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CESÀRO SUMMABILITY OF ONE- AND TWO-DIMENSIONAL TRIGONOMETRIC-FOURIER SERIES

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We introduce p-quasilocal operators and prove that, if a sublinear operator T is p-quasilocal and bounded from L_∞ to L_∞, then it is also bounded from the classical Hardy space $H_p(\mathbf{T})$ to L_p $(0 < p \le 1)$. As an application it is shown that the maximal operator of the one-parameter Cesàro means of a distribution is bounded from $H_p(\mathbf{T})$ to L_p $(3/4 < p \le \infty)$ and is of weak type (L_1, L_1). We define the two-dimensional dyadic hybrid Hardy space $H_1^\sharp(\mathbf{T}^2)$ and verify that the maximal operator of the Cesàro means of a two-dimensional function is of weak type $(H_1^\sharp(\mathbf{T}^2), L_1)$. So we deduce that the two-parameter Cesàro means of a function $f \in H_1^\sharp(\mathbf{T}^2) \supset L \log L$ converge a.e. to the function in question.

1. Introduction. It can be found in Zygmund [23] that the Cesàro means $\sigma_n f$ of a function $f \in L_1(\mathbf{T})$ converge a.e. to f as $n \to \infty$ and that if $f \in L \log^+ L(\mathbf{T}^2)$ then the two-parameter Cesàro summability holds. Analogous results for Walsh–Fourier series are due to Fine [11] and Móricz, Schipp and Wade [15].

The Hardy–Lorentz spaces $H_{p,q}$ of distributions on the unit circle are introduced with the $L_{p,q}$ Lorentz norms of the non-tangential maximal function. Of course, $H_p = H_{p,p}$ are the usual Hardy spaces $(0 < p \le \infty)$.

In the one-dimensional case it is known (see Zygmund [23] and Torchinsky [20]) that the maximal operator of the Cesàro means $\sup_{n\in\mathbb{N}} |\sigma_n|$ is of weak type (L_1, L_1), i.e.

$$\sup_{\gamma>0} \gamma\lambda(\sup_{n\in\mathbb{N}} |\sigma_n f| > \gamma) \le C\|f\|_1 \qquad (f \in L_1(\mathbf{T}))$$

(for the Walsh case see Schipp [17]). Also, for Walsh–Fourier series, the

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boundedness of the operator $\sup_{n\in\mathbb{N}}|\sigma_n|$ from H_p to L_p was shown by Fujii [12] ($p=1$) and by Weisz [21] ($1/2<p\le 1$).

In this paper we generalize these results for trigonometric-Fourier series with the help of the so-called p-quasilocal operators. An operator T is *p-quasilocal* ($0<p\le 1$) if for all p-atoms a the integral of $|Ta|^p$ over $\mathbf{T}\setminus I$ is less than an absolute constant where I is the support of the atom a. We shall verify that a sublinear, p-quasilocal operator T which is bounded from L_∞ to L_∞ is also bounded from H_p to L_p ($0<p\le 1$). By interpolation we find that T is bounded from $H_{p,q}$ to $L_{p,q}$ as well ($0<p<\infty,\ 0<q\le\infty$) and is of weak type (L_1,L_1).

It will be shown that $\sup_{n\in\mathbb{N}}|\sigma_n|$ is p-quasilocal for each $3/4<p\le 1$. Consequently, $\sup_{n\in\mathbb{N}}|\sigma_n|$ is bounded from $H_{p,q}$ to $L_{p,q}$ for $3/4<p<\infty$ and $0<q\le\infty$ and is of weak type (L_1,L_1). We will extend this result also to (C,β) means.

For two-dimensional trigonometric-Fourier series we will verify that $\sup_{n,m\in\mathbb{N}}|\sigma_{n,m}|$ is of weak type $(H_1^\sharp,L_1)$ where $H_1^\sharp$ is defined by the L_1-norm of the two-dimensional hybrid non-tangential maximal function. Recall that $L\log L(\mathbf{T}^2)\subset H_1^\sharp$ (see Zygmund [23]). A usual density argument implies then that $\sigma_{n,m}f\to f$ a.e. as $\min(n,m)\to\infty$ whenever $f\in H_1^\sharp$.

2. Preliminaries and notations. For a set $\mathbf{X}\ne\emptyset$ let $\mathbf{X}^2$ be the Cartesian product $\mathbf{X}\times\mathbf{X}$; moreover, let $\mathbf{T}:=[-\pi,\pi)$ and λ be the Lebesgue measure. We also use the notation $|I|$ for the Lebesgue measure of the set I. We briefly write L_p or $L_p(\mathbf{T}^j)$ instead of the real $L_p(\mathbf{T}^j,\lambda)$ space ($j=1,2$), and the norm (or quasinorm) of this space is defined by $\|f\|_p := (\int_{\mathbf{T}^j}|f|^p\,d\lambda)^{1/p}$ ($0<p\le\infty$). For simplicity, we assume that for a function $f\in L_1$ we have $\int_{\mathbf{T}} f\,d\lambda=0$.

The distribution function of a Lebesgue-measurable function f is defined by

$$\lambda(\{|f|>\gamma\}):=\lambda(\{x:|f(x)|>\gamma\})\qquad(\gamma\ge 0).$$

The *weak* L_p space L_p^* ($0<p<\infty$) consists of all measurable functions f for which

$$\|f\|_{L_p^*}:=\sup_{\gamma>0}\gamma\lambda(\{|f|>\gamma\})^{1/p}<\infty;$$

moreover, we set $L_\infty^*=L_\infty$.

The spaces L_p^* are special cases of the more general Lorentz spaces $L_{p,q}$. In their definition another concept is used. For a measurable function f the *non-increasing rearrangement* is defined by

$$\widetilde{f}(t):=\inf\{\gamma:\lambda(\{|f|>\gamma\})\le t\}.$$

The Lorentz space $L_{p,q}$ is defined as follows: for $0 < p < \infty$, $0 < q < \infty$,

$$\|f\|_{p,q} := \left(\int_0^\infty \widetilde{f}(t)^q t^{q/p} \, \frac{dt}{t} \right)^{1/q}$$

while for $0 < p \le \infty$,

$$\|f\|_{p,\infty} := \sup_{t>0} t^{1/p} \widetilde{f}(t).$$

Let

$$L_{p,q} := L_{p,q}(\mathbf{T}^j, \lambda) := \{f : \|f\|_{p,q} < \infty\} \quad (j = 1, 2).$$

One can show the following equalities:

$$L_{p,p} = L_p, \quad L_{p,\infty} = L_p^* \quad (0 < p \le \infty)$$

(see e.g. Bennett–Sharpley [1] or Bergh–Löfström [2]).

Let f be a distribution on $C^\infty(\mathbf{T})$ (briefly $f \in \mathcal{D}'(\mathbf{T}) = \mathcal{D}'$). The nth Fourier coefficient is defined by $\widehat{f}(n) := f(e^{-\imath nx})$ where $\imath = \sqrt{-1}$. In the special case when f is an integrable function,

$$\widehat{f}(n) = \frac{1}{2\pi} \int_{\mathbf{T}} f(x) e^{-\imath nx} \, dx.$$

Denote by $s_n f$ the nth partial sum of the Fourier series of a distribution f, namely,

$$s_n f(x) := \sum_{k=-n}^{n} \widehat{f}(k) e^{\imath kx}.$$

For $f \in \mathcal{D}'$ and $z := re^{\imath x}$ $(0 < r < 1)$ let

$$u(z) = u(re^{\imath x}) := f * P_r(x)$$

where $*$ denotes the convolution and

$$P_r(x) := \sum_{k=-\infty}^{\infty} r^{|k|} e^{\imath kx} = \frac{1 - r^2}{1 + r^2 - 2r \cos x} \quad (x \in \mathbf{T})$$

is the Poisson kernel. It is easy to show that $u(z)$ is a harmonic function on the unit disc and

$$u(re^{\imath x}) = \sum_{k=-\infty}^{\infty} \widehat{f}(k) r^{|k|} e^{\imath kx}$$

with absolute and uniform convergence (see e.g. Kashin–Saakyan [13], Edwards [8]).

Let $0 < \alpha < 1$ be an arbitrary number. We denote by $\Omega_\alpha(x)$ $(x \in \mathbf{T})$ the region bounded by two tangents to the circle $|z| = \alpha$ from $e^{\imath x}$ and the longer arc of the circle included between the points of tangency. The non-tangential maximal function is defined by

$$u_\alpha^*(x) := \sup_{z\in\Omega_\alpha(x)} |u(z)| \qquad (0<\alpha<1).$$

For $0<p,q\le\infty$ the *Hardy–Lorentz space* $H_{p,q}(\mathbf{T}) = H_{p,q}$ consists of all distributions f for which $u_\alpha^* \in L_{p,q}$; we set

$$\|f\|_{H_{p,q}} := \|u_{1/2}^*\|_{p,q}.$$

The equivalence $\|u_\alpha^*\|_{p,q} \sim \|u_{1/2}^*\|_{p,q}$ $(0<p,q<\infty,\ 0<\alpha<1)$ was proved in Burkholder–Gundy–Silverstein [3] and Fefferman–Stein [10]. Note that in case $p=q$ the usual definition of Hardy spaces $H_{p,p}=H_p$ is obtained. For other equivalent definitions we also refer to the previous two papers. It is known that if $f\in H_p$ then $f(x)=\lim_{r\to1} u(re^{\imath x})$ in the sense of distributions (see Fefferman–Stein [10]). Recall that $L_1\subset H_{1,\infty}$ and $L\log L\subset H_1$; more exactly,

$$\|f\|_{H_{1,\infty}} = \sup_{\gamma>0}\gamma\lambda(u_{1/2}^*>\gamma) \le \|f\|_1 \qquad (f\in L_1) \tag{1}$$

and

$$\|f\|_{H_1} \le C + CE(|f|\log^+|f|) \qquad (f\in L\log L) \tag{2}$$

where $\log^+ u = 1_{\{u>1\}}\log u$. Moreover, $H_{p,q}\sim L_{p,q}$ for $1<p<\infty$, $0<q\le\infty$ (see Fefferman–Stein [10], Stein [19], Fefferman–Rivière–Sagher [9]).

The following interpolation result concerning Hardy–Lorentz spaces will be used several times in this paper (see Fefferman–Rivière–Sagher [9]).

Theorem A. *If a sublinear operator T is bounded from H_{p_0} to L_{p_0} and from L_∞ to L_∞ then it is also bounded from $H_{p,q}$ to $L_{p,q}$ if $p_0<p<\infty$ and $0<q\le\infty$.*

3. Quasilocal operators. The atomic decomposition is a useful characterization of Hardy spaces. To demonstrate this let us introduce first the concept of an atom. A *generalized interval* on $\mathbf{T}$ is either an interval $I\subset\mathbf{T}$ or $I=[-\pi,x)\cup[y,\pi)$. A bounded measurable function a is a *p-atom* if there exists a generalized interval I such that

(i) $\int_I a(x)x^\alpha\,dx = 0$ where $\alpha\in\mathbb{N}$ and $\alpha\le[1/p-1]$, the integer part of $1/p-1$,
(ii) $\|a\|_\infty \le |I|^{-1/p}$,
(iii) $\{a\ne0\}\subset I$.

The basic result on the atomic decomposition is stated as follows (see Coifman [4], Coifman–Weiss [5] and also Weisz [22]).

Theorem B. *A distribution f is in H_p $(0<p\le1)$ if and only if there exist a sequence $(a_k, k\in\mathbb{N})$ of p-atoms and a sequence $(\mu_k, k\in\mathbb{N})$ of real*

numbers such that

$$\sum_{k=0}^{\infty} \mu_k a_k = f \quad \textit{in the sense of distributions},$$
(3)
$$\sum_{k=0}^{\infty} |\mu_k|^p < \infty.$$

Moreover, the following equivalence of norms holds:

$$\|f\|_{H_p} \sim \inf \Big(\sum_{k=0}^{\infty} |\mu_k|^p \Big)^{1/p} \tag{4}$$

where the infimum is taken over all decompositions of f of the form (3).

Motivated by the definition in Móricz–Schipp–Wade [15] we introduce the quasilocal operators. Their definition is weakened and extended here.

An operator T which maps the set of distributions into the collection of measurable functions will be called *p-quasilocal* if there exists a constant $C_p > 0$ such that

$$\int_{\mathbf{T}\setminus 4I} |Ta|^p \, d\lambda \le C_p$$

for every p-atom a where I is the support of the atom and $4I$ is the generalized interval with the same center as I and with length $4|I|$.

The quasilocal operators were defined in [15] only for $p = 1$ and for L_1 functions instead of atoms.

The following result gives sufficient conditions for T to be bounded from H_p to L_p. For the sake of completeness it is verified here.

THEOREM 1. *Suppose that the operator T is sublinear and p-quasilocal for some $0 < p \le 1$. If T is bounded from L_∞ to L_∞ then*

$$\|Tf\|_p \le C_p \|f\|_{H_p} \qquad (f \in H_p).$$

Proof. Suppose that a is a p-atom with support I. By the p-quasilocality and L_∞ boundedness of T we obtain

$$\begin{aligned} \int_{\mathbf{T}} |Ta|^p \, d\lambda &= \int_{4I} |Ta|^p \, d\lambda + \int_{\mathbf{T}\setminus 4I} |Ta|^p \, d\lambda \\ &\le \|T\|_\infty^p \|a\|_\infty^p 4|I| + C_p = C_p \end{aligned}$$

where the symbol C_p may denote different constants in different contexts. Applying Theorem B, we get

$$\|Tf\|_p^p \le \sum_{k=0}^{\infty} |\mu_k|^p \|Ta_k\|_p^p \le C_p \|f\|_{H_p}^p,$$

which proves the theorem. ∎

Taking into account Theorem A and (1) we have

Corollary 1. *Suppose that the operator T is sublinear and p-quasilocal for each $p_0 < p \le 1$. If T is bounded from L_∞ to L_∞ then*

$$\|Tf\|_{p,q} \le C_{p,q}\|f\|_{H_{p,q}} \qquad (f \in H_{p,q})$$

for every $p_0 < p < \infty$ and $0 < q \le \infty$. In particular, T is of weak type $(1,1)$, i.e. if $f \in L_1$ then

$$\|Tf\|_{1,\infty} = \sup_{\gamma>0} \gamma\lambda(|Tf| > \gamma) \le C_1\|f\|_{H_{1,\infty}} \le C_1\|f\|_1.$$

4. Cesàro summability of one-dimensional trigonometric-Fourier series. For $n \in \mathbb{N}$ and a distribution f the *Cesàro mean* of order n of the Fourier series of f is given by

$$\sigma_n f := \frac{1}{n+1}\sum_{k=0}^{n} s_k f = f * K_n \qquad (n \in \mathbb{N})$$

where K_n is the Fejér kernel of order n. It is shown in Zygmund [23] that

$$0 \le K_n(t) \le \frac{\pi^2}{(n+1)t^2} \qquad (0 < |t| < \pi) \tag{5}$$

and

$$\int_{\mathbf{T}} K_n(t)\,dt = \pi. \tag{6}$$

As an application of Theorem 1 we have the following result.

Theorem 2. *There are absolute constants C and $C_{p,q}$ such that*

$$\|\sup_{n\in\mathbb{N}} |\sigma_n f|\|_{p,q} \le C_{p,q}\|f\|_{H_{p,q}} \qquad (f \in H_{p,q}) \tag{7}$$

for every $3/4 < p < \infty$ and $0 < q \le \infty$. In particular, if $f \in L_1$ then

$$\lambda(\sup_{n\in\mathbb{N}} |\sigma_n f| > \gamma) \le \frac{C}{\gamma}\|f\|_1 \qquad (\gamma > 0). \tag{8}$$

P r o o f. By Corollary 1 the proof of Theorem 2 will be complete if we show that the operator $\sup_{n\in\mathbb{N}} |\sigma_n|$ is p-quasilocal for each $3/4 < p \le 1$ and bounded from L_∞ to L_∞.

The boundedness follows from (6). To verify the p-quasilocality for $3/4 < p \le 1$ let a be an arbitrary p-atom with support I and $2^{-K-1} < |I|/\pi \le 2^{-K}$ $(K \in \mathbb{N})$. We can suppose that the center of I is zero. In this case

$$[-\pi 2^{-K-2}, \pi 2^{-K-2}] \subset I \subset [-\pi 2^{-K-1}, \pi 2^{-K-1}].$$

Obviously,

$$\begin{aligned}\int_{\mathbf{T}\setminus 4I} \sup_{n\in\mathbb{N}} |\sigma_n a(x)|^p\,dx &\le \sum_{|i|=1}^{2^K-1} \int_{\pi i 2^{-K}}^{\pi(i+1)2^{-K}} \sup_{n\in\mathbb{N}} |\sigma_n a(x)|^p\,dx \\ &\le \sum_{|i|=1}^{2^K-1} \int_{\pi i 2^{-K}}^{\pi(i+1)2^{-K}} \sup_{n\ge r_i} |\sigma_n a(x)|^p\,dx \\ &\quad + \sum_{|i|=1}^{2^K-1} \int_{\pi i 2^{-K}}^{\pi(i+1)2^{-K}} \sup_{n< r_i} |\sigma_n a(x)|^p\,dx \\ &= (A)+(B)\end{aligned}$$

where $r_i := [2^K/i^\alpha]$ $(i\in\mathbb{N})$ with $\alpha>0$ to be chosen later.

It follows from (5) and from the definition of the atom that

$$|\sigma_n a(x)| = \Big|\int_{\mathbf{T}} a(t)K_n(x-t)\,dt\Big| \le C_p 2^{K/p}\int_I \frac{1}{(n+1)(x-t)^2}\,dt.$$

By a simple calculation we get

$$\int_{-\pi 2^{-K-1}}^{\pi 2^{-K-1}} \frac{1}{(x-t)^2}\,dt \le \frac{C2^{-K}}{(\pi|i|2^{-K}-\pi 2^{-K-1})^2} \le \frac{C2^K}{i^2}$$

if $x\in[\pi i2^{-K},\pi(i+1)2^{-K})$ $(|i|\ge 1)$. Hence

$$|\sigma_n a(x)|^p \le C_p 2^{K+Kp}\frac{1}{(n+1)^p i^{2p}}.$$

Using the value of r_i we can conclude that

$$(A)\le C_p\sum_{i=1}^{2^K-1} 2^{-K}2^{K+Kp}\frac{1}{(r_i+1)^p i^{2p}} \le C_p\sum_{i=1}^{2^K-1}\frac{1}{i^{2p-\alpha p}}.$$

This series is convergent if

$$\alpha < \frac{2p-1}{p}\ (\le 1). \tag{9}$$

Now let us consider (B). It is well-known that

$$\sigma_n a(x) = \sum_{|j|=1}^{n}\Big(1-\frac{|j|}{n+1}\Big)\widehat{a}(j)e^{\imath jx}.$$

If $n<r_i$ then

$$|\sigma_n a(x)| \le \sum_{|j|=1}^{n}\Big(\frac{n+1-|j|}{|j|}\Big)|\widehat{a}(j)| \le \sum_{|j|=1}^{r_i}\Big(\frac{r_i-|j|}{|j|}\Big)|\widehat{a}(j)|.$$

On the other hand, by the definition of the atom,

$$|\widehat{a}(j)| = \left| \frac{1}{2\pi} \int_I a(x)(e^{-\imath jx} - 1)\, dx \right| \le \frac{1}{2\pi} \int_I |a(x)| \cdot |jx|\, dx \le \frac{|j| \cdot |I|^{2-1/p}}{4\pi}.$$

Therefore

$$\sup_{n<r_i} |\sigma_n a(x)| \le C_p \sum_{j=1}^{r_i} (r_i - j) 2^{-K(2-1/p)} \le C_p r_i^2 2^{-K(2-1/p)}.$$

Finally, we can estimate (B):

$$(B) \le C_p \sum_{i=1}^{2^K-1} 2^{-K} \left(\frac{2^K}{i^\alpha} \right)^{2p} 2^{-K(2-1/p)p} = C_p \sum_{i=1}^{2^K-1} \frac{1}{i^{2\alpha p}}.$$

The last series converges if

$$\alpha > \frac{1}{2p}. \tag{10}$$

The number α satisfies (9) and (10) if and only if $3/4 < p \le 1$. The proof of the theorem is complete. ∎

Note that (8) can be found in Zygmund [23] or in Torchinsky [20], however, (7) was known only for Walsh–Fourier series (see Weisz [21]).

5. (C, β) **summability of one-dimensional trigonometric-Fourier series.** In this section we generalize Theorem 2. For $0 < \beta \le 1$ let

$$A_j^\beta := \binom{j+\beta}{j} = \frac{(\beta+1)(\beta+2)\dots(\beta+j)}{j!} = O(j^\beta) \qquad (j \in \mathbb{N})$$

(see Zygmund [23]). The (C, β) *means* of a distribution f are defined by

$$\sigma_n^\beta f := \frac{1}{A_n^\beta} \sum_{k=0}^{n} A_{n-k}^{\beta-1} s_k f = f * K_n^\beta$$

where the K_j^β kernel satisfies the conditions

$$|K_j^\beta(t)| \le \frac{C_\beta}{j^\beta t^{\beta+1}} \qquad (0 < |t| < \pi)$$

and

$$\int_{\mathbf{T}} |K_j^\beta(t)|\, dt = C_\beta \qquad (j \in \mathbb{N})$$

(see Zygmund [23]). In case $\beta = 1$ we get the Cesàro means.

The following result can be proved with the same method as Theorem 2.

THEOREM 3. *If $0<\beta\le 1$ then there are absolute constants C and $C_{p,q}$ such that*

$$\|\sup_{n\in\mathbb{N}} \sigma_n^\beta f\|_{p,q} \le C_{p,q}\|f\|_{H_{p,q}} \qquad (f\in H_{p,q})$$

for every $(\beta+2)/2(\beta+1)<p\le\infty$ and $0<q\le\infty$. In particular, if $f\in L_1$ then

$$\lambda(\sup_{n\in\mathbb{N}} \sigma_n^\beta f>\gamma) \le \frac{C}{\gamma}\|f\|_1 \qquad (\gamma>0).$$

The latter weak type inequality implies the next convergence result.

COROLLARY 2. *If $0<\beta\le 1$ and $f\in L_1$ then*

$$\sigma_n^\beta f \to f \quad \text{a.e.} \quad \text{as } n\to\infty.$$

We remark that this corollary can also be found in Zygmund [23].

6. Cesàro summability of two-dimensional trigonometric-Fourier series. For $f\in L_1(\mathbf{T}^2)$ and $z:=re^{\imath x}$ $(0<r<1)$ let

$$u(z,y)=u(re^{\imath x},y):=\frac{1}{2\pi}\int_{\mathbf{T}} f(t,y)P_r(x-t)\,dt$$

and

$$u_\alpha^*(x,y):=\sup_{z\in\Omega_\alpha(x)}|u(z,y)| \qquad (0<\alpha<1).$$

We say that $f\in L_1(\mathbf{T}^2)$ is in the *hybrid Hardy space* $H_1^\sharp(\mathbf{T}^2)=H_1^\sharp$ if

$$\|f\|_{H_1^\sharp}:=\|u_{1/2}^*\|_1<\infty.$$

The Fourier coefficients of a two-dimensional integrable function are defined by

$$\widehat{f}(n,m)=\frac{1}{(2\pi)^2}\int_{\mathbf{T}}\int_{\mathbf{T}} f(x,y)e^{-\imath nx}e^{-\imath my}\,dx\,dy.$$

We can introduce the Cesàro means $\sigma_{n,m}f$ again as the arithmetic mean of the rectangle partial sums of the Fourier series of f and can prove that

$$\sigma_{n,m}f=f*(K_n\times K_m).$$

We generalize (8) in the following way.

THEOREM 4. *If $f\in H_1^\sharp$ then*

$$\lambda(\sup_{n,m\in\mathbb{N}}|\sigma_{n,m}f|>\gamma)\le\frac{C}{\gamma}\|f\|_{H_1^\sharp} \qquad (\gamma>0).$$

Proof. Applying Fubini's theorem, (8) and the positivity of K_m (see (5)) we have

$$\lambda\Big((x,y): \sup_{n,m\in\mathbb{N}}\Big|\int_{\mathbf{T}}\int_{\mathbf{T}} f(t,u)K_n(x-t)K_m(y-u)\,dt\,du\Big| > \gamma\Big)$$

$$\le \lambda\Big((x,y): \sup_{m\in\mathbb{N}}\int_{\mathbf{T}}\Big(\sup_{n\in\mathbb{N}}\Big|\int_{\mathbf{T}} f(t,u)K_n(x-t)\,dt\Big|\Big)K_m(y-u)\,du > \gamma\Big)$$

$$= \int_{\mathbf{T}}\int_{\mathbf{T}} 1_{\{\sup_{m\in\mathbb{N}}\int_{\mathbf{T}}(\sup_{n\in\mathbb{N}}|\int_{\mathbf{T}} f(t,u)K_n(\cdot-t)\,dt|)K_m(\cdot-u)\,du>\gamma\}}(x,y)\,dy\,dx$$

$$\le \frac{C}{\gamma}\int_{\mathbf{T}}\int_{\mathbf{T}}\sup_{n\in\mathbb{N}}\Big|\int_{\mathbf{T}} f(t,y)K_n(x-t)\,dt\Big|\,dy\,dx.$$

For a fixed $y\in\mathbf{T}$ we deduce by (7) that

$$\int_{\mathbf{T}}\sup_{n\in\mathbb{N}}\Big|\int_{\mathbf{T}} f(t,y)K_n(x-t)\,dt\Big|\,dx \le C\int_{\mathbf{T}} u^*_{1/2}(x,y)\,dx.$$

Theorem 4 follows from Fubini's theorem. ∎

Note that we can verify with the same method that the operator $\sup_{n,m\in\mathbb{N}}|\sigma_{n,m}|$ is bounded from $L_p(\mathbf{T}^2)$ to $L_p(\mathbf{T}^2)$ if $1<p\le\infty$.

It is easy to show that the two-dimensional trigonometric polynomials are dense in $H_1^\sharp$. Hence Theorem 4 and the usual density argument (see Marcinkiewicz–Zygmund [14]) imply

COROLLARY 3. *If* $f\in H_1^\sharp$ *then*

$$\sigma_{n,m}f\to f \quad a.e. \quad as\ \min(n,m)\to\infty.$$

Note that $H_1^\sharp\supset L\log L(\mathbf{T}^2)$ by (2). Corollary 3 for $L\log L$ functions can be found in Zygmund [23], and, for Walsh–Fourier series in Móricz–Schipp–Wade [15].

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