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EXTREME NON-ARENS REGULARITY OF QUOTIENTS OF THE FOURIER ALGEBRA $A(G)$

BY

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1. Introduction. Let $\mathcal{A}$ be a Banach algebra. As is well known, on the second dual $\mathcal{A}^{**}$ of $\mathcal{A}$ there exist two Banach algebra multiplications extending that of $\mathcal{A}$ (see [1]). When these two multiplications coincide on $\mathcal{A}^{**}$, the algebra $\mathcal{A}$ is said to be *Arens regular*. Let $\mathrm{WAP}(\mathcal{A}^*)$ denote the space of all weakly almost periodic functionals on $\mathcal{A}$. Then the equality $\mathrm{WAP}(\mathcal{A}^*) = \mathcal{A}^*$ is equivalent to the Arens regularity of $\mathcal{A}$ (cf. [21]). Recently, Granirer introduced the concept "extreme non-Arens regularity". $\mathcal{A}$ is called *extremely non-Arens regular* (or *ENAR* for short) if $\mathcal{A}^*/\mathrm{WAP}(\mathcal{A}^*)$ is as big as $\mathcal{A}^*$, namely if $\mathcal{A}^*/\mathrm{WAP}(\mathcal{A}^*)$ contains a closed subspace which has $\mathcal{A}^*$ as a continuous linear image (see [13]).

Let G be a locally compact group and $A(G)$ the Fourier algebra of G. Lau proved that if G is amenable then $A(G)$ is Arens regular if and only if G is finite (see [18, Proposition 3.3]). Generally, Forrest showed that if $A(G)$ is Arens regular then G is discrete ([8, Theorem 3.2]). He further showed in [9] that $A(G)$ is not Arens regular if G contains an infinite abelian subgroup. Lately, Granirer investigated the non-Arens regularity of quotients of $A(G)$. Let J be a closed ideal of $A(G)$ with zero set $Z(J) = F$. Granirer proved that $A(G)/J$ is not Arens regular if there exist $a, b \in G$ such that one of the following conditions holds:

(1) $\mathrm{int}_{aHb}(F) \neq \emptyset$ for some non-discrete subgroup H of G;

(2) G contains $\mathbb{R}$ (or $\mathbb{T}$) as a closed subgroup and there is a symmetric set $S \subset \mathbb{R}$ (or $\mathbb{T}$) satisfying $aSb \subseteq F$ ([14, Corollary 8]).

Furthermore, if G is second countable, Granirer showed that $A(G)/J$ is ENAR ([13, Corollaries 6 and 7]). He asked if this is the case when G is not second countable.

In this paper, we attempt to deal with non-second countable groups. Some conditions on G and $Z(J)$ are proposed which guarantee the extreme

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non-Arens regularity of $A(G)/J$. In particular, we show that $A(G)/J$ is ENAR if G is any σ-compact non-discrete locally compact group and J is a closed ideal of $A(G)$ such that

(∗) $Z(J)$ contains a non-empty intersection B of $\aleph$ many open subsets of G with $\aleph < b(G)$,

where $b(G)$ denotes the smallest cardinality of an open basis at the unit e of G (condition (∗) is satisfied if $\operatorname{int} Z(J) \neq \emptyset$).

It is worth noting that our discussion on the extreme non-Arens regularity of $A(G)/J$ is primarily based on our understanding of the extreme non-ergodicity of $(A(G)/J)^*$. Let $VN(G)$ be the von Neumann algebra generated by the left regular representation of G. Let $\mathbb{P} = J^\perp = \{T \in \mathrm{VN}(G) : \langle T, u\rangle = 0 \text{ if } u \in J\}$. Then $\mathbb{P}$ is linear isometric to $(A(G)/J)^*$. For $x \in G$, let $E_{\mathbb{P}}(x)$ be the norm closure of $\{T \in \mathbb{P} : x \notin \operatorname{supp} T\}$ and let $W_{\mathbb{P}}(x) = \mathbb{C}\delta_x + E_{\mathbb{P}}(x)$. Denote by μ the first ordinal with $|\mu| = b(G)$ and let $X = \{\alpha : \alpha < \mu\}$. We show that if G is any non-discrete locally compact group and J is a closed ideal of $A(G)$ such that $Z(J)$ satisfies condition (∗), then $\mathbb{P}$ is extremely non-ergodic at each $x \in B$, namely $\mathbb{P}/W_{\mathbb{P}}(x)$ has $l^\infty(X)$ as a continuous linear image and $\mathrm{TIM}_{\mathbb{P}}(x)$ contains $\mathcal{F}(X)$, where $\mathrm{TIM}_{\mathbb{P}}(x) = \{\phi \in \mathbb{P}^*; \|\phi\| = \langle \phi, \delta_x\rangle = 1 \text{ and } \phi = 0 \text{ on } E_{\mathbb{P}}(x)\}$ and $\mathcal{F}(X) = \{\phi \in l^\infty(X)^* : \|\phi\| = \phi(\mathbf{1}) = 1 \text{ and } \phi(f) = 0 \text{ if } f \in l^\infty(X) \text{ and } \lim_{\alpha\in X} f(\alpha) = 0\}$. Moreover, if G is non-metrizable, then $\mathbb{P}/W_{\mathbb{P}}(x)$ contains an isomorphic copy of $l^\infty(X)$ for each $x \in B$ (Theorem 3.4 combined with Remark 3.5(iii)). These results extend and improve some of those in [13] and [17].

It is our pleasure to thank Professor E. E. Granirer for his valuable comments and also for providing the preprint of his paper [14].

2. Preliminaries and notations. Let G be a locally compact group with identity e and a fixed left Haar measure $\lambda = dx$, and let $L^2(G)$ be the usual Hilbert space with the inner product $(f, g) = \int_G f(x)\overline{g(x)}\,dx$, for $f, g \in L^2(G)$.

Let $\mathrm{VN}(G)$ denote the von Neumann algebra generated by the left regular representation of G, i.e. the closure of the linear span of $\{\varrho(a) : a \in G\}$ in the weak operator topology, where $[\varrho(a)f](x) = f(a^{-1}x)$, for $x \in G$, $f \in L^2(G)$. Let $A(G)$ denote the subalgebra of $C_0(G)$ (bounded continuous complex-valued functions on G vanishing at infinity) consisting of all functions of the form $f * \widetilde{g}$, where $f, g \in L^2(G)$ and $\widetilde{g}(x) = \overline{g(x^{-1})}$. Then each $\phi = f * \widetilde{g}$ in $A(G)$ can be regarded as an ultraweakly continuous functional on $\mathrm{VN}(G)$ defined by $\phi(T) = (Tf, g)$ for $T \in \mathrm{VN}(G)$. Furthermore, as shown by P. Eymard in [6, pp. 210 and 218], each ultraweakly continuous functional on $\mathrm{VN}(G)$ is of the form $f * \widetilde{g}$ with f, $g \in L^2(G)$. Also, $A(G)$ with pointwise multiplication and the norm $\|\phi\| = \sup\{|\phi(T)| : T \in \mathrm{VN}(G) \text{ and } \|T\| \le 1\}$ forms a commutative Banach algebra called the *Fourier algebra* of G.

There is a natural action of $A(G)$ on $\mathrm{VN}(G)$ given by

$$\langle u \cdot T, v\rangle = \langle T, uv\rangle, \qquad \text{for } u, v \in A(G),\ T \in \mathrm{VN}(G).$$

Under this action, $\mathrm{VN}(G)$ becomes a Banach $A(G)$-module. Let $T \in \mathrm{VN}(G)$. We say that $x \in G$ is in the *support* of T, denoted by $\operatorname{supp} T$, if $\varrho(x)$ is the ultraweak limit of operators of the form $u \cdot T$, $u \in A(G)$.

An $m \in \mathrm{VN}(G)^*$ is called a *topologically invariant mean* on $\mathrm{VN}(G)$ if

(i) $\|m\| = \langle m, I\rangle = 1$, where $I = \varrho(e)$ denotes the identity operator,
(ii) $\langle m, u\cdot T\rangle = \langle m, T\rangle$ for $T \in \mathrm{VN}(G)$ and $u \in A(G)$ with $\|u\| = u(e) = 1$.

Let $\mathrm{TIM}(\widehat{G})$ be the set of topologically invariant means on $\mathrm{VN}(G)$. Denote by $F(\widehat{G})$ the space of all $T \in \mathrm{VN}(G)$ such that $m(T)$ equals a fixed constant $d(T)$ as m runs through $\mathrm{TIM}(\widehat{G})$. Then $F(\widehat{G})$ is a norm closed self-adjoint $A(G)$-submodule of $\mathrm{VN}(G)$.

The space $\{T \in \mathrm{VN}(G) : u \mapsto u \cdot T$ is a weakly compact operator of $A(G)$ into $\mathrm{VN}(G)\}$ is called the *space of weakly almost periodic functionals* on $A(G)$ and denoted by $W(\widehat{G})$. It turns out that $W(\widehat{G})$ is a self-adjoint closed $A(G)$-submodule of $\mathrm{VN}(G)$. Also, it is known that $W(\widehat{G}) \subseteq F(\widehat{G})$ (see [5] and [10]).

Let $M(G)$ denote the algebra of finite regular Borel measures on G with convolution as multiplication. $M(G)$ can be considered as a subspace of $\mathrm{VN}(G)$ by virtue of

$$\langle \mu, u\rangle = \int_G u\, d\mu, \quad \text{for } u \in A(G).$$

In particular, $\langle \delta_x,\, u\rangle = u(x)$, $x \in G$, $u \in A(G)$, where δ_x denotes the point measure at x.

Let $\mathbb{P}$ be a norm closed $A(G)$-submodule of $\mathrm{VN}(G)$ and $x \in G$. Following notations and definitions of Granirer [12], we put

$$\begin{aligned}
\sigma(\mathbb{P}) &= \{z \in G : \delta_z \in \mathbb{P}\},\\
\mathbb{P}_{\mathrm{c}} &= \text{the norm closure of } \{T \in \mathbb{P} : \operatorname{supp} T \text{ is compact}\},\\
E_{\mathbb{P}}(x) &= \text{the norm closure of } \{T \in \mathbb{P} : x \notin \operatorname{supp} T\},\\
W_{\mathbb{P}}(x) &= \mathbb{C}\delta_x + E_{\mathbb{P}}(x).
\end{aligned}$$

It is shown that $E_{\mathbb{P}}(x)$ is the norm closure of $\{T - u \cdot T : T \in \mathbb{P},\ u \in A(G)$ and $\|u\| = u(x) = 1\}$ (see Granirer [12, Proposition 1]). Furthermore, if $x \in \sigma(\mathbb{P})$, let $\mathrm{TIM}_{\mathbb{P}}(x)$ denote the set of all topologically invariant means on $\mathbb{P}$ at x, i.e.

$$\mathrm{TIM}_{\mathbb{P}}(x) = \{\phi \in \mathbb{P}^* : \|\phi\| = \phi(\delta_x) = 1 \text{ and } \phi = 0 \text{ on } E_{\mathbb{P}}(x)\}.$$

When $\mathbb{P} = \mathrm{VN}(G)$, $W_{\mathbb{P}}(e) = F(\widehat{G})$ and $\mathrm{TIM}_{\mathbb{P}}(e) = \mathrm{TIM}(\widehat{G})$.

For a closed ideal J of $A(G)$, $Z(J)$ denotes the set $\{x \in G : u(x) = 0$ for all $u \in J\}$. If F is a closed subset of G, let $I(F) = \{u \in A(G) : u = 0$ on $F\}$. F is called a *set of spectral synthesis*, or simply an *s-set*, if $I(F)$ is the only closed ideal I of $A(G)$ with $Z(I) = F$.

Let E_1 and E_2 be two Banach spaces. We say that E_2 contains an *isomorphic* (*isometric*) *copy* of E_1 if there is a linear mapping $L : E_1 \to E_2$ and some positive constants γ_1, γ_2 ($\gamma_1 = \gamma_2 = 1$) such that $\gamma_1\|x\| \le \|Lx\| \le \gamma_2\|x\|$ for all $x \in E_1$; further, E_2 *has* E_1 *as a quotient* if there is a bounded linear mapping from E_2 onto E_1. Also, for a Banach space Y, we denote by $\mathcal{D}(Y)$ the *density character* of Y, i.e. the smallest cardinality such that there exists a norm dense subset of Y having that cardinality.

For any set A, $|A|$ denotes the cardinality of A. If μ is an ordinal, then $|\mu|$ denotes the cardinality of the set $\{\alpha : \alpha < \mu\}$. For a locally compact group G with identity e, we denote by $b(G)$ the smallest cardinality of an open basis at e.

Let $\mathcal{A}$ be a Banach algebra. It is well known that there exist two Banach algebra multiplications on $\mathcal{A}^{**}$ extending that of $\mathcal{A}$. When these two multiplications coincide on $\mathcal{A}^{**}$, $\mathcal{A}$ is said to be *Arens regular*. Details of the construction of these multiplications can be found in many places, including the pioneering paper [1], the book [2] and the survey article [4]. $T \in \mathcal{A}^*$ is called *weakly almost periodic* if the set $\{u \cdot T : u \in \mathcal{A}$ and $\|u\| \le 1\}$ is a relatively weakly compact subset of $\mathcal{A}^*$, where $u \cdot T \in \mathcal{A}^*$ is defined by $\langle u \cdot T, v\rangle = \langle T, uv\rangle$, $v \in \mathcal{A}$. The space of all weakly almost periodic functionals on $\mathcal{A}$ is denoted by $\mathrm{WAP}(\mathcal{A}^*)$. Then $\mathrm{WAP}(\mathcal{A}^*) = \mathcal{A}^*$ if and only if $\mathcal{A}$ is Arens regular ([21]). $\mathcal{A}$ is called *extremely non-Arens regular* (or *ENAR* for short) if $\mathcal{A}^*/\mathrm{WAP}(\mathcal{A}^*)$ is as big as $\mathcal{A}^*$, namely if $\mathcal{A}^*/\mathrm{WAP}(\mathcal{A}^*)$ contains a closed subspace which has $\mathcal{A}^*$ as a quotient. The definition of ENAR was made by Granirer in [13] where he first investigated the extreme non-Arens regularity for quotients of $A(G)$.

LEMMA 2.1. *Let $\mathcal{A}$ be a Banach algebra and Γ be a set. If $l^\infty(\Gamma)$ contains an isomorphic copy of $\mathcal{A}^*$ (in particular, if $\mathcal{D}(\mathcal{A}) \le |\Gamma|$) and $\mathcal{A}^*/\mathrm{WAP}(\mathcal{A}^*)$ has $l^\infty(\Gamma)$ as a quotient, then $\mathcal{A}$ is ENAR.*

P r o o f. Let t be a linear isomorphism of $\mathcal{A}^*$ into $l^\infty(\Gamma)$ and r a bounded linear map of $\mathcal{A}^*/\mathrm{WAP}(\mathcal{A}^*)$ onto $l^\infty(\Gamma)$. Let $Y = r^{-1}[t(\mathcal{A}^*)]$. Then Y ($\subseteq \mathcal{A}^*/\mathrm{WAP}(\mathcal{A}^*)$) has $\mathcal{A}^*$ as a quotient. Therefore, $\mathcal{A}$ is ENAR. If $\mathcal{D}(\mathcal{A}) \le |\Gamma|$, then there exists a subset Γ_0 of Γ such that $\mathcal{D}(\mathcal{A}) = |\Gamma_0|$. Let $\{x_\gamma\}_{\gamma\in\Gamma_0}$ be norm dense in the unit ball of $\mathcal{A}$. Define $h : \mathcal{A}^* \to l^\infty(\Gamma_0)$ by $(h\Phi)(\gamma) = \langle \Phi, x_\gamma\rangle$, $\Phi \in \mathcal{A}^*$, $\gamma \in \Gamma_0$. Then h is a linear isometry of $\mathcal{A}^*$ into $l^\infty(\Gamma_0) \subseteq l^\infty(\Gamma)$. ■

3. Extreme non-ergodicity of $A(G)$-submodules of $\mathrm{VN}(G)$. This section is partially motivated by Granirer [12] and [13]. The basic idea used in the proof of our main theorem (Theorem 3.4) is similar to that used in [17]. Let G be a locally compact group with identity e. We begin this section with the following property of $A(G)$-submodules of $\mathrm{VN}(G)$, which is needed in the proof of Theorem 3.4.

PROPOSITION 3.1. *Let $\mathbb{P}$ be a norm closed $A(G)$-submodule of* $\mathrm{VN}(G)$ *and $e \in \sigma(\mathbb{P})$. Then $W_{\mathbb{P}}(e) = F(\widehat{G}) \cap \mathbb{P}$.*

Proof. Since $e \in \sigma(\mathbb{P})$, $W_{\mathbb{P}}(e) \subseteq \mathbb{P}$. Let $S = \{u \in A(G) : \|u\| = u(e) = 1\}$. By [12, Proposition 1], $E_{\mathbb{P}}(e)$ equals the norm closure of $\{T - u \cdot T : T \in \mathbb{P}$ and $u \in S\}$. So $W_{\mathbb{P}}(e) = \mathbb{C}I + E_{\mathbb{P}}(e) \subseteq F(\widehat{G})$. Therefore, $W_{\mathbb{P}}(e) \subseteq F(\widehat{G}) \cap \mathbb{P}$.

Conversely, let $T \in F(\widehat{G}) \cap \mathbb{P}$. Then there exists a constant a such that $m(T) = a$ for all $m \in \mathrm{TIM}(\widehat{G})$. We now follow an argument of Granirer [14, Proposition 3] to show that $T - aI \in E_{\mathbb{P}}(e)$. If $T - aI \notin E_{\mathbb{P}}(e)$, then, by the Hahn–Banach theorem, there exists a $\phi \in \mathrm{VN}(G)^*$ such that $\langle \phi, T - aI\rangle \neq 0$, but $\langle \phi, \Phi - u \cdot \Phi\rangle = 0$ for all $\Phi \in \mathbb{P}$ and $u \in S$. Note that the pointwise multiplication in $A(G)$ makes S into an abelian semigroup. Let $M \in l^\infty(S)^*$ be a translation invariant mean. Define $\psi \in \mathrm{VN}(G)^*$ by

$$\langle \psi, \Phi\rangle = \langle M, \phi(u \cdot \Phi)\rangle, \qquad \Phi \in \mathrm{VN}(G),$$

where $\phi(u\cdot\Phi)$ is considered as a bounded function on S (i.e. it is in $l^\infty(S)$). It is easy to check that ψ extends ϕ, and $\langle \psi, v \cdot \Phi\rangle = \langle \psi, \Phi\rangle$ for all $\Phi \in \mathrm{VN}(G)$ and $v \in S$. Therefore, ψ is topologically invariant and $\langle \psi, T - aI\rangle \neq 0$. According to Chou [3, Lemma 4.2], there exists an $m_0 \in \mathrm{TIM}(\widehat{G})$ such that $\langle m_0, T - aI\rangle \neq 0$, or $\langle m_0, T\rangle \neq a$. We have thus reached a contradiction. It follows that $T - aI \in E_{\mathbb{P}}(e)$ and hence $T \in W_{\mathbb{P}}(e)$. ∎

In the following, G is always a non-discrete locally compact group. Recall that $b(G)$ denotes the smallest cardinality of an open basis at e. Let μ be the initial ordinal with $|\mu| = b(G)$ and let $X = \{\alpha : \alpha$ is an ordinal and $\alpha < \mu\}$. Let $l^\infty(X)$ be the Banach space of all bounded complex-valued functions on X with the supremum norm and $c(X)$ the subspace of $l^\infty(X)$ consisting of all $f \in l^\infty(X)$ such that $\lim_{\alpha\in X} f(\alpha)$ exists. In [16], we defined a subset of $l^\infty(X)^*$ as follows

$$\mathcal{F}(X) = \{\phi \in l^\infty(X)^* : \|\phi\| = \phi(\mathbf{1}) = 1 \text{ and } \phi(f) = 0 \text{ if } \lim_{\alpha\in X} f(\alpha) = 0\}.$$

It is shown that $|\mathcal{F}(X)| = 2^{2^{|X|}}$ (see [16, Proposition 3.3]). If Y is a Banach space and $K \subseteq Y^*$, we say that K *contains* $\mathcal{F}(X)$ if there is an onto bounded linear map $t : Y \to l^\infty(X)$ such that $t^* : l^\infty(X)^* \to Y^*$ satisfies $t^*(\mathcal{F}(X)) \subseteq K$ (it is easily seen that t^* is a w^*-w^* continuous norm isomorphism into).

Definition 3.2. Let $\aleph > 0$ be a cardinal. A non-empty subset B of G is called a *$G_\aleph$-set* if B is an intersection of $\aleph$ many open subsets of G.

Theorem 3.3. *Let G be a non-discrete locally compact group. Let $\mathbb{P}$ and $\mathbb{Q}$ be $A(G)$-submodules of* $\mathrm{VN}(G)$ *such that $\mathbb{P}$ is w^*-closed, $\mathbb{Q}$ is norm closed, $\mathbb{P}_{\mathrm{c}} \subseteq \mathbb{Q} \subseteq \mathbb{P}$, and $\sigma(\mathbb{P}) = F$. Assume that*

(∗) *F contains a $G_\aleph$-set B with $\aleph < b(G)$,*

and $e \in B$. Then $\mathbb{Q}/W_{\mathbb{Q}}(e)$ has $l^\infty(X)$ as a quotient.

If G is further assumed to be non-metrizable, then $\mathbb{Q}/W_{\mathbb{Q}}(e)$ contains an isomorphic copy of $l^\infty(X)$.

Proof. By the definition, B is a non-empty intersection of $\aleph$ many open subsets of G. If G is metrizable, then B is open and $e \in B \subseteq \mathrm{int}(F)$. By Granirer [13, Corollary 7], $\mathbb{Q}/W_{\mathbb{Q}}(e)$ has l^∞ as a quotient.

We now assume that G is non-metrizable. By the injectivity of $l^\infty(X)$ (see [19, p. 105]), we only need to show that $\mathbb{Q}/W_{\mathbb{Q}}(e)$ contains an isomorphic copy of $l^\infty(X)$. We may also assume that $\aleph$ is infinite and ν is the initial ordinal satisfying $|\nu| = \aleph$. Then $\nu < \mu$.

Suppose first that G is compactly generated. Let $(N_\alpha)_{0<\alpha\le\mu}$ be the decreasing net of compact normal subgroups of G as in [16, Proposition 4.3]. According to the construction of $(N_\alpha)_{0<\alpha\le\mu}$, this net can be chosen so that $N_\nu \subseteq B \subseteq F$ (see [16]). Let λ_α be the normalized Haar measure of N_α. Let $Q_0 = \lambda_1$ and $Q_\alpha = \lambda_{\alpha+1} - \lambda_\alpha$ $(0 < \alpha < \mu)$. Then $(Q_\alpha)_{\alpha<\mu}$ is an orthogonal net of projections in $\mathrm{VN}(G)$ (see [16]). For each $\nu \le \alpha < \mu$, $Q_\alpha \in \mathbb{P}$ (since $\mathbb{P} = ({}^\perp\mathbb{P})^\perp$ and $\langle Q_\alpha, u\rangle = \int_G u\, d(\lambda_{\alpha+1} - \lambda_\alpha) = 0$ if $u \in A(G)$ and $u = 0$ on F). Also, $\mathrm{supp}\, Q_\alpha (\subseteq N_\alpha)$ is compact. Therefore, $Q_\alpha \in \mathbb{P}_{\mathrm{c}} \subseteq \mathbb{Q}$ for all $\nu \le \alpha < \mu$. If $f \in l^\infty(X)$, let $\sum_{\alpha<\mu} f(\alpha)Q_{\nu+\alpha}$ denote the w^*-limit of $\{\sum_{\alpha\in\Lambda} f(\alpha)Q_{\nu+\alpha} : \Lambda \subset X \text{ is finite}\}$ in $\mathrm{VN}(G)$. Then $\sum_{\alpha<\mu} f(\alpha)Q_{\nu+\alpha} \in \mathbb{P}$ (since $\mathbb{P}$ is w^*-closed) and $\mathrm{supp}[\sum_{\alpha<\mu} f(\alpha)Q_{\nu+\alpha}]$ $(\subseteq N_\nu)$ is compact. So $\sum_{\alpha<\mu} f(\alpha)Q_{\nu+\alpha} \in \mathbb{P}_{\mathrm{c}} \subseteq \mathbb{Q}$ for all $f \in l^\infty(X)$. Define $\tau : l^\infty(X) \to \mathbb{Q}$ by

$$\tau(f) = \sum_{\alpha<\mu} f(\alpha)Q_{\nu+\alpha}, \qquad f \in l^\infty(X).$$

By [17, Lemmas 4.4 and 4.5], τ is a linear isometry of $l^\infty(X)$ into $\mathbb{Q}$ and $\tau(c(X)) \subseteq F(\widehat{G}) \cap \mathbb{Q} = W_{\mathbb{Q}}(e)$ (Proposition 3.1 above). For $f \in l^\infty(X)$, define $\widetilde{f} \in l^\infty(X)$ by

$$\widetilde{f}(\alpha) = \begin{cases} 0 & \text{if } \alpha < \nu, \\ f(\beta) & \text{if } \alpha = \nu + \beta. \end{cases}$$

Then $\tau(f) = \sum_{\alpha<\mu} \widetilde{f}(\alpha)Q_\alpha$. By [17, Lemma 5.8], $\|\widetilde{f} + c(X)\| = \|\tau(f) +$

$F(\widehat{G})\|$. Also, notice that $\|f+c(X)\| = \|\widetilde{f}+c(X)\|$. It follows that

$$\begin{aligned}\|\widetilde{f}+c(X)\| &= \|\tau(f)+F(\widehat{G})\| \\ &\le \|\tau(f)+W_{\mathbb{Q}}(e)\| && \text{(by Proposition 3.1)} \\ &\le \|f+c(X)\| && (\text{since } \tau(c(X)) \subseteq W_{\mathbb{Q}}(e)) \\ &= \|\widetilde{f}+c(X)\|,\end{aligned}$$

i.e. $\|\tau(f)+W_{\mathbb{Q}}(e)\| = \|f+c(X)\|$ for all $f \in l^\infty(X)$. Therefore, $\mathbb{Q}/W_{\mathbb{Q}}(e)$ contains an isometric copy of $l^\infty(X)/c(X)$. But $l^\infty(X)$ can be isomorphically embedded into $l^\infty(X)/c(X)$ ([17, Lemma 3.2]). Consequently, $\mathbb{Q}/W_{\mathbb{Q}}(e)$ contains an isomorphic copy of $l^\infty(X)$.

Generally, let G_0 be a compactly generated open subgroup of G. We may assume that $B \subseteq G_0$ (since we may assume that the closure of B is compact). Now G_0 is also non-metrizable with $b(G_0) = b(G)$. Let $r : A(G) \to A(G_0)$ be the restriction map. Then r^* is isometric (see Eymard [6]). Granirer showed that $r^*[\mathrm{TIM}(\widehat{G})] = \mathrm{TIM}(\widehat{G}_0)$ (see [10]) and hence $r^*[F(\widehat{G}_0)] \subseteq F(\widehat{G})$. Let $\mathbb{Q}_0 = \{T \in \mathrm{VN}(G_0) : \operatorname{supp} T \subseteq \overline{B}\}$. Then $\mathbb{Q}_0$ is a w^*-closed $A(G_0)$-submodule of $\mathrm{VN}(G_0)$ with $\sigma(\mathbb{Q}_0) = \overline{B}$. Let $\tau : l^\infty(X) \to \mathbb{Q}_0$ be the same linear isometry as in the previous paragraph. We claim that $r^* \circ \tau[l^\infty(X)] \subseteq \mathbb{Q}$. In fact, let $f \in l^\infty(X)$, then $\operatorname{supp}[r^* \circ \tau(f)]$ ($\subseteq \operatorname{supp}[\tau(f)] \subseteq N_\nu$) is compact and $r^* \circ \tau(f) \in \mathbb{P} = ({}^\perp\mathbb{P})^\perp$ (by the definitions of r and τ, $\langle r^* \circ \tau(f), u\rangle = \langle \tau(f), r(u)\rangle = 0$ if $u \in {}^\perp\mathbb{P}$). Therefore, $r^* \circ \tau(f) \in \mathbb{P}_{\mathrm{c}} \subseteq \mathbb{Q}$. Also, since $r^*[\tau(c(X))] \subseteq r^*[F(\widehat{G}_0)] \subseteq F(\widehat{G})$, we have $r^* \circ \tau(c(X)) \subseteq F(\widehat{G}) \cap \mathbb{Q} = W_{\mathbb{Q}}(e)$ (Proposition 3.1). Consequently,

$$\begin{aligned}\|f+c(X)\| &= \|\widetilde{f}+c(X)\| \\ &= \|\tau(f)+F(\widehat{G}_0)\| && \text{(by [17, Lemma 5.8])} \\ &= \|r^*[\tau(f)]+F(\widehat{G})\| && \text{(by [17, Lemma 5.9])} \\ &\le \|r^*[\tau(f)]+W_{\mathbb{Q}}(e)\| && \text{(by Proposition 3.1)} \\ &\le \|f+c(X)\| && (\text{since } r^* \circ \tau(c(X)) \subseteq W_{\mathbb{Q}}(e)),\end{aligned}$$

i.e. $\|r^*[\tau(f)]+W_{\mathbb{Q}}(e)\| = \|f+c(X)\|$ for all $f \in l^\infty(X)$. It follows that $\mathbb{Q}/W_{\mathbb{Q}}(e)$ contains an isometric copy of $l^\infty(X)/c(X)$ and hence it contains an isomorphic copy of $l^\infty(X)$ (by [17, Lemma 3.2]). ∎

The main result of this section is Theorem 3.4. The crux of its proof is actually contained in the proof of Theorem 3.3.

THEOREM 3.4. *With assumptions on $\mathbb{P}$ and $\mathbb{Q}$ as in Theorem* 3.3, *if*

($*$) *F contains a $G_\aleph$-set B with $\aleph < b(G)$,*

then $\mathbb{Q}/W_{\mathbb{Q}}(x)$ has $l^\infty(X)$ as a quotient for all $x \in B$.

Furthermore, if G is non-metrizable, then $\mathbb{Q}/W_{\mathbb{Q}}(x)$ contains an isomorphic copy of $l^\infty(X)$ for all $x \in B$.

Proof. Let $x \in B$ and $y = x^{-1}$. Let L_y be the left translation on $A(G)$ by y (i.e. $u \mapsto {}_y u$, $u \in A(G)$). Then L_y^* is a w^*-w^* continuous linear isometry of $\mathrm{VN}(G)$ onto itself. Define $\mathbb{P}' = L_y^*(\mathbb{P})$, $\mathbb{Q}' = L_y^*(\mathbb{Q})$, $F' = {}_yF$ and $B' = {}_yB$. Then $\mathbb{P}'$ and $\mathbb{Q}'$ are $A(G)$-submodules of $\mathrm{VN}(G)$ such that $\mathbb{P}'$ is w^*-closed and $\mathbb{Q}'$ is norm closed.

Also, B' is a $G_\aleph$-set with $e \in B' \subseteq F'$ and $F' = \sigma(\mathbb{P}')$. It is easy to check that $[\mathbb{P}']_{\mathrm{c}} = L_y^*(\mathbb{P}_{\mathrm{c}})$ and hence $[\mathbb{P}']_{\mathrm{c}} \subseteq \mathbb{Q}' \subseteq \mathbb{P}'$. But $L_y^*[W_{\mathbb{Q}}(x)] = W_{\mathbf{Q}'}(e)$. Therefore, $\mathbb{Q}/W_{\mathbb{Q}}(x)$ is linear isometric to $\mathbb{Q}'/W_{\mathbb{Q}'}(e)$. It follows that $\mathbb{Q}/W_{\mathbb{Q}}(x)$ has $l^\infty(X)$ as a quotient (or contains an isomorphic copy of $l^\infty(X)$ when G is non-metrizable) because so does $\mathbb{Q}'/W_{\mathbb{Q}'}(e)$ (by Theorem 3.3). ■

Remark 3.5. (i) Theorem 3.3 improves [17, Theorem 6.9]. In [17], we only considered the case when $\mathbb{P} = \{T \in \mathrm{VN}(G) : \operatorname{supp} T \subseteq F\}$ and $\mathbb{Q} = \{T \in \mathrm{UCB}(\widehat{G}) : \operatorname{supp} T \subseteq F\}$ for some closed subset F of G satisfying condition $(*)$, where $\mathrm{UCB}(\widehat{G})$ is the norm closure of $\{T \in \mathrm{VN}(G) : \operatorname{supp} T$ is compact$\}$.

(ii) Note that if $\mathcal{D}(A(G)) = b(G)$ (e.g. if G is non-discrete and σ-compact) then $\mathrm{VN}(G)$ is isometric to a subspace of $l^\infty(X)$. Hence the assertion "$\mathbb{Q}/W_{\mathbb{Q}}(x)$ has $l^\infty(X)$ as a quotient" means that the space $\mathbb{Q}/W_{\mathbb{Q}}(x)$ is as big as it can be.

(iii) Let G be non-metrizable and let $\pi : \mathrm{VN}(G) \to l^\infty(X)$ be the bounded onto linear mapping as in [17, Theorem 5.1]. With the assumptions of Theorem 3.3, if we define $\pi' : \mathbb{Q} \to l^\infty(X)$ by

$$\pi'(T)(\alpha) = \pi(T)(\nu + \alpha), \qquad T \in \mathbb{Q},\ \alpha \in X,$$

where ν is the first ordinal with $|\nu| = \aleph$ (we may assume that $\aleph$ is infinite), then it can be seen that π' is onto, $(\pi')^*$ is a linear isometry into, $\pi'(W_{\mathbb{Q}}(e)) \subseteq c(X)$ and $(\pi')^*(\mathcal{F}(X)) \subseteq \mathrm{TIM}_{\mathbb{Q}}(e)$. Also, $L_y^{**}(\mathrm{TIM}_{\mathbb{Q}'}(e)) = \mathrm{TIM}_{\mathbb{Q}}(y^{-1})$, where L_y is the left translation on $A(G)$ by y and $\mathbb{Q}' = L_y^*(\mathbb{Q})$. Therefore, we can add to the conclusion of Theorem 3.4 that $\mathrm{TIM}_{\mathbb{Q}}(x)$ contains $\mathcal{F}(X)$ for all $x \in B$ (this is also true if G is metrizable and non-discrete, see the following (iv)). In this situation, we have $|\mathrm{TIM}_{\mathbb{Q}}(x)| = 2^{2^{b(G)}}$ because $|\mathcal{F}(X)| = 2^{2^{b(G)}} = |\mathrm{TIM}(\widehat{G})|$ (see [16]) and $|\mathrm{TIM}_{\mathbb{Q}}(x)| \le |\mathrm{TIM}(\widehat{G})|$ (see [14, Corollary 4]).

(iv) Granirer in [12]–[14] investigated operators in $\mathrm{PM}_p(G)$ $(1 < p < \infty)$ with thin support. In particular, under the same assumptions on $\mathbb{P}$ and $\mathbb{Q}$ as in our Theorem 3.4, he showed that $|\mathrm{TIM}_{\mathbb{Q}}(x)| \ge 2^{\mathfrak{c}}$ if there exist $a, b \in G$ such that one of the following two conditions is satisfied:

(1) $\mathbb{R}$ (or $\mathbb{T}$) is a closed subgroup of G and there is a symmetric set $S \subset \mathbb{R}$ (or $\mathbb{T}$) such that $x \in aSb \subseteq F$;

(2) $x \in \mathrm{int}_{aHb}(F)$ for some non-discrete subgroup of G

(see [14, Theorems 6 and 7]). Furthermore, if F is first countable, then it is proved that $\mathbb{Q}/W_{\mathbb{Q}}(x)$ has l^∞ as a quotient and $\mathrm{TIM}_{\mathbb{Q}}(x)$ contains $\mathcal{F}(\mathbb{N})$ (see Granirer [13, Corollaries 6 and 7]). In this case, Granirer called $\mathbb{Q}$ *extremely non-ergodic* at $x \in \sigma(\mathbb{Q}) = F$. Notice that if G is metrizable, then condition $(*)$ of Theorem 3.4 implies that $B \subseteq \mathrm{int}(F)$; if G is non-metrizable and F satisfies $(*)$, then condition (2) holds for all $x \in B$ but F is not first countable at each $x \in B$. Therefore, Theorem 3.4 combined with the above (iii) extends Granirer's results on extreme non-ergodicity of $\mathbb{Q}$ to non-metrizable $\sigma(\mathbb{Q})$ with l^∞ replaced by $l^\infty(X)$ and condition (2) by condition $(*)$.

Recall that a Banach space Y is said to have the *weak Radon–Nikodym property* (or *WRNP* for short) if every Y-valued measure ξ on a finite complete measure space (S, Σ, η) which is η-continuous and of σ-finite variation has a Pettis-integrable derivative $f : S \to Y$ (i.e. $\xi(E) = P\text{-}\int_E f\, d\eta$ for each $E \in \Sigma$). See [20] for more information on the WRNP. It is known that if Y has the WRNP then Y does not contain any isomorphic copy of l^∞ ([20, Proposition 4]). So, our isomorphic embedding results yield the following

Corollary 3.6. *Let G be a non-discrete locally compact group. Then*

(i) $\mathrm{VN}(G)$ *does not have the WRNP*;

(ii) $\mathbb{Q}$ *and* $\mathbb{Q}/W_{\mathbb{Q}}(x)$ *do not have the WRNP if G is non-metrizable and* $\mathbb{Q}$ *and x are the same as in Theorem* 3.4.

Proof. By [17, Theorem 5.1], $\mathrm{VN}(G)$ contains an isometric copy of $l^\infty(X)$. Also, according to Theorems 3.3–3.4 and their proofs, $\mathbb{Q}$ and $\mathbb{Q}/W_{\mathbb{Q}}(x)$ contain an isomorphic copy of $l^\infty(X)$ if G is non-metrizable and $\mathbb{Q}$ and x are the same as in Theorem 3.4. Consequently, all the spaces considered in Corollary 3.6 contain an isomorphic copy of l^∞. It follows that they do not have the WRNP. ■

Remark 3.7. (a) Corollary 3.6(i) is included in Granirer [11, Theorem 5(a)], where he showed that if G in non-discrete then any nonzero ideal of $A_p(G)$ contains an isomorphic copy of l^1 and hence $PM_p(G)$ does not have the WRNP.

(b) A particular case of Granirer [12, Theorem 1], namely $p = 2$, implies that $\mathbb{P}$ does not have the WRNP if G is amenable as a discrete group, $\mathbb{P}$ is a w^*-closed $A(G)$-submodule of $\mathrm{VN}(G)$ and $\sigma(\mathbb{P})$ contains some compact perfect metrizable set.

4. Extreme non-Arens regularity of quotients of $A(G)$**.** Let G be a locally compact group. For a closed ideal J of $A(G)$, let $\mathcal{A} = A(G)/J$ and let $q : A(G) \to \mathcal{A}$ be the quotient map. Then $\mathcal{A}$ is a commutative Banach algebra and $q^* : \mathcal{A}^* \to \mathrm{VN}(G)$ is a linear isometry of $\mathcal{A}^*$ onto $J^\perp = \{T \in \mathrm{VN}(G) : \langle T, u\rangle = 0 \text{ for all } u \in J\}$. In the following, we will identify $\mathcal{A}^*$ with $J^\perp$. It is easily seen that $\mathrm{WAP}(\mathcal{A}^*) = W(\widehat{G}) \cap J^\perp \subseteq F(\widehat{G}) \cap J^\perp$.

Granirer in [14, Corollary 8] showed that if $F = Z(J)$ satisfies (1) or (2) of Remark 3.4(iv) then $A(G)/J$ is not Arens regular. If G is further assumed to be second countable, then $A(G)/J$ is extremely non-Arens regular (ENAR) (see Granirer [13, Corollaries 6 and 7]). Granirer asked if this is the case when G is not second countable (see [14]). In this section, we will propose some conditions on G and $Z(J)$ which guarantee that $A(G)/J$ is ENAR.

Let μ be the first ordinal satisfying $|\mu| = b(G)$ and let $X = \{\alpha : \alpha < \mu\}$. Also, recall that for a Banach space Y, $\mathcal{D}(Y)$ denotes the density character of Y, i.e. the smallest cardinality of a norm dense subset of Y.

THEOREM 4.1. *Let G be a non-discrete locally compact group with $\mathcal{D}(A(G)) = b(G)$. If J is is a closed ideal of $A(G)$ such that*

($*$) *$Z(J)$ contains a $G_\aleph$-set with $\aleph < b(G)$,*

then $A(G)/J$ is ENAR.

Proof. Let $\mathcal{A} = A(G)/J$. Then $\mathcal{D}(\mathcal{A}) \le \mathcal{D}(A(G)) = b(G) = |X|$. By Lemma 2.1, we only need to show that $\mathcal{A}^*/\mathrm{WAP}(\mathcal{A}^*)$ has $l^\infty(X)$ as a quotient.

For $x \in G$, let L_x be the left translation on $A(G)$ by x. Then L_x is an isometric algebra isomorphism of $A(G)$ and $Z(L_x(J)) = {}_{x^{-1}}Z(J)$. So we may assume that $e \in B \subseteq Z(J)$ for some $G_\aleph$-set B.

Let $\mathbb{P} = J^\perp$. Then $\mathbb{P}$ is a w^*-closed $A(G)$-submodule of $\mathrm{VN}(G)$ with $\sigma(\mathbb{P}) = Z(J)$. By Theorem 3.3, $\mathbb{P}/W_{\mathbb{P}}(e)$ has $l^\infty(X)$ as a quotient. But $\mathcal{A}^* = J^\perp = \mathbb{P}$ and $\mathrm{WAP}(\mathcal{A}^*) \subseteq F(\widehat{G}) \cap \mathbb{P} = W_{\mathbb{P}}(e)$ (Proposition 3.1). It follows that the quotient Banach space $\mathcal{A}^*/\mathrm{WAP}(\mathcal{A}^*)$ has $l^\infty(X)$ as a quotient. ∎

In Theorem 4.1, if $\mathrm{int}(Z(J)) \ne \emptyset$, then condition ($*$) is automatically satisfied. In particular, we have

COROLLARY 4.2. *Let G be a non-discrete locally compact group with $\mathcal{D}(A(G)) = b(G)$. Then $A(G)$ is ENAR.*

COROLLARY 4.3. *Let G be a σ-compact non-discrete locally compact group. Let J be a closed ideal of $A(G)$ such that $Z(J)$ satisfies condition ($*$). Then $A(G)/J$ is ENAR.*

In particular, $A(G)$ is ENAR for any σ-compact non-discrete locally compact group G.

Proof. According to Theorem 4.1, it suffices to prove that $\mathcal{D}(A(G)) = b(G)$.

If G is metrizable, then G is second countable (since G is σ-compact) and hence $A(G)$ is norm separable. In this situation, $\mathcal{D}(A(G)) = \aleph_0 = b(G)$.

If G is non-metrizable, from [16, Lemma 5.2] we deduce that there exist $b(G)$ many elements in $A(G)$ such that the distance between any two of them is 2. So, $\mathcal{D}(A(G)) \geq b(G)$. On the other hand, $\mathcal{D}(L^2(G)) \leq b(G)$ because G is σ-compact. Therefore, $\mathcal{D}(A(G)) \leq b(G)$. ∎

Recall that, for a closed subset F of G, $I(F)$ denotes the closed ideal of $A(G)$ consisting of all $u \in A(G)$ such that $u = 0$ on F. When $F = H$ is a closed subgroup of G, we have the following

Corollary 4.4. *Let G be a locally compact group and H a σ-compact non-discrete closed subgroup of G. Then $A(G)/I(H)$ is ENAR.*

Proof. This follows from Corollary 4.3 because $A(G)/I(H)$ is isometrically algebra-isomorphic to $A(H)$ (see [9, Lemma 3.8]). ∎

For any non-discrete locally compact group G, let G_0 be a compactly generated open subgroup of G. Since $A(G_0)$ can be isometrically embedded into $A(G)$, $\mathcal{D}(A(G)) \geq \mathcal{D}(A(G_0))$. From the proof of Corollary 4.3 we see that $\mathcal{D}(A(G_0)) = b(G_0) = b(G)$. Therefore, $\mathcal{D}(A(G)) \geq b(G)$ for any locally compact group G. It is natural to ask whether Theorem 4.1 holds when $\mathcal{D}(A(G)) > b(G)$. We will see that the answer to this question is positive for some closed ideals of $A(G)$, such as those ideals J with $Z(J)$ being a compact s-set. In this case, $J = I(Z(J)) = \{u \in A(G) : u = 0 \text{ on } Z(J)\}$.

Theorem 4.5. *Let G be a non-discrete locally compact group and J a closed ideal of $A(G)$. If $Z(J)$ is an s-set satisfying condition $(*)$ and is contained in some σ-compact open subgroup G_0 of G, then $A(G)/J$ is ENAR.*

Proof. Let $\mathcal{A} = A(G)/J$. An analogous argument to the proof of Theorem 4.1 yields that $\mathcal{A}^*/\operatorname{WAP}(\mathcal{A}^*)$ has $l^\infty(X)$ as a quotient. So, to complete the present proof, it suffices to establish a linear isometry of $J^\perp$ into $l^\infty(X)$ (by Lemma 2.1).

Let $r : A(G) \to A(G_0)$ be the restriction map and let $t : A(G_0) \to A(G)$ be the extension map defined by $tv = \mathring{v}$, where $\mathring{v} = v$ on G_0 and 0 outside G_0. Then t is a linear isometry of $A(G_0)$ into $A(G)$ and $\|r\| \leq 1$ (see [6]). Notice that $\mathcal{D}(A(G_0)) = b(G_0) = b(G) = |X|$ (see the proof of Corollary 4.3). Let $\{u_\alpha\}_{\alpha\in X}$ be norm dense in the unit ball of $A(G_0)$. Define $\Lambda : J^\perp \to l^\infty(X)$ by

$$\Lambda(T)(\alpha) = \langle T, tu_\alpha\rangle, \quad T \in J^\perp,\ \alpha \in X.$$

For each $u \in A(G)$, $u - t(ru) \in J$ because $u - t(ru) = 0$ on $G_0 \supseteq Z(J)$ and $Z(J)$ is an s-set. Then

$$\langle T, u\rangle = \langle T, t(ru)\rangle, \quad \text{for all } T \in J^\perp \text{ and } u \in A(G).$$

It follows that $\|\Lambda(T)\| = \|T\|$ for all $T \in J^\perp$, i.e. Λ is a linear isometry of $J^\perp$ into $l^\infty(X)$. ∎

COROLLARY 4.6. *Let G be a non-discrete locally compact group and J a closed ideal of $A(G)$. If $Z(J)$ is a compact s-set satisfying condition* $(*)$, *then $A(G)/J$ is ENAR.*

Proof. Since $Z(J)$ is compact, there exists a compactly generated open subgroup G_0 of G such that $Z(J) \subseteq G_0$ (see [15, (5.14)]). It follows from Theorem 4.5 that $A(G)/J$ is ENAR. ∎

Remark 4.7. Let $d(G)$ denote the smallest cardinality of a covering of G by compact sets. It can be seen that $d(G) \le b(G)$ implies $\mathcal{D}(A(G)) = b(G)$. Therefore, 4.1 and 4.2 remain true if $\mathcal{D}(A(G)) = b(G)$ is replaced by $d(G) \le b(G)$. Also, 4.3, 4.4, and 4.5 hold true if the σ-compactness of M is replaced by $d(M) \le b(M)$, where $M = G$, H, and G_0, respectively.

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