\nu+1}}\\&+(4g-1)\nu S_g\phi\frac{S_g-C_g\cos(\pi/(2g))+\nu S_g\sin^2(\pi/(2g))}{(C_g-S_g\cos(\pi/(2g)))^{\nu+2}}\end{aligned}$$

so all we have to check is

$$\frac{(\sin\phi)/\phi}{(C_g-S_g\cos\phi)^{\nu+1}}\ge(4g-1)\frac{S_g-C_g\cos(\pi/(2g))+\nu S_g\sin^2(\pi/(2g))}{(C_g-S_g\cos(\pi/(2g)))^{\nu+2}}$$

for all $\phi\in[0,\delta_g]$.

As $\cos\phi\ge\cos(\pi/(2g))$, so $(C_g-S_g\cos\phi)^\nu\le(C_g-S_g\cos(\pi/(2g)))^\nu$, we may tighten the inequality to

$$\frac{(\sin\phi)/\phi}{C_g-S_g\cos\phi}\ge(4g-1)\frac{S_g-C_g\cos(\pi/(2g))+\nu S_g\sin^2(\pi/(2g))}{(C_g-S_g\cos(\pi/(2g)))^2}\doteq R_g(\nu);$$

as the right hand side is constant in $\phi$ while the left hand side decreases monotonically, we let $\phi=\delta_g$. Finally, we set $\nu=1$ to maximize the right hand side. Our goal is now to show

$$\frac{(\sin\delta_g)/\delta_g}{C_g-S_g\cos\delta_g}\ge R_g(1).$$

But, by definition of $\delta_g$ (see Lemma 4), one has $C_g-S_g\cos\delta_g=1/C_g$ and $C_g\sin\delta_g=\sqrt{C_g^2-S_g^2}=1$, so that our goal reduces to showing

$$1/\delta_g\ge R_g(1).$$

That this is true for $g\le 27$ can in turn be checked on a pocket calculator. Thus when $g\le 27$ and $\nu\in[0,1]$ the function $F_g(\nu,-)$ is monotonically decreasing on $[0,\pi/(4g)]$; its maxima are at $0+j\pi/(2g)$ and its minima at $\pi/(4g)+j\pi/(2g)$. ∎

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