COLLOQUIUM MATHEMATICUM

VOL. 72 1997 NO. 2

LIPSCHITZ DIFFERENCES AND LIPSCHITZ FUNCTIONS

BY

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Introduction. Let $f:\mathbb{R}\to\mathbb{R}$ be continuous and such that the difference function $\Delta_h f(x)=f(x+h)-f(x)$ is bounded for every $h\in\mathbb{R}$. In a recent paper [T], S. I. Trofimchuk proved that if $\Delta_h f$ is uniformly continuous for every $h\in\mathbb{R}$ then f is also uniformly continuous. In this note we prove that in this theorem uniform continuity can be replaced by the Lipschitz property. More exactly, we investigate the following question. Suppose that f is continuous and $\Delta_h f$ is Lipschitz for every h belonging to a given subset, A, of $\mathbb{R}$. We show that this condition implies that f is Lipschitz if and only if A cannot be covered by a proper F_σ group of $\mathbb{R}$. We also discuss the analogous problem for uniform Lipschitz functions and for functions defined on the circle group $\mathbb{T}=\mathbb{R}/\mathbb{Z}$.

We shall use the following notation. We set $\mathbb{N}=\{1,2,\ldots\}$. Let $\mathbb{G}$ be any of the groups $\mathbb{R}$ or $\mathbb{T}$. If $A,B\subset\mathbb{G}$ then we define $A+B=\{a+b: a\in A, b\in B\}$. The sets $A-B$ and $-A$ are defined similarly. If $k\in\mathbb{N}$, the k-fold sum $A+\ldots+A$ is denoted by kA. By closed (open) intervals in $\mathbb{T}$ we mean closed (open) connected sets. For every $L>0$ we denote by Lip_L the set of functions $f:\mathbb{G}\to\mathbb{R}$ satisfying

$$|f(x)-f(y)|\le L|x-y|$$

for every $x,y\in\mathbb{G}$. In the case of $\mathbb{G}=\mathbb{T}$, by $|x|$ we mean $\min\{|x|,1-|x|\}$, when we identify $\mathbb{T}$ with $[0,1)$. We put $\mathrm{Lip}=\bigcup_{L>0}\mathrm{Lip}_L$. For $H\subset\mathbb{G}$, the closure and the Lebesgue outer measure of H are denoted by $\mathrm{cl}\,H$ and $|H|$.

The identity $\Delta_{h_1+h_2}f(x)=\Delta_{h_1}f(x+h_2)-\Delta_{h_2}f(x)$ gives

LEMMA 0.1. *Assume that* $L_1,L_2>0$, $f:\mathbb{G}\to\mathbb{R}$, $B_1,B_2\subset\mathbb{G}$. *If* $\Delta_h f\in\mathrm{Lip}_{L_i}$ *for every* $h\in B_i$ $(i=1,2)$ *then* $\Delta_h f\in\mathrm{Lip}_{L_1+L_2}$ *for* $h\in B_1+B_2$.

It is well known that if $F_1,F_2\subset\mathbb{G}$ are closed sets of positive measure then the interior of F_1+F_2 is non-empty. This easily implies

1991 *Mathematics Subject Classification*: Primary 26A16.

Key words and phrases: Lipschitz function, difference function, circle group.

LEMMA 0.2. *For every $A \subset \mathbb{G}$ the following statements are equivalent*:

(i) *kA is nowhere dense for every $k \in \mathbb{N}$.*
(ii) $|\mathrm{cl}(kA)| = 0$ *for every $k \in \mathbb{N}$.*

1. Functions defined on the circle group $\mathbb{T}$

THEOREM 1.1. *Let $L > 0$ and let A be a subset of $\mathbb{T}$ such that $A = -A$. Then the following statements are equivalent*:

(i) *If $f : \mathbb{T} \to \mathbb{R}$ is continuous and $\Delta_h f \in \mathrm{Lip}_L$ for each $h \in A$, then f is Lipschitz.*
(ii) *There is an $n \in \mathbb{N}$ such that nA is dense in $\mathbb{T}$.*

Proof. Suppose (ii), and let $f : \mathbb{T} \to \mathbb{R}$ be a continuous function such that $\Delta_h f \in \mathrm{Lip}_L$ for every $h \in A$. By Lemma 0.1, this implies that $\Delta_h f \in \mathrm{Lip}_{nL}$ for a set of h's everywhere dense in $\mathbb{T}$. Since f is continuous, we have $\Delta_h f \in \mathrm{Lip}_{nL}$ for every $h \in \mathbb{T}$, that is,

$$|f(x+h) - f(x) - f(y+h) + f(y)| \le nL|x-y| \tag{1}$$

for every x, y and h. Using $\int_{\mathbb{T}}(f(x+h) - f(y+h))\,dh = 0$ and (1) we obtain

$$|f(y) - f(x)| = \Big|\int_{\mathbb{T}} [f(x+h) - f(x) - f(y+h) + f(y)]\,dh\Big| \le nL|x-y|,$$

and this proves the implication (ii)⇒(i). To prove the converse we need the following lemma.

LEMMA 1.2. *Assume that $A \subset \mathbb{T}$ and $|\mathrm{cl}(kA)| = 0$ for any $k \in \mathbb{N}$. Then there is a closed set $H \subset \mathbb{T}$ such that $|H| > 0$ and $H + \mathrm{cl}(kA)$ is nowhere dense for any $k \in \mathbb{N}$.*

Proof. Denote the rationals in $\mathbb{T}$ by $\mathbb{Q}$. Clearly, $|\mathbb{Q} - \mathrm{cl}(kA)| = 0$ for any $k \in \mathbb{N}$. Let $B = \bigcup_{k\in\mathbb{N}}(\mathbb{Q} - \mathrm{cl}(kA))$; then $|B| = 0$. Choose a closed set $H \subset \mathbb{T} \setminus B$ such that $|H| > 0$. Then $H + \mathrm{cl}(kA)$ is closed. Suppose that $x \in (H + \mathrm{cl}(kA)) \cap \mathbb{Q}$. Then there exist $h \in H$ and $y \in \mathrm{cl}(kA)$ with $h + y = x \in \mathbb{Q}$, that is, $h = x - y \in \mathbb{Q} - \mathrm{cl}(kA) \subset B$, contradicting $h \in H \subset \mathbb{T} \setminus B$. This implies that $\mathbb{Q} \cap (H + \mathrm{cl}(kA)) = \emptyset$, and hence the closed set $H + \mathrm{cl}(kA)$ is nowhere dense.

Now we turn to the proof of the implication (i)⇒(ii). We may assume that $L = 1$, and $0 \in A$. Suppose that (ii) is not true; this easily implies that kA is nowhere dense for every $k \in \mathbb{N}$. We shall construct a continuous non-Lipschitz function $f : \mathbb{T} \to \mathbb{R}$ such that $\Delta_h f \in \mathrm{Lip}_1$ for each $h \in A$. We shall define f as $\int_0^x g(t)\,dt$, where $g : \mathbb{T} \to \mathbb{R}$ is summable and $\int_0^1 g(t)\,dt = 0$. (In this proof we identify $\mathbb{T}$ with $[0, 1)$.) Then f will be continuous on $\mathbb{T}$ and will satisfy $f(0) = \lim_{x\to 1-} f(x) = f(1)$.

By Lemma 1.2 we can choose a closed set $H \subset \mathbb{T}$ such that $|H| > 0$ and $H + \mathrm{cl}(kA)$ is nowhere dense for any $k \in \mathbb{N}$. Put $H_{-1} = \emptyset$, $H_0 = H$ and $H_j = H + \mathrm{cl}(jA) = H_{j-1} + \mathrm{cl}(A)$ for $j = 1, 2, \ldots$ From $0 \in A$ it follows that $H_{j-1} \subset H_j$. Put $H_\infty = \bigcup_{j\in\mathbb{N}} H_j$; then $\lim_{j\to\infty} |H_\infty \setminus H_{j-1}| = 0$ by $|\mathbb{T}| = 1$. Let $j_0 = 1$. If j_{k-1} is defined for a $k \in \mathbb{N}$, choose j_k such that $j_k > j_{k-1}$ and

$$|H_\infty \setminus H_{j_k-1}| < 1/(k2^k). \tag{2}$$

For $j_{k-1} < j \le j_k$ we put $c_j = k$. Thus by induction we have defined j_k for all k, and c_j for all j. We put $g_1(x) = c_j$ if $x \in H_j \setminus H_{j-1}$ ($j \in \mathbb{N}$), and $g_1(x) = 0$ for $x \in \mathbb{T} \setminus H_\infty$. From (2) it follows that $c = \int_{\mathbb{T}} g_1 < \infty$. Let $g(x) = g_1(x) - c$ for $x \in \mathbb{T}$; then $\int_{\mathbb{T}} g = 0$.

Let $x \in H_\infty$ and $h \in A$. Then $y = x+h \in H_\infty$, and thus $x \in H_{j_x} \setminus H_{j_x-1}$ and $y \in H_{j_y} \setminus H_{j_y-1}$ with suitable j_x and j_y. If $j_x \le j_y$, then $y = x + h \in H_{j_x} + \mathrm{cl}(A) = H_{j_x+1}$, and hence $j_y = j_x$ or $j_y = j_x + 1$. Thus, in this case, $|g(y)-g(x)| = 0$ or $|g(y)-g(x)| = |(c_{j_x+1}-c)-(c_{j_x}-c)| \le 1$. If, on the other hand, $j_x > j_y$ then, using $A = -A$, $x = y-h$, and interchanging the roles of x and y, we reach the same conclusion. Therefore, $|g(x+h)-g(x)| \le 1$ holds for any $x \in H_\infty$ and $h \in A$. If $x \in \mathbb{T} \setminus H_\infty$ and $h \in A$ then $x + h \in \mathbb{T} \setminus H_\infty$. Indeed, from $x+h \in H_\infty$ it follows that $x+h \in H_j$ for some $j \ge 0$, and then $A = -A$ implies $x = (x+h)-h \in H_j + A \subset H_\infty$, contradicting $x \in \mathbb{T}\setminus H_\infty$. Therefore $|g(x+h) - g(x)| = c - c = 0$ holds for any $x \in \mathbb{T} \setminus H_\infty$ and $h \in A$. Thus $|g(x + h) - g(x)| \le 1$ for $x \in \mathbb{T}$ and $h \in A$. Let $f(x) = \int_0^x g(t)\,dt$ for $x \in \mathbb{T}$. To show that $\Delta_h f \in \mathrm{Lip}_1$ for $h \in A$, let $x, d \in \mathbb{T}$ be given. We have

$$\begin{aligned}|\Delta_h f(x + d) - \Delta_h f(x)| &= |\Delta_d f(x + h) - \Delta_d f(x)| \\ &= \Big|\int_x^{x+d} (g(t + h) - g(t))\,dt\Big| \le |d|.\end{aligned}$$

That is, $\Delta_h f \in \mathrm{Lip}_1$. Observe that we may replace A by $A \cup \{1/n : n \in \mathbb{N}\} \cup \{-(1/n) : n \in \mathbb{N}\}$. Then $A_\infty = \bigcup_{k\in\mathbb{N}} \mathrm{cl}(kA)$ is dense in $\mathbb{T}$. Thus, for any subinterval J of $\mathbb{T}$, we have $0 < |H_\infty \cap J| = |(H + A_\infty) \cap J|$. Since the H_j's are nowhere dense, there are infinitely many j's for which $|H_j \setminus H_{j-1}| > 0$. Hence, putting $S_K = \{x \in \mathbb{T} : |g(x)| > K\}$ ($K > 0$), we have $|S_K| > 0$ for all $K > 0$. Since $f' = (\int_0^x g(t)\,dt)' = g(x)$ almost everywhere on $\mathbb{T}$, it follows that, for any K, the inequality $|f'(x)| > K$ holds for almost every $x \in S_K$. Thus f cannot be Lipschitz and hence (i) does not hold. This completes the proof of Theorem 1.1.

Remark 1.3. Since $\Delta_{-h} f \in \mathrm{Lip}_L$ follows from $\Delta_h f \in \mathrm{Lip}_L$, the assumption $A = -A$ is natural. We show that this assumption cannot be deleted from the implication (i)$\Rightarrow$(ii) of Theorem 1.1.

Indeed, by a result of Haight [H], there exists an F_σ subset B of the positive real line such that $B - B = \mathbb{R}$ but kB has zero Lebesgue measure for any positive integer k. Choose closed compact sets F_n of measure zero such that $B = \bigcup_{n=1}^{\infty} F_n$ and $F_1 \subset F_2 \subset \ldots$ Since $B - B = \mathbb{R}$, it follows that $F_n - F_n$ contains an interval for a suitable $n \in \mathbb{N}$. Taking this F_n "mod1" we obtain a nowhere dense compact set A such that kA is nowhere dense for every $k \in \mathbb{N}$ and $A - A$ contains an interval. It is easy to see, following the proof of Theorem 1.1, that if $\Delta_h f \in \mathrm{Lip}_L$ for every $h \in A$ then f is Lipschitz.

Next we turn to the non-uniform case, i.e. to the case when the difference functions are Lipschitz but not necessarily with the same constant.

THEOREM 1.4. *For every $A \subset \mathbb{T}$ the following statements are equivalent*:

(i) *If $f : \mathbb{T} \to \mathbb{R}$ is continuous and $\Delta_h f \in \mathrm{Lip}$ for every $h \in A$ then f is Lipschitz.*

(ii) *There is no proper F_σ subgroup of $\mathbb{T}$ containing A.*

Proof. (ii)⇒(i). Suppose (ii), and let $f : \mathbb{T} \to \mathbb{R}$ be a continuous function such that $\Delta_h f \in \mathrm{Lip}$ for every $h \in A$. Put $G = \{h \in \mathbb{T} : \Delta_h f \in \mathrm{Lip}\}$ and $G_n = \{h \in \mathbb{T} : \Delta_h f \in \mathrm{Lip}_n\}$ for $n \in \mathbb{N}$. Then $G = \bigcup_{n\in\mathbb{N}} G_n$. Since f is continuous, it is easy to verify that the sets G_n are closed and G is an F_σ set. The identities $\Delta_{-h_1} f(x) = f(x - h_1) - f(x) = -\Delta_{h_1} f(x - h_1)$ and $\Delta_{h_1+h_2} f(x) = \Delta_{h_2} f(x + h_1) - \Delta_{h_1} f(x)$ show that G is a group. Since $A \subset G$, (ii) implies that $G = \mathbb{T}$. Therefore, by the Baire category theorem, there exists $n \in \mathbb{N}$ such that G_n contains a subinterval of $\mathbb{T}$. Then $kG_n = \mathbb{T}$ for some $k \in \mathbb{N}$. By Theorem 1.1, this implies that f is Lipschitz.

(i)⇒(ii). Suppose that there exists an F_σ group C such that $A \subset C \subset \mathbb{T}$ and $C \neq \mathbb{T}$. Then we can choose nowhere dense closed sets C_n such that $C = \bigcup_{n\in\mathbb{N}} C_n$. Since $C = -C$, we may assume $C_n = -C_n$. Setting $D/n = \{x \in \mathbb{T} : |x| < 1/n,\ n \cdot x \in D\}$ for every $D \subset \mathbb{T}$, we define $B = \{0\} \cup \bigcup_{n\in\mathbb{N}} C_n/n$. Then $B = -B$ and B is a nowhere dense closed set. Thus, for each $k \in \mathbb{N}$, the set kB is a closed subset of $\bigcup_{n\in\mathbb{N}} C/n$, as $kC = C$. Since C is of first category, so is $\bigcup_{n\in\mathbb{N}} C/n$. Therefore kB is a closed set of first category and thus it is nowhere dense. By Theorem 1.1, there exists a non-Lipschitz and continuous function f for which $\Delta_h f \in \mathrm{Lip}_1$ if $h \in B$. It is clear that the group generated by B contains all C_n's and hence all of C. Thus $\Delta_h f \in \mathrm{Lip}$ for $h \in C$; that is, (i) does not hold.

2. Functions defined on the real line

THEOREM 2.1. *Let $L > 0$ and let A be a bounded subset of $\mathbb{R}$ such that $A = -A$. Then the following statements are equivalent*:

(i) *If $f : \mathbb{R} \to \mathbb{R}$ is continuous, $\Delta_h f$ is bounded for some $h \neq 0$, and $\Delta_h f \in \mathrm{Lip}_L$ for each $h \in A$ then f is Lipschitz.*

(i′) *If* $f : \mathbb{R} \to \mathbb{R}$ *is continuous*, $\Delta_h f$ *is bounded for every* $h \in \mathbb{R}$ *and* $\Delta_h f \in \mathrm{Lip}_L$ *for each* $h \in A$ *then* f *is Lipschitz.*

(ii) *There is an* $n \in \mathbb{N}$ *such that* nA *is dense in a nondegenerate interval.*

Proof. (ii)⇒(i). We can assume that $\Delta_h f$ is bounded for some $h > 0$. Fix such an $h = h_0$, and let B be a constant such that $|\Delta_{h_0} f(x)| \le B$ for all x. Fix also an n with nA dense in an interval $[m, M]$, for some $m < M$. By making n larger, if necessary, we can assume that $m + h_0 \le M$. Since f is continuous, $\Delta_h f \in \mathrm{Lip}_{nL}$ for all h in $[m, M]$. Now, for $x < y$, $\int_m^{m+h_0} (f(y+h) - f(x+h))\,dh = \int_{x+m}^{y+m} (f(h+h_0) - f(h))\,dh$, so we get

$$
\begin{aligned}
&|f(y) - f(x)| \\
&= \Big|(1/h_0) \int_m^{m+h_0} (f(x+h) - f(x) - f(y+h) + f(y) \\
&\qquad\qquad + f(y+h) - f(x+h))\,dh\Big| \\
&\le (1/h_0) \int_m^{m+h_0} |\Delta_h f(x) - \Delta_h f(y)|\,dh + (1/h_0) \int_{x+m}^{y+m} |f(h+h_0) - f(h)|\,dh \\
&\le nL|y-x| + (1/h_0) \int_{x+m}^{y+m} |\Delta_{h_0} f(h)|\,dh \le nL|y-x| + (B/h_0)|y-x|.
\end{aligned}
$$

Thus, $f \in \mathrm{Lip}_{nL+B/h_0}$.

(i)⇒(i′) is obvious.

(i′)⇒(ii). Suppose that (ii) is not true. Let ν denote the canonical homomorphism which maps $\mathbb{R}$ onto $\mathbb{R}/\mathbb{Z} = \mathbb{T}$. Since A is bounded and kA is nowhere dense in $\mathbb{R}$, it is easy to see that kB is nowhere dense in $\mathbb{T}$, where $B = \nu(A) \subset \mathbb{T}$. Applying Theorem 1.1, we find a continuous non-Lipschitz function $g : \mathbb{T} \to \mathbb{R}$ such that $\Delta_h g \in \mathrm{Lip}_L$ for each $h \in B$. Extending this function g from $\mathbb{T}$ onto $\mathbb{R}$ periodically, that is, taking $f = g \circ \nu$, we obtain a periodic continuous non-Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ such that $\Delta_h f \in \mathrm{Lip}_L$ for each $h \in A \subset \nu^{-1}(B)$. Since f is obviously bounded, $\Delta_h f$ is also bounded for each $h \in \mathbb{R}$.

Remark 2.2. 1. The condition on the boundedness of the differences $\Delta_h f$ cannot be deleted. Indeed, for $f(x) = x^2$, $\Delta_h f \in \mathrm{Lip}_2$ for every $h \in [0, 1]$, but f is not Lipschitz.

2. The boundedness of A was not used in (ii)⇒(i). On the other hand, we do not know whether or not (i′)⇒(ii) is true for each $A \subset \mathbb{R}$ satisfying $A = -A$.

In the non-uniform case we obtain

THEOREM 2.3. *For every* $A \subset \mathbb{R}$, *the following two statements are equivalent*:

(i) *If* $f : \mathbb{R} \to \mathbb{R}$ *is continuous*, $\Delta_h f$ *is bounded for every* $h \in \mathbb{R}$ *and* $\Delta_h f \in \mathrm{Lip}$ *for each* $h \in A$ *then* f *is Lipschitz.*

(ii) *There is no proper* F_σ *subgroup of* $\mathbb{R}$ *containing* A.

Proof. (ii)⇒(i). Suppose (ii) and let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\Delta_h f$ is bounded for every $h \in \mathbb{R}$ and $\Delta_h f \in \mathrm{Lip}$ for each $h \in A$. Let $G = \{h \in \mathbb{R} : \Delta_h f \in \mathrm{Lip}\}$ and $G_n = \{h \in \mathbb{R} : \Delta_h f \in \mathrm{Lip}_n\}$ for $n \in \mathbb{N}$. Then $G = \bigcup_{n\in\mathbb{N}} G_n$ and G is an F_σ group containing A. Hence $G = \mathbb{R}$. By the Baire category theorem, G_n is dense in an interval for some $n \in \mathbb{N}$. Next it suffices to apply Theorem 2.1.

(i)⇒(ii). Assume that there exists an F_σ group $C \neq \mathbb{R}$ containing A. Since C must be of the first category, we can choose closed nowhere dense sets $C_n \subset [-n, n]$ such that $C_n = -C_n$, and $C = \bigcup_{n\in\mathbb{N}} C_n$. Setting $D/n = \{x/n : x \in D\}$ for every $D \subset \mathbb{R}$, we define $B = \{0\} \cup \bigcup_{n\in\mathbb{N}} (C_n/n^2)$. Then $B = -B$ is bounded, closed and nowhere dense. The rest of the proof is similar to the (i)⇒(ii) part of the proof of Theorem 1.4.

Acknowledgements. We would like to thank the referee, who has simplified the proof of Theorem 2.1. The first author thanks Z. Buczolich and M. Laczkovich for inviting him to the Eötvös University in December '93 when the work on this paper started.

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Received 27 February 1995;
revised 27 May 1996