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A generalization of the uniform ergodic theorem to poles of arbitrary order

by

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Abstract. We obtain a generalization of the uniform ergodic theorem to the sequence $(1/n^p)\sum_{k=0}^{n-1}T^k$, where T is a bounded linear operator on a Banach space and p is a positive integer. Indeed, we show that uniform convergence of the sequence above, together with an additional condition which is automatically satisfied for $p=1$, is equivalent to 1 being a pole of the resolvent of T plus convergence to zero of $\|T^n\|/n^p$. Furthermore, we show that the two conditions above, together, are also equivalent to 1 being a pole of order less than or equal to p of the resolvent of T, plus a certain condition $\mathcal{E}(k,p)$, which is less restrictive than convergence to zero of $\|T^n\|/n^p$ and generalizes the condition (called condition ($\mathcal{E}$-k)) introduced by K. B. Laursen and M. Mbekhta in their paper [LM2] (dealing with the case $p=1$).

Introduction. Throughout this paper, when the scalar field is not specified we assume it may be either $\mathbb{R}$ or $\mathbb{C}$. If X and Y are Banach spaces over the same scalar field, let $L(X,Y)$ denote the Banach space of all bounded linear operators from X into Y, endowed with the canonical norm induced by the norms of X and Y. Then convergence in $L(X,Y)$ of a sequence $(T_n)_{n\in\mathbb{N}}$ of bounded linear operators from X into Y means uniform convergence of $T_n x$ on the closed unit ball of X. For the Banach algebra of all bounded linear operators on X we use the notation $L(X)$ instead of $L(X,X)$. We denote by I_X the identity operator on X, which is the identity element of $L(X)$.

Let X be a complex Banach space. For each $T\in L(X)$ we denote the spectrum of T by $\sigma(T)$, the spectral radius of T by $r(T)$ and the resolvent set of T by $\varrho(T)$ (that is, $\varrho(T)=\mathbb{C}\setminus\sigma(T)$). It is well known that the resolvent function

$$R(\cdot,T):\varrho(T)\ni\lambda\mapsto(\lambda I_X-T)^{-1}\in L(X)$$

(where, for each $\lambda\in\varrho(T)$, $(\lambda I_X-T)^{-1}$ denotes the inverse of λI_X-T in $L(X)$) is holomorphic on $\varrho(T)$. By $\mathbb{N}$ and $\mathbb{Z}_+$ we denote the sets of all nonnegative and positive integers, respectively. If $p\in\mathbb{Z}_+$, we recall that

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a *pole of order* p of $R(\cdot,T)$ is an isolated point λ_0 of $\sigma(T)$ such that the coefficient of index $-p$ of the Laurent expansion of $R(\cdot,T)$ in a punctured neighborhood of λ_0 is nonzero and the coefficient of index $-n$ is zero for every $n > p$. By a *pole of order zero* of $R(\cdot,T)$ we mean an element of $\varrho(T)$. Thus, according to our terminology, a pole of $R(\cdot,T)$ may not belong to $\sigma(T)$ (this happens exactly when the order of the pole is zero).

In the paper [D1] by N. Dunford, several theorems about convergence of $f_n(T)$, where T is a bounded linear operator on a complex Banach space and $(f_n)_{n\in\mathbb{N}}$ is a sequence of complex-valued functions, each of which is holomorphic on some open neighborhood of $\sigma(T)$, are obtained. Different kinds of convergence (namely, convergence in $L(X)$, strong and weak convergence, and, in the case of a space of measurable functions, almost everywhere convergence) are treated in [D1]. A special case of the results in [D1] about convergence in $L(X)$ is the uniform ergodic theorem, concerning convergence of $(1/n)\sum_{k=0}^{n-1}T^k$ in $L(X)$, which in particular turns out to be equivalent, under the hypothesis of convergence to zero of $\|T^n\|/n$, to 1 being a pole of order less than or equal to one of the resolvent of T (see [D1], 3.16; see also [D2], comments following Theorem 8). Dunford's uniform ergodic theorem has been improved by M. Lin in [Li], and further improvements have recently been obtained by M. Mbekhta and J. Zemánek in [MZ], and later by K. B. Laursen and M. Mbekhta in [LM2]. Dunford's uniform ergodic theorem and the further contributions by Lin, by Mbekhta and Zemánek and by Laursen and Mbekhta are put together in Theorem 1.5 below.

A partial generalization of the uniform ergodic theorem to the sequence $(1/n^p)\sum_{k=0}^{n-1}T^k$, where p is a positive integer, has been provided by H.-D. Wacker in [W]: indeed, in [W], Wacker proves that, if 1 is a pole of order less than or equal to p of the resolvent of T and $\|T^n\|/n^p$ converges to zero, then $(1/n^p)\sum_{k=0}^{n-1}T^k$ converges in $L(X)$. Also, an example showing that the converse implication does not hold (more precisely, showing that convergence in $L(X)$ of $(1/n^p)\sum_{k=0}^{n-1}T^k$ does not imply that 1 is a pole of the resolvent of T when $p > 1$) is given in [W] (p. 543, Beispiel).

This paper contains (the developments of) part of the results announced in [B]. Indeed, we are concerned here with providing a converse of Wacker's result.

Section 1 presents some preliminaries, in order to make this paper as self-contained as possible. In Section 2 we introduce condition $\mathcal{E}(k,p)$ (where k and p are positive integers), which is a generalization of the relaxed version, provided in [LM2] and called condition ($\mathcal{E}$-k), of the condition $\|T^n\|/n \to 0$. Condition $\mathcal{E}(k,p)$ coincides with condition ($\mathcal{E}$-k) for $p = 1$ and turns out to be equivalent to convergence to zero of $\|T^n\|/n^p$ for $k = 1$. Furthermore, we prove that $\mathcal{E}(k,p)$ implies $\mathcal{E}(k-j,p+j)$ for all $j = 0,\ldots,k-1$ (Theorem 2.4), whereas the converse does not hold (Example 2.7). In Section 3 we obtain our main result (Theorem 3.4), namely a complete generalization of the uniform ergodic theorem to the sequence $(1/n^p)\sum_{k=0}^{n-1}T^k$: among other things, we prove that convergence of this sequence in $L(X)$, plus the additional requirement that the sum of the kernel of $I_X - T$ and the range of $(I_X - T)^{p-1}$ be a closed subspace of X (which is automatically satisfied when $p = 1$, and thus is a "hidden" condition in the uniform ergodic case), is equivalent to 1 being a pole of order less than or equal to p of the resolvent of T, plus condition $\mathcal{E}(k,p)$ for some $k \in \mathbb{Z}_+$. Several other equivalent conditions are given, involving either condition $\mathcal{E}(k,p)$ or convergence to zero of $\|T^n\|/n^p$. We also provide some examples in order to illustrate the role of the hypotheses of Theorem 3.4.

We remark that Theorem 3.4 is used by H. C. Rönnefarth in [R3] in order to obtain an extension to the sequence $(n+1)^{-p+1}M_n^{(k)}(T)$, where $k \in \mathbb{Z}_+$ and the $M_n^{(k)}(T)$, $n \in \mathbb{N}$, are convenient operator means, generalizing the Cesàro means $M_n^{(1)}(T)$ $(= (n+1)^{-1}\sum_{k=0}^{n}T^k)$. We also recall that the asymptotic behavior of the generalized Cesàro means $M_n^{(k)}(T)$ as $n \to \infty$, in connection with 1 being a pole of order less than or equal to 1 of $R(\cdot,T)$, is studied in [R2]. More generally, in [R2]—as well as in [R1] and in [Z]—several results and a list of references concerning the iterates of a bounded linear operator (or an element of a Banach algebra) can be found.

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1. Preliminaries. In this section we recall the uniform ergodic theorem and the generalization by H.-D. Wacker, as well as some other known results which we shall frequently use in the sequel.

If X and Y are Banach spaces, for each $T \in L(X,Y)$ we denote the kernel and range of T by $\mathcal{N}(T)$ and $\mathcal{R}(T)$, respectively. We begin with the following closed range theorem.

THEOREM 1.1 (see [TL], IV, 5.10). *Let X and Y be Banach spaces and let $T \in L(X,Y)$. If there exists a closed subspace Z of Y such that $\mathcal{R}(T)\cap Z = \{0\}$ and $\mathcal{R}(T)\oplus Z$ is closed, then $\mathcal{R}(T)$ is closed.*

By a *projection* of a Banach space X we mean an element P of $L(X)$ satisfying $P^2 = P$. We recall that, if P is a projection of X, then $\mathcal{R}(P)$ is a closed subspace of X and in addition $X = \mathcal{N}(P)\oplus\mathcal{R}(P)$. Conversely, for every direct-sum decomposition $X = Y \oplus Z$, where Y and Z are closed subspaces of X, there exists a unique projection P of X such that $\mathcal{R}(P) = Y$ and $\mathcal{N}(P) = Z$: we call P the *projection of X onto Y along Z*.

Following [LM2], by a *complement* of a linear subspace V of X we mean a subspace W of X such that $X = V \oplus W$ (where neither V nor W are necessarily closed).

For each $T \in L(X)$, let $\alpha(T)$ and $\delta(T)$ denote the *ascent* and *descent* of T, respectively. Namely,

$$\alpha(T) = \inf\{n \in \mathbb{N} : \mathcal{N}(T^n) = \mathcal{N}(T^{n+1})\},$$
$$\delta(T) = \inf\{n \in \mathbb{N} : \mathcal{R}(T^n) = \mathcal{R}(T^{n+1})\}.$$

Then $\alpha(T)$ and $\delta(T)$ belong to $\mathbb{N} \cup \{\infty\}$. Furthermore, we have $\alpha(T) = \infty$ (respectively, $\delta(T) = \infty$) when $\mathcal{N}(T^n)$ is strictly contained in $\mathcal{N}(T^{n+1})$ (respectively, $\mathcal{R}(T^{n+1})$ is strictly contained in $\mathcal{R}(T^n)$) for every $n \in \mathbb{N}$. If, on the contrary, $\mathcal{N}(T^n) = \mathcal{N}(T^{n+1})$ (respectively, $\mathcal{R}(T^n) = \mathcal{R}(T^{n+1})$) for some $n \in \mathbb{N}$, then $\alpha(T)$ (respectively, $\delta(T)$) is the minimum of all natural numbers for which the equality above is satisfied. We also recall that, if $\alpha(T) < \infty$ (respectively, $\delta(T) < \infty$), then $\mathcal{N}(T^n) = \mathcal{N}(T^{\alpha(T)})$ for every $n \geq \alpha(T)$ (respectively, $\mathcal{R}(T^n) = \mathcal{R}(T^{\delta(T)})$ for every $n \geq \delta(T)$). Notice that $\alpha(T) = 0$ (respectively, $\delta(T) = 0$) if and only if T is one-to-one (respectively, onto).

It is well known that finiteness of the ascent and descent of a bounded linear operator on a Banach space X is equivalent to a certain decomposition of X, as the following result shows.

THEOREM 1.2 (see [TL], V, 6.2, 6.3 and 6.4). *Let X be a Banach space and let $T \in L(X)$. If both $\alpha(T)$ and $\delta(T)$ are finite, then $\alpha(T) = \delta(T)$ and the following decomposition holds, where p denotes the common value of $\alpha(T)$ and $\delta(T)$:*

$$X = \mathcal{N}(T^p) \oplus \mathcal{R}(T^p)$$

(which implies that $\mathcal{R}(T^p)$ is closed, in virtue of Theorem 1.1). Conversely, if the decomposition above holds for some $p \in \mathbb{Z}_+$, then $\alpha(T) = \delta(T) \leq p$.

Now let T be a bounded linear operator on a complex Banach space and let $\lambda_0 \in \mathbb{C}$ either be in $\varrho(T)$ or be an isolated point of $\sigma(T)$. We recall that then the coefficient of index -1 of the Laurent expansion of $R(\cdot, T)$ in a punctured neighborhood of λ_0 is a projection P of X, which is nonzero if and only if $\lambda_0 \in \sigma(T)$ (see [TL], V.10): we call P the *spectral projection* of T associated with λ_0. The following classical result relates ascent and descent of complex Banach space operators to poles of the resolvent.

THEOREM 1.3 (see [TL], V, 10.1 and 10.2). *Let X be a complex Banach space, let $T \in L(X)$ and let $\lambda_0 \in \mathbb{C}$. If λ_0 is a pole of order p of $R(\cdot, T)$ for some $p \in \mathbb{N}$, then $\alpha(\lambda_0 I_X - T) = \delta(\lambda_0 I_X - T) = p$ and the spectral projection associated with λ_0 coincides with the projection of X onto $\mathcal{N}((\lambda_0 I_X - T)^p)$ along $\mathcal{R}((\lambda_0 I_X - T)^p)$. Conversely, if both $\alpha(\lambda_0 I_X - T)$ and $\delta(\lambda_0 I_X - T)$ are finite, then λ_0 is a pole of $R(\cdot, T)$.*

Let X be a Banach space. We recall (see for instance [LM2], Definition 5) that $T \in L(X)$ is said to be *quasi-Fredholm* if there exist two closed subspaces M and N of X, invariant under T, such that $X = N \oplus M$, the operator $T_0 : N \to N$ defined by $T_0 x = Tx$ for every $x \in N$ is nilpotent, and finally $T(M)$ is closed in X and contains $\mathcal{N}(T^n) \cap M$ for every $n \in \mathbb{N}$. As remarked in [LM2], it is well known (see [K], Theorem 4) that the bounded linear semi-Fredholm operators on X are quasi-Fredholm.

The following result is implicitly contained in the proof of [LM2], Theorem 6 (see [LM2], Theorem 6, proof of "(a) and (b) imply (c)", in which condition ($\mathcal{E}$-k) is not used).

THEOREM 1.4. *Let X be a complex Banach space and let $T \in L(X)$. If $\lambda_0 \in \mathbb{C}$ is a pole of $R(\cdot, T)$, then $\lambda_0 I_X - T$ is a quasi-Fredholm operator.*

Finally, let $k \in \mathbb{Z}_+$. Following [LM2], Definition 2, we say that a bounded linear operator T on a Banach space X satisfies *condition* ($\mathcal{E}$-k) if

$$\left\| \frac{1}{n}(I_X - T)^k \sum_{j=0}^{n-1} T^j \right\| \to 0 \quad \text{as } n \to \infty.$$

We recall that condition ($\mathcal{E}$-k) implies ($\mathcal{E}$-h) for every $h \geq k$. We also recall (see [LM2], Lemma 3; notice that $-(I_X - T)^{k-1} T^n/n$ should be replaced by $(1/n)(I_X - T)^{k-1}(I_X - T^n)$ in the proof) that T satisfies condition ($\mathcal{E}$-k) if and only if

$$\left\| \frac{1}{n}(I_X - T)^{k-1} T^n \right\| \to 0 \quad \text{as } n \to \infty.$$

Hence, as remarked in [LM2], T satisfies condition ($\mathcal{E}$-1) if and only if $\|T^n\|/n \to 0$ as $n \to \infty$.

By also taking Theorems 1.2 and 1.3 into account, the uniform ergodic theorems in [D1], [Li], [MZ] and [LM2] can be rephrased in the following way (see [D1], 3.16, [D2], comments following Theorem 8, [Li], [MZ], Théorème 1 and final Remark 1, and [LM2], Theorems 6 and 9).

THEOREM 1.5. *Let X be a complex Banach space and let $T \in L(X)$. Then the following conditions are equivalent:*

(1.5.1) $(1/n)\sum_{k=0}^{n-1} T^k$ *converges in* $L(X)$;

(1.5.2) $\|T^n\|/n \to 0$ *as* $n \to \infty$ *and* 1 *is a pole of* $R(\cdot, T)$;

(1.5.3) *T satisfies condition ($\mathcal{E}$-k) for some $k \in \mathbb{Z}_+$ and 1 is a pole of $R(\cdot, T)$, of order less than or equal to 1*;

(1.5.4) *T satisfies condition ($\mathcal{E}$-k) for some $k \in \mathbb{Z}_+$ and $X = \mathcal{N}(I_X - T) \oplus \mathcal{R}(I_X - T)$*;

(1.5.5) $\|T^n\|/n \to 0$ *as* $n \to \infty$ *and* $\delta(I_X - T) < \infty$;

(1.5.6) T *satisfies condition* $(\mathcal{E}\text{-}k)$ *for some* $k \in \mathbb{Z}_+$ *and* $\delta(I_X - T) \le 1$;

(1.5.7) T *satisfies condition* $(\mathcal{E}\text{-}k)$ *for some* $k \in \mathbb{Z}_+$, $\delta(I_X - T) < \infty$ *and* $\mathcal{N}(I_X - T)$ *has a complement which is invariant under* T;

(1.5.8) $\|T^n\|/n \to 0$ *as* $n \to \infty$ *and* $I_X - T$ *is a quasi-Fredholm operator*;

(1.5.9) $\|T^n\|/n \to 0$ *as* $n \to \infty$ *and* $\mathcal{R}((I_X - T)^k)$ *is closed for some positive integer* k;

(1.5.10) $\|T^n\|/n \to 0$ *as* $n \to \infty$ *and* $\mathcal{R}((I_X - T)^k)$ *is closed for every positive integer* k;

(1.5.11) $\|T^n\|/n \to 0$ *as* $n \to \infty$ *and* $\mathcal{N}(I_X - T) + \mathcal{R}(I_X - T)$ *is closed.*

Furthermore, if (1.5.1)–(1.5.11) *are satisfied and* E *is the projection of* X *onto* $\mathcal{N}(I_X - T)$ *along* $\mathcal{R}(I_X - T)$, *then* $\|(1/n)\sum_{k=0}^{n-1} T^k - E\| \to 0$ *as* $n \to \infty$.

We now recall H.-D. Wacker's generalization of the uniform ergodic theorem.

THEOREM 1.6 ([W], Satz 4). *Let* $p \in \mathbb{Z}_+$, *let* X *be a complex Banach space and let* $T \in L(X)$ *be such that* $\|T^n\|/n^p \to 0$ *as* $n \to \infty$. *Then the following two conditions are equivalent*:

(1.6.1) 1 *is a pole of* $R(\cdot, T)$, *of order less than or equal to* p;

(1.6.2) $\mathcal{R}((I_X - T)^{p+1})$ *is closed.*

Furthermore, if (1.6.1) *and* (1.6.2) *are satisfied and* P *is the projection of* X *onto* $\mathcal{N}((I_X - T)^p)$ *along* $\mathcal{R}((I_X - T)^p)$, *then*

$$\left\| \frac{1}{n^p} \sum_{k=0}^{n-1} T^k - \frac{1}{p!}(T - I_X)^{p-1} P \right\| \to 0 \quad \text{as } n \to \infty.$$

As remarked by H.-D. Wacker ([W], p. 543, Bemerkung), $(T - I_X)^{p-1}P$ is the coefficient of order $-p$ of the Laurent expansion of $R(\cdot, T)$ in a punctured neighborhood of 1.

Also the following two results by H.-D. Wacker will be useful in the sequel.

THEOREM 1.7 ([W], Satz 2). *Let* X *be a complex Banach space, let* $T \in L(X)$ *and let* $p \in \mathbb{N}$ *be such that* $\lim_{n\to\infty}(1/n^p)T^n x = 0$ *for every* $x \in X$. *Then* $\alpha(\lambda I_X - T) \le p$ *for every* $\lambda \in \mathbb{C}$ *such that* $|\lambda| = 1$.

THEOREM 1.8 ([W], Satz 5). *Let* X *be a complex Banach space, let* $p \in \mathbb{Z}_+$ *and let* $T \in L(X)$ *be such that* $r(T) = 1$ *and the intersection of* $\sigma(T)$ *and the unit circle consists of a finite number of poles of* $R(\cdot, T)$, *each of order less than or equal to* p. *Then the sequence* $(\|T^n\|/n^{p-1})_{n\in\mathbb{N}}$ *is bounded and* $\|T^n\|/n^p \to 0$ *as* $n \to \infty$.

Finally, we will need the following well known identity (which is not difficult to prove by induction on n):

$$\sum_{k=j}^{n-1} \binom{k}{j} = \binom{n}{j+1} \quad \text{for every } j \in \mathbb{N} \text{ and for all } n \ge j+1. \tag{1.9}$$

2. A generalization of condition $(\mathcal{E}\text{-}k)$

DEFINITION 2.1. Let X be a Banach space, let $T \in L(X)$ and let $k, p \in \mathbb{Z}_+$. We say that T satisfies *condition* $\mathcal{E}(k,p)$ if

$$\left\| \frac{1}{n^p}(I_X - T)^k \sum_{j=0}^{n-1} T^j \right\| \to 0 \quad \text{as } n \to \infty.$$

We say that T satisfies *condition* $\mathcal{S}(k,p)$ if

$$\left\| \frac{1}{n^p}(I_X - T)^k \sum_{j=0}^{n-1} T^j x \right\| \to 0 \quad \text{as } n \to \infty \text{ for every } x \in X.$$

Notice that $\mathcal{E}(k,1)$ coincides with $(\mathcal{E}\text{-}k)$ for each $k \in \mathbb{Z}_+$. Notice also that, for every $k, p \in \mathbb{Z}_+$, condition $\mathcal{E}(k,p)$ implies $\mathcal{S}(k,p)$ (and is equivalent to it in the special case of a finite-dimensional space X). Furthermore, we remark that $\mathcal{E}(k,p)$ (respectively, $\mathcal{S}(k,p)$) implies $\mathcal{E}(h,q)$ (respectively, $\mathcal{S}(h,q)$) for every $h \ge k$ and for every $q \ge p$, whereas $\mathcal{E}(h,q)$ (respectively, $\mathcal{S}(h,q)$) for some $h \ge k$ and some $q \ge p$ does not imply $\mathcal{E}(k,p)$ (respectively, $\mathcal{S}(k,p)$) if one of the two equalities is strict: for instance, the operator in the example on p. 3444 of [LM2]—represented by the matrix $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$—satisfies both $\mathcal{E}(2,1)$ and $\mathcal{E}(1,2)$, but does not satisfy $\mathcal{S}(1,1)$.

The following result generalizes Lemma 3 of [LM2].

PROPOSITION 2.2. *Let* X *be a Banach space, let* $T \in L(X)$ *and let* $k, p \in \mathbb{Z}_+$. *Then*:

(2.2.1) T *satisfies condition* $\mathcal{E}(k,p)$ *if and only if*

$$\left\| \frac{1}{n^p}(I_X - T)^{k-1} T^n \right\| \to 0 \quad \textit{as } n \to \infty;$$

(2.2.2) T *satisfies condition* $\mathcal{S}(k,p)$ *if and only if*

$$\left\| \frac{1}{n^p}(I_X - T)^{k-1} T^n x \right\| \to 0 \quad \textit{as } n \to \infty \textit{ for every } x \in X.$$

Proof. As in the proof of [LM2], Lemma 3, it suffices to remark that, for each $n \in \mathbb{Z}_+$, we have

$$(I_X - T)\sum_{j=0}^{n-1} T^j = I_X - T^n,$$

and consequently

$$\frac{1}{n^p}(I_X - T)^k \sum_{j=0}^{n-1} T^j = \frac{1}{n^p}(I_X - T)^{k-1}(I_X - T^n).$$

Since clearly $(1/n^p)(I_X - T)^{k-1} \to 0$ in $L(X)$ as $n \to \infty$, we obtain the desired result. ∎

Let X be a Banach space, $T \in L(X)$ and $p \in \mathbb{Z}_+$. Notice that, from Proposition 2.2, it follows that T satisfies $\mathcal{E}(1,p)$ (respectively, $\mathcal{S}(1,p)$) if and only if $\|T^n\|/n^p \to 0$ as $n \to \infty$ (respectively, $\|T^n x\|/n^p \to 0$ as $n \to \infty$ for every $x \in X$).

LEMMA 2.3. *Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be sequences of nonnegative numbers. If $(a_n)_{n\in\mathbb{N}}$ is nonincreasing and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_n b_n = 0$, then*

$$a_n \max\{b_k : k = 0,\ldots,n\} \to 0 \quad \text{as } n \to \infty.$$

Proof. For each $n \in \mathbb{N}$, set

$$m_n = \max\{m \in \{0,\ldots,n\} : b_m = \max\{b_k : k = 0,\ldots,n\}\}.$$

Notice that $(m_n)_{n\in\mathbb{N}}$ is a nondecreasing sequence of natural numbers. Hence either $\lim_{n\to\infty} m_n = \infty$, or $(m_n)_{n\in\mathbb{N}}$ is eventually constant. In the latter case, the desired result follows from convergence to zero of $(a_n)_{n\in\mathbb{N}}$.

Now suppose that $\lim_{n\to\infty} m_n = \infty$. Then $\lim_{n\to\infty} a_{m_n} b_{m_n} = 0$. Since $m_n \le n$ for every $n \in \mathbb{N}$, and $(a_n)_{n\in\mathbb{N}}$ is nonincreasing, it follows that

$$a_n b_{m_n} \le a_{m_n} b_{m_n} \quad \text{for every } n \in \mathbb{N}.$$

Hence

$$0 = \lim_{n\to\infty} a_n b_{m_n} = \lim_{n\to\infty} a_n \max\{b_k : k = 0,\ldots,n\}. \ ∎$$

THEOREM 2.4. *Let X be a Banach space and let $T \in L(X)$. If T satisfies condition $\mathcal{E}(q,p)$ (respectively, $\mathcal{S}(q,p)$) for some $q,p \in \mathbb{Z}_+$, then T satisfies condition $\mathcal{E}(q-j,p+j)$ (respectively, $\mathcal{S}(q-j,p+j)$) for $j = 0,\ldots,q-1$. In particular, it follows that $\|T^n\|/n^{p+q-1} \to 0$ as $n \to \infty$ (respectively, $\|T^n x\|/n^{p+q-1} \to 0$ as $n \to \infty$ for every $x \in X$).*

Proof. We begin by remarking that for each $n \in \mathbb{Z}_+$ we have

$$\frac{1}{n}(T^n - I_X) = \frac{1}{n}((I_X - (I_X - T))^n - I_X) = \frac{1}{n}\sum_{k=1}^{n}(-1)^k\binom{n}{k}(I_X - T)^k$$

$$\begin{aligned}
&= \sum_{k=1}^{n}(-1)^k\frac{(n-1)!}{k!(n-k)!}(I_X - T)^k \\
&= \sum_{k=1}^{n}\frac{(-1)^k}{k}\binom{n-1}{k-1}(I_X - T)^k \\
&= -(I_X - T)\sum_{k=0}^{n-1}\frac{(-1)^k}{k+1}\binom{n-1}{k}(I_X - T)^k \\
&= -(I_X - T)\sum_{k=0}^{n-1}(-1)^k\Big(\int_0^1 t^k\,dt\Big)\binom{n-1}{k}(I_X - T)^k \\
&= -(I_X - T)\int_0^1\Big(\sum_{k=0}^{n-1}(-1)^k t^k\binom{n-1}{k}(I_X - T)^k\Big)dt \\
&= -(I_X - T)\int_0^1(I_X - t(I_X - T))^{n-1}dt \\
&= -(I_X - T)\int_0^1((1-t)I_X + tT)^{n-1}dt \\
&= -(I_X - T)\int_0^1\Big(\sum_{k=0}^{n-1}\binom{n-1}{k}(1-t)^{n-1-k}t^kT^k\Big)dt.
\end{aligned}$$

Now we prove that, if T satisfies $\mathcal{E}(q,p)$ for some $p,q \in \mathbb{Z}_+$, then T satisfies $\mathcal{E}(q-j,p+j)$ for $j = 0,\ldots,q-1$. In virtue of Proposition 2.2, this holds if and only if, for $j = 0,\ldots,q-1$, we have

$$\left\|\frac{1}{n^{p+j}}(I_X - T)^{q-j-1}T^n\right\| \to 0 \quad \text{as } n \to \infty. \tag{2.4.1}$$

We proceed by induction. From Proposition 2.2 it follows that (2.4.1) holds for $j = 0$. Now suppose $q \ge 2$ and (2.4.1) to be satisfied for some $j \in \{0,\ldots,q-2\}$. From the formula above for $(1/n)(T^n - I_X)$ it follows that for each $n \in \mathbb{Z}_+$ we have

$$\begin{aligned}
&\frac{1}{n^{p+j+1}}(I_X - T)^{q-j-2}T^n \\
&\quad= \frac{1}{n^{p+j+1}}(I_X - T)^{q-j-2} + \frac{1}{n^{p+j}}(I_X - T)^{q-j-2}\Big(\frac{1}{n}(T^n - I_X)\Big) \\
&\quad= \frac{1}{n^{p+j+1}}(I_X - T)^{q-j-2} \\
&\qquad - \frac{1}{n^{p+j}}(I_X - T)^{q-j-1}\int_0^1\Big(\sum_{k=0}^{n-1}\binom{n-1}{k}(1-t)^{n-1-k}t^kT^k\Big)dt
\end{aligned}$$

$$= \frac{1}{n^{p+j+1}}(I_X - T)^{q-j-2}$$
$$- \int_0^1 \left(\sum_{k=0}^{n-1} \binom{n-1}{k} (1-t)^{n-1-k} t^k \left(\frac{1}{n^{p+j}} (I_X - T)^{q-j-1} T^k \right) \right) dt,$$

which gives (by setting $M_n = \max\{\|(I_X - T)^{q-j-1}T^k\| : k = 0, \ldots, n-1\}$)

$$\left\| \frac{1}{n^{p+j+1}} (I_X - T)^{q-j-2} T^n \right\|$$
$$\le \frac{1}{n^{p+j+1}} \|(I_X - T)^{q-j-2}\|$$
$$+ \int_0^1 \left(\sum_{k=0}^{n-1} \binom{n-1}{k} (1-t)^{n-1-k} t^k \left\| \frac{1}{n^{p+j}} (I_X - T)^{q-j-1} T^k \right\| \right) dt$$
$$\le \frac{1}{n^{p+j+1}} \|(I_X - T)^{q-j-2}\| + \frac{M_n}{n^{p+j}} \int_0^1 \left(\sum_{k=0}^{n-1} \binom{n-1}{k} (1-t)^{n-1-k} t^k \right) dt$$
$$= \frac{1}{n^{p+j+1}} \|(I_X - T)^{q-j-2}\| + \frac{M_n}{n^{p+j}} \int_0^1 (t + (1-t))^{n-1}\, dt$$
$$= \frac{1}{n^{p+j+1}} \|(I_X - T)^{q-j-2}\| + \frac{M_n}{n^{p+j}}.$$

From Lemma 2.3 it follows that $M_n/n^{p+j} \to 0$ as $n \to \infty$. Hence

$$\left\| \frac{1}{n^{p+j+1}} (I_X - T)^{q-j-2} T^n \right\| \to 0 \quad \text{as } n \to \infty,$$

which gives the desired result. The same arguments (replacing M_n, for each $x \in X$, by $\max\{\|(I_X - T)^{q-j-1}T^k x\| : k = 0, \ldots, n-1\}$) prove that $\mathcal{S}(q,p)$ implies $\mathcal{S}(q-j, p+j)$ for $j = 0, \ldots, q-1$. ■

The following result, which generalizes Proposition 4 of [LM2] as well as Theorem 1.7, is a consequence of Theorems 2.4 and 1.7.

COROLLARY 2.5. *Let X be a complex Banach space, let $T \in L(X)$ and let $k, p \in \mathbb{Z}_+$ be such that T satisfies condition $\mathcal{S}(k,p)$. Then $\alpha(\lambda I_X - T) \le p + k - 1$ for every $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$.*

The following example shows that the converse of Theorem 2.4 does not hold, namely, for each $p \ge 2$, there exist operators which satisfy $\mathcal{E}(1,p)$ and fail to satisfy even $\mathcal{S}(2, p-1)$.

EXAMPLE 2.6. Let p be a positive integer, $p \ge 2$. We consider the Banach space $\mathbb{K}^p$ (where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$). For each $A \in L(\mathbb{K}^p)$, let A_{jk} ($j, k \in \{1, \ldots, p\}$) denote the coefficients of the $p \times p$ matrix representing A with respect to the canonical basis of $\mathbb{K}^p$. Now let $T \in L(\mathbb{K}^p)$ be defined by

$$T_{jk} = \begin{cases} -1 & \text{if } j \le k, \\ 0 & \text{if } j > k. \end{cases}$$

We prove that for each $n \in \mathbb{Z}_+$ we have

$$(T^n)_{jk} = \begin{cases} (-1)^n \binom{n+k-j-1}{n-1} & \text{if } j \le k, \\ 0 & \text{if } j > k. \end{cases}$$

This is clearly true for $n = 1$. Now, proceeding by induction, suppose this holds for some $n \in \mathbb{Z}_+$. Then, for $j, k \in \{1, \ldots, p\}$, we have

$$(T^{n+1})_{jk} = \sum_{h=1}^{p} (T^n)_{jh} T_{hk},$$

which gives $(T^{n+1})_{jk} = 0$ if $j > k$. If $j \le k$, then

$$(T^{n+1})_{jk} = \sum_{h=j}^{k} (T^n)_{jh} T_{hk} = -\sum_{h=j}^{k} (-1)^n \binom{n+h-j-1}{n-1}$$
$$= (-1)^{n+1} \sum_{m=n-1}^{n+k-j-1} \binom{m}{n-1} = (-1)^{n+1} \binom{n+k-j}{n},$$

by (1.9). Thus the desired result holds for $n+1$.

We remark that, for every $n \in \mathbb{Z}_+$ and for all $j, k \in \{1, \ldots, p\}$, we have

$$|(T^n)_{jk}| \le \max\left\{ \binom{n-1+h}{n-1} : h = 0, \ldots, p-1 \right\}$$
$$= \binom{n+p-2}{n-1} = \frac{1}{(p-1)!} \prod_{h=0}^{p-2} (n+h).$$

Consequently, $|(T^n)_{jk}|/n^p \to 0$ as $n \to \infty$ for $j, k \in \{1, \ldots, p\}$. Hence $\|T^n\|/n^p \to 0$ as $n \to \infty$, that is, T satisfies $\mathcal{E}(1,p)$. We could also have obtained this by remarking that the spectrum of every element of $L(\mathbb{C}^p)$ consists of a finite number of poles of the resolvent, each of order less than or equal to p, and consequently, by Theorem 1.8, $\|A^n\|/n^p \to 0$ as $n \to \infty$ for every $A \in L(\mathbb{C}^p)$ satisfying $r(A) = 1$: this gives the desired result in the complex case, as clearly $\sigma(T) = \{-1\}$, and the real case follows by remarking that the norms of the iterates of T do not depend on $\mathbb{K}$.

Now we prove that T does not satisfy $\mathcal{S}(2, p-1)$. By Proposition 2.2, T satisfies $\mathcal{S}(2, p-1)$ if and only if $(1/n^{p-1})(T^n - T^{n+1})x \to 0$ in $\mathbb{K}^p$ for every $x \in \mathbb{K}^p$. Since, for every $n \in \mathbb{Z}_+$, we have

$$\left\|\frac{1}{n^{p-1}}(T^n - T^{n+1})(0,\ldots,0,1)\right\|$$
$$= \frac{1}{n^{p-1}}\|((T^n)_{1p} - (T^{n+1})_{1p},\ldots,(T^n)_{pp} - (T^{n+1})_{pp})\|$$
$$\geq \frac{1}{n^{p-1}}|(T^n)_{1p} - (T^{n+1})_{1p}|$$
$$= \frac{1}{n^{p-1}(p-1)!}\Big(\prod_{h=0}^{p-2}(n+h) + \prod_{h=1}^{p-1}(n+h)\Big)$$
$$\geq \frac{1}{n^{p-1}(p-1)!}\prod_{h=0}^{p-2}(n+h)$$

and

$$\lim_{n\to\infty}\left(\frac{1}{n^{p-1}(p-1)!}\prod_{h=0}^{p-2}(n+h)\right) = \frac{1}{(p-1)!} > 0,$$

it follows that T does not satisfy $\mathcal{S}(2,p-1)$. Notice that, from Theorem 2.4, it follows that T satisfies $\mathcal{S}(j+1,p-j)$ for no $j=1,\ldots,p-1$. ∎

3. A generalization of the uniform ergodic theorem. We begin with some auxiliary results. Lemma 3.1 below is implicitly proved in [ṀZ] (proof of Théorème 1) for $m=0$ and $j=1$, and is proved on p. 128 of [LM1] for $n=2$ and $m=0$.

LEMMA 3.1. *Let X be a Banach space, let $T\in L(X)$ and let $\mathcal{R}(T^n)+\mathcal{N}(T^m)$ be closed for some $(n,m)\in\mathbb{N}\times\mathbb{N}$. Then $\mathcal{R}(T^{n-j})+\mathcal{N}(T^{m+j})$ is closed for every $j\in\{0,\ldots,n\}$.*

Proof. It suffices to remark that $\mathcal{R}(T^{n-j})+\mathcal{N}(T^{m+j}) = (T^j)^{-1}(\mathcal{R}(T^n)+\mathcal{N}(T^m))$ for every $j\in\{0,\ldots,n\}$. Now the desired result follows from continuity of T. ∎

We shall also need the following immediate consequence of stability of semi-Fredholm operators under small perturbations (see [K], §3, Theorem 1).

LEMMA 3.2. *Let X and Y be Banach spaces and let $T\in L(X,Y)$. If $\mathcal{R}(T)$ is closed, then there exists $\varepsilon>0$ such that $\mathcal{R}(S)=\mathcal{R}(T)$ for every $S\in L(X,Y)$ satisfying $\|S-T\|<\varepsilon$ and $\mathcal{R}(S)\subset\mathcal{R}(T)$.*

In the following lemma we collect some consequences of the convergence in $L(X)$ of $(1/n^p)\sum_{k=0}^{n-1}T^k$.

LEMMA 3.3. *Let $p\in\mathbb{Z}_+$, let X be a Banach space and let $T\in L(X)$. If*

$$\left\|\frac{1}{n^p}\sum_{k=0}^{n-1}T^k - E\right\| \to 0 \quad \text{as } n\to\infty \text{ for some } E\in L(X),$$

then

(3.3.1) $\|T^n\|/n^p \to 0$ *as* $n\to\infty$;

(3.3.2) $$\left\|\frac{1}{n^p}\sum_{j=p}^{n-1}\binom{n}{j+1}(T-I_X)^j - E + \frac{1}{p!}(T-I_X)^{p-1}\right\| \to 0 \quad \text{as } n\to\infty;$$

(3.3.3) $\mathcal{R}(E)\subset\mathcal{N}(I_X-T)\cap\overline{\mathcal{R}((I_X-T)^{p-1})}$.

Proof. In order to prove (3.3.1), it is sufficient to observe that

$$\frac{T^n}{n^p} = \frac{(n+1)^p}{n^p}\left(\frac{1}{(n+1)^p}\sum_{k=0}^{n}T^k\right) - \frac{1}{n^p}\sum_{k=0}^{n-1}T^k,$$

for every $n\in\mathbb{Z}_+$, and $(n+1)^p/n^p\to 1$ as $n\to\infty$.

Now we prove (3.3.2). For every $n\in\mathbb{Z}_+$, the following equalities hold (in virtue of (1.9)):

$$\sum_{k=0}^{n-1}T^k = \sum_{k=0}^{n-1}[I_X+(T-I_X)]^k = \sum_{k=0}^{n-1}\sum_{j=0}^{k}\binom{k}{j}(T-I_X)^j$$
$$= \sum_{j=0}^{n-1}(T-I_X)^j\sum_{k=j}^{n-1}\binom{k}{j} = \sum_{j=0}^{n-1}\binom{n}{j+1}(T-I_X)^j$$
$$= \sum_{j=0}^{n-1}\frac{1}{(j+1)!}\Big[\prod_{k=0}^{j}(n-k)\Big](T-I_X)^j.$$

Hence for every $n\geq p+1$ we have

$$\frac{1}{n^p}\sum_{j=p}^{n-1}\binom{n}{j+1}(T-I_X)^j$$
$$= \frac{1}{n^p}\sum_{k=0}^{n-1}T^k - \sum_{j=0}^{p-1}\frac{1}{(j+1)!}\left[\frac{1}{n^p}\prod_{k=0}^{j}(n-k)\right](T-I_X)^j.$$

Since, for every $j\in\{0,\ldots,p-1\}$, $\prod_{k=0}^{j}(n-k)$ is a monic polynomial of degree $j+1$ in n, it follows that $(1/n^p)\prod_{k=0}^{j}(n-k)$ converges to zero (as $n\to\infty$) when $j<p-1$ and to 1 when $j=p-1$. Then

$$\left\|\frac{1}{n^p}\sum_{j=p}^{n-1}\binom{n}{j+1}(T-I_X)^j - \left(E - \frac{1}{p!}(T-I_X)^{p-1}\right)\right\| \to 0 \quad \text{as } n\to\infty,$$

which establishes (3.3.2).

Finally, we prove (3.3.3). Since we have $\mathcal{R}((T-I_X)^j)\subset\mathcal{R}((I_X-T)^p)\subset\mathcal{R}((I_X-T)^{p-1})$ for every $j\in\{p,\ldots,n-1\}$ and every $n>p$, from (3.3.2) it follows that $\mathcal{R}(E) \subset \overline{\mathcal{R}((I_X - T)^{p-1})}$. Furthermore, since, by (3.3.1),

$$\|(I_X - T)E\| = \lim_{n\to\infty} \left\|(I_X - T)\frac{1}{n^p}\sum_{k=0}^{n-1} T^k\right\| = \lim_{n\to\infty}\left\|\frac{1}{n^p}(I_X - T^n)\right\| = 0,$$

it follows that $\mathcal{R}(E) \subset \mathcal{N}(I_X - T)$. ■

We can now formulate our generalization of the uniform ergodic theorem.

THEOREM 3.4. *Let $p \in \mathbb{Z}_+$, let X be a complex Banach space and let $T \in L(X)$. Then the following conditions are equivalent:*

(3.4.1) *$(1/n^p)\sum_{k=0}^{n-1} T^k$ converges in $L(X)$ and $\mathcal{R}((I_X - T)^{p-1}) + \mathcal{N}(I_X - T)$ is closed;*

(3.4.2) *$\|T^n\|/n^p \to 0$ as $n \to \infty$ and 1 is a pole of $R(\cdot, T)$;*

(3.4.3) *T satisfies condition $\mathcal{E}(k,p)$ for some $k \in \mathbb{Z}_+$ and 1 is a pole of $R(\cdot,T)$, of order less than or equal to p;*

(3.4.4) *T satisfies condition $\mathcal{E}(k,p)$ for some $k \in \mathbb{Z}_+$ and*

$$X = \mathcal{N}((I_X - T)^p) \oplus \mathcal{R}((I_X - T)^p);$$

(3.4.5) *$\|T^n\|/n^p \to 0$ as $n \to \infty$ and $\delta(I_X - T) < \infty$;*

(3.4.6) *T satisfies condition $\mathcal{E}(k,p)$ for some $k \in \mathbb{Z}_+$ and $\delta(I_X - T) \le p$;*

(3.4.7) *T satisfies condition $\mathcal{E}(k,p)$ for some $k \in \mathbb{Z}_+$, $\delta(I_X - T) < \infty$ and $\mathcal{N}((I_X - T)^p)$ has a complement which is invariant under T;*

(3.4.8) *$\|T^n\|/n^p \to 0$ as $n \to \infty$ and $I_X - T$ is a quasi-Fredholm operator;*

(3.4.9) *$\|T^n\|/n^p \to 0$ as $n \to \infty$ and $\mathcal{R}((I_X - T)^k) + \mathcal{N}((I_X - T)^j)$ is closed for some $(k,j) \in \mathbb{N}\times\mathbb{N}$ with $k \ge p$;*

(3.4.10) *$\|T^n\|/n^p \to 0$ as $n \to \infty$ and $\mathcal{R}((I_X - T)^k) + \mathcal{N}((I_X - T)^j)$ is closed for every $(k,j) \in \mathbb{N}\times\mathbb{N}$ satisfying $k + j \ge p$.*

Furthermore, if (3.4.1)–(3.4.10) are satisfied and P is the projection of X onto $\mathcal{N}((I_X - T)^p)$ along $\mathcal{R}((I_X - T)^p)$, then

$$\left\|\frac{1}{n^p}\sum_{k=0}^{n-1} T^k - \frac{1}{p!}(T - I_X)^{p-1}P\right\| \to 0 \quad \text{as } n \to \infty.$$

Proof. Conditions (3.4.3) and (3.4.4) are equivalent in virtue of Theorems 1.2 and 1.3. From Theorem 1.2 it also follows that they imply (3.4.7). In virtue of [LM2], Proposition 1, condition (3.4.7) implies $\delta((I_X - T)^p) \le 1$, which gives $\delta(I_X - T) \le p$. Hence (3.4.7) implies (3.4.6). Furthermore, from Corollary 2.5 it follows that $\mathcal{E}(k,p)$ implies $\alpha(I_X - T) < \infty$. Then (3.4.6) implies (3.4.3) and (3.4.4) by Theorem 1.3. We have thus proved that (3.4.3), (3.4.4), (3.4.6) and (3.4.7) are equivalent.

Since $\|T^n\|/n^p \to 0$ as $n \to \infty$ implies $\alpha(I_X - T) \le p$ by Theorem 1.7, from Theorem 1.3 it also follows that (3.4.2) and (3.4.5) are equivalent and imply (3.4.3), (3.4.4), (3.4.6) and (3.4.7).

Now we prove that (3.4.3), (3.4.4), (3.4.6) and (3.4.7) imply (3.4.2) and (3.4.5). If T satisfies the former set of conditions, then $\mathcal{R}((I_X - T)^p)$ is closed by Theorem 1.2. Furthermore, $\mathcal{R}((I_X - T)^n) = \mathcal{R}((I_X - T)^p)$ for every $n \ge p$. Consequently, in virtue of [TL], IV, 5.9, if we set $m = \max\{p, k-1\}$ there exists $\delta > 0$ such that, for each $y \in \mathcal{R}((I_X - T)^p)$, there exists $u_y \in X$ satisfying $(I_X - T)^m u_y = y$ and $\|u_y\| \le \delta\|y\|$. Since T satisfies $\mathcal{E}(k,p)$, from Proposition 2.2 it follows that

$$\left\|\frac{1}{n^p}(I_X - T)^{k-1}T^n\right\| \to 0 \quad \text{as } n \to \infty.$$

Hence for every $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\left\|\frac{1}{n^p}(I_X - T)^{k-1}T^n\right\| < \frac{\varepsilon}{\delta M} \quad \text{for all } n \ge n_\varepsilon$$

(where $M > 0$ satisfies $M \ge \|I_X - P\| \cdot \|(I_X - T)^{m-k+1}\|$). Then for all $n \ge n_\varepsilon$ and for each $x \in X$ we have

$$\begin{aligned}
\left\|\frac{1}{n^p}T^n(I_X - P)x\right\| &= \left\|\frac{1}{n^p}T^n(I_X - T)^m u_{(I_X-P)x}\right\| \\
&= \left\|\frac{1}{n^p}(I_X - T)^{k-1}T^n(I_X - T)^{m-k+1}u_{(I_X-P)x}\right\| \\
&\le \frac{\varepsilon}{\delta M}\|(I_X - T)^{m-k+1}u_{(I_X-P)x}\| \\
&\le \frac{\varepsilon}{\delta M}\|(I_X - T)^{m-k+1}\| \cdot \|u_{(I_X-P)x}\| \\
&\le \frac{\varepsilon}{M}\|(I_X - T)^{m-k+1}\| \cdot \|I_X - P\| \cdot \|x\| \le \varepsilon\|x\|,
\end{aligned}$$

which gives

$$\left\|\frac{1}{n^p}T^n(I_X - P)\right\| \le \varepsilon \quad \text{for all } n \ge n_\varepsilon.$$

Hence $(1/n^p)T^n(I_X - P)$ converges to zero in $L(X)$ as $n \to \infty$. Since 1 is a pole of order less than or equal to p of the resolvent of the bounded linear operator

$$T_0 : \mathcal{N}((I_X - T)^p) \ni x \mapsto Tx \in \mathcal{N}((I_X - T)^p),$$

and moreover we clearly have $\sigma(T_0) \subset \{1\}$, we may apply Theorem 1.8 and conclude that $(1/n^p)T^nP \to 0$ in $L(X)$ as $n \to \infty$. Then $(1/n^p)T^n \to 0$ in $L(X)$ as $n \to \infty$, that is, T satisfies (3.4.2) and (3.4.5).

We have thus proved that (3.4.2)–(3.4.7) are equivalent. Notice that they imply (3.4.8) by Theorem 1.4.

Now we prove that (3.4.8) implies (3.4.10). If T satisfies (3.4.8), then $I_X - T$ is a quasi-Fredholm operator, that is, there exist two closed subspaces

M and N of X, invariant under T, such that $X = M \oplus N$, the operator

$$T_1 : N \ni x \mapsto (I_X - T)x \in N$$

is nilpotent, $(I_X - T)(M)$ is closed in X and $\mathcal{N}((I_X - T)^n) \cap M \subset (I_X - T)(M)$ for every $n \in \mathbb{N}$. Then $(I_X - T)^n(M)$ is closed for all $n \in \mathbb{N}$ by [MO], 2.5, and [TL], IV, 5.9 (or [MO], 1.1). Furthermore, since $\|T^n\|/n^p \to 0$ as $n \to \infty$, and consequently $\alpha(I_X - T) \le p$ by Theorem 1.7, it follows that $N \subset \mathcal{N}((I_X - T)^p)$. Hence for all $(k, j) \in \mathbb{N} \times \mathbb{N}$ satisfying $k + j \ge p$ we have $\mathcal{R}((I_X - T)^{k+j}) = (I_X - T)^{k+j}(M)$, which is closed. Now from Lemma 3.1 it follows that $\mathcal{R}((I_X - T)^k) + \mathcal{N}((I_X - T)^j)$ is closed. We conclude that T satisfies (3.4.10).

From what we have proved above, it follows that, if T satisfies (3.4.2)–(3.4.7), then $\mathcal{R}((I_X - T)^{p-1}) + \mathcal{N}(I_X - T)$ is closed. Theorem 1.6 provides the remaining part of the proof of the assertion "(3.4.2)–(3.4.7) imply (3.4.1)", as well as the proof of convergence of $(1/n^p)\sum_{k=0}^{n-1} T^k$ to $(1/p!)(T - I_X)^{p-1}P$ when conditions (3.4.2)–(3.4.7) are assumed.

Clearly, (3.4.10) implies (3.4.9). Now it suffices to prove that (3.4.1) implies (3.4.9), and (3.4.9) implies (3.4.2)–(3.4.7).

We prove that (3.4.1) implies (3.4.9). We first recall that convergence in $L(X)$ of $(1/n^p)\sum_{k=0}^{n-1} T^k$ implies convergence to zero of $\|T^n\|/n^p$ by (3.3.1). Now we prove that, if T satisfies (3.4.1), then $\mathcal{R}((I_X - T)^p) + \mathcal{N}(I_X - T)$ is closed. Let $E \in L(X)$ be such that $\|(1/n^p)\sum_{k=0}^{n-1} T^k - E\| \to 0$ as $n \to \infty$. Then

$$\left\| \frac{1}{n^p} \sum_{j=p}^{n-1} \binom{n}{j+1} (T - I_X)^j - \left(E - \frac{1}{p!}(T - I_X)^{p-1}\right) \right\| \to 0 \quad \text{as } n \to \infty$$

by (3.3.2) and $\mathcal{R}(E) \subset \mathcal{N}(I_X - T)$ in view of (3.3.3). Hence, if $Q \in L(X, X/\mathcal{N}(I_X - T))$ is the canonical quotient map, we have

$$\left\| \frac{1}{n^p} \sum_{j=p}^{n-1} \binom{n}{j+1} Q(T - I_X)^j + \frac{1}{p!} Q(T - I_X)^{p-1} \right\| \to 0 \quad \text{as } n \to \infty.$$

Notice that $\mathcal{R}(Q(T - I_X)^{p-1}) = (\mathcal{R}((I_X - T)^{p-1}) + \mathcal{N}(I_X - T))/\mathcal{N}(I_X - T)$, which is a closed subspace of $X/\mathcal{N}(I_X - T)$ as $\mathcal{R}((I_X - T)^{p-1}) + \mathcal{N}(I_X - T)$ is a closed subspace of X and contains $\mathcal{N}(I_X - T)$. Furthermore, since for every $n > p$ we have

$$\mathcal{R}\left(\frac{1}{n^p} \sum_{j=p}^{n-1} \binom{n}{j+1} (T - I_X)^j\right) \subset \mathcal{R}((I_X - T)^p) \subset \mathcal{R}((I_X - T)^{p-1}),$$

it follows that

$$\mathcal{R}\left(\frac{1}{n^p} \sum_{j=p}^{n-1} \binom{n}{j+1} Q(T - I_X)^j\right) \subset (\mathcal{R}((I_X - T)^{p-1}) + \mathcal{N}(I_X - T))/\mathcal{N}(I_X - T)$$

for all $n > p$. Thus from Lemma 3.2 we conclude that

$$\mathcal{R}\left(\frac{1}{n^p} \sum_{j=p}^{n-1} \binom{n}{j+1} Q(T - I_X)^j\right) = (\mathcal{R}((I_X - T)^{p-1}) + \mathcal{N}(I_X - T))/\mathcal{N}(I_X - T)$$

for sufficiently large n. Since

$$\begin{aligned}\mathcal{R}\left(\frac{1}{n^p} \sum_{j=p}^{n-1} \binom{n}{j+1} Q(T - I_X)^j\right) & \\ &\subset \mathcal{R}(Q(I_X - T)^p) \\ &= (\mathcal{R}((I_X - T)^p) + \mathcal{N}(I_X - T))/\mathcal{N}(I_X - T)\end{aligned}$$

for all $n > p$, we have

$$\begin{aligned}(\mathcal{R}((I_X - T)^{p-1}) + \mathcal{N}(I_X - T))/\mathcal{N}(I_X - T) & \\ &= (\mathcal{R}((I_X - T)^p) + \mathcal{N}(I_X - T))/\mathcal{N}(I_X - T)\end{aligned}$$

and consequently

$$\mathcal{R}((I_X - T)^p) + \mathcal{N}(I_X - T) = \mathcal{R}((I_X - T)^{p-1}) + \mathcal{N}(I_X - T).$$

Hence $\mathcal{R}((I_X - T)^p) + \mathcal{N}(I_X - T)$ is closed. We have thus proved that T satisfies (3.4.9) for $k = p$ and $j = 1$.

Now we prove that (3.4.9) implies (3.4.2)–(3.4.7). Suppose that T satisfies (3.4.9). Then $\|T^n\|/n^p \to 0$ as $n \to \infty$ and consequently, as we remarked above, $\alpha(I_X - T) \le p$.

Now let $k \ge p$ and $j \in \mathbb{N}$ be such that $\mathcal{R}((I_X - T)^k) + \mathcal{N}((I_X - T)^j)$ is closed. Since the vector space $\mathcal{R}((I_X - T)^k) \cap \mathcal{N}((I_X - T)^j)$ is algebraically isomorphic to $\mathcal{N}((I_X - T)^{k+j})/\mathcal{N}((I_X - T)^k)$ (see [TL], V, 6.3) and since $\alpha(I_X - T) \le p$, it follows that $\mathcal{R}((I_X - T)^k) \cap \mathcal{N}((I_X - T)^j) = \{0\}$. Then, since $\mathcal{R}((I_X - T)^k) \oplus \mathcal{N}((I_X - T)^j)$ is closed, it follows from Theorem 1.1 that $\mathcal{R}((I_X - T)^k)$ is closed.

Suppose first $k > p$. Since $\|T^n\|/n^p \to 0$ as $n \to \infty$, it follows that $r(T) \le 1$. Then $1 \in \varrho(T)$. Since $\alpha(I_X - T) \le p$, it follows from [La], 2.7 that 1 is a pole of $R(\cdot, T)$. Thus T satisfies (3.4.2)–(3.4.7).

Now suppose $k = p$. Let $A \in L(\mathcal{R}((I_X - T)^p), X)$ be defined by

$$Ax = (T - I_X)x$$

for every $x \in \mathcal{R}((I_X - T)^p)$. Then $\mathcal{R}(A) = \mathcal{R}((I_X - T)^{p+1})$.

Since $\mathcal{R}((I_X - T)^p)$ is closed, there exists $\tau > 0$ such that, for each $y \in \mathcal{R}((I_X - T)^p)$, there exists $x_y \in X$ which satisfies $(T - I_X)^p x_y = y$ and

$\|x_y\| \le \tau\|y\|$ (see [TL], IV, 5.9). Then, for every $y \in \mathcal{R}((I_X - T)^p)$ and for all $n \ge p+1$, we have

$$\begin{aligned}
\frac{1}{n^p}\Big(\sum_{k=p+1}^{n}\binom{n}{k}(T-I_X)^{k-p-1}\Big)(T-I_X)y
&= \frac{1}{n^p}\sum_{k=p+1}^{n}\binom{n}{k}(T-I_X)^k x_y\\
&= \frac{1}{n^p}\Big(T^n x_y - \sum_{k=0}^{p}\binom{n}{k}(T-I_X)^k x_y\Big)\\
&= \frac{T^n x_y}{n^p} - \sum_{k=0}^{p-1}\frac{1}{n^p}\binom{n}{k}(T-I_X)^k x_y - \frac{1}{n^p}\binom{n}{p}y.
\end{aligned}$$

Since $\|T^n\|/n^p \to 0$ as $n \to \infty$ and $\lim_{n\to\infty}(1/n^p)\binom{n}{k}$ is equal to zero if $k \le p-1$ and to $1/p!$ if $k = p$, there exists $\nu > p$ such that

$$\Big\|\frac{1}{\nu^p}T^\nu - \sum_{k=0}^{p-1}\frac{1}{\nu^p}\binom{\nu}{k}(T-I_X)^k\Big\| < \frac{1}{p!\cdot 4\tau} \quad\text{and}\quad \frac{1}{\nu^p}\binom{\nu}{p} > \frac{1}{p!\cdot 2}.$$

Consequently, for each $y \in \mathcal{R}((I_X - T)^p)$ we have

$$\begin{aligned}
&\Big\|\frac{1}{\nu^p}\Big(\sum_{k=p+1}^{\nu}\binom{\nu}{k}(T-I_X)^{k-p-1}\Big)Ay\Big\|\\
&\quad\ge \frac{1}{\nu^p}\binom{\nu}{p}\|y\| - \Big\|\frac{1}{\nu^p}T^\nu - \sum_{k=0}^{p-1}\frac{1}{\nu^p}\binom{\nu}{k}(T-I_X)^k\Big\|\,\|x_y\|\\
&\quad\ge \Big(\frac{1}{\nu^p}\binom{\nu}{p} - \tau\Big\|\frac{1}{\nu^p}T^\nu - \sum_{k=0}^{p-1}\frac{1}{\nu^p}\binom{\nu}{k}(T-I_X)^k\Big\|\Big)\|y\| \ge \frac{\|y\|}{p!\cdot 4}.
\end{aligned}$$

Hence $(1/\nu^p)(\sum_{k=p+1}^{\nu}\binom{\nu}{k}(T-I_X)^{k-p-1})A$ is one-to-one and has closed range, which implies that also A is one-to-one and has closed range. Thus $\mathcal{R}((I_X - T)^{p+1})$ is closed. Now, in order to conclude that T satisfies (3.4.2)–(3.4.7), we can either remark that we have reduced the situation to the case $k > p$, or appeal to Theorem 1.6.

We have thus completed the proof of the theorem. ∎

We remark that the condition "$\mathcal{R}((I_X - T)^{p-1}) + \mathcal{N}(I_X - T)$ is closed" is automatically satisfied when $p = 1$, as in this case $\mathcal{R}((I_X - T)^{p-1}) = \mathcal{R}((I_X - T)^0) = X$. Thus each of conditions (3.4.1)–(3.4.8) generalizes the corresponding condition of Theorem 1.5. Notice that conditions (3.4.9) and (3.4.10) generalize conditions (1.5.9) and (1.5.11) and condition (1.5.10), respectively, also when $p = 1$: indeed, when $p = 1$, conditions (1.5.9) and (1.5.11) correspond to the special cases $j = 0$ and $k = j = 1$, respectively, of (3.4.9), and condition (1.5.10) corresponds to the special case $j = 0$ of (3.4.10).

We also recall that in [LM2], Theorem 6, several equivalent conditions to 1 being a pole of the resolvent of a bounded linear operator T, satisfying condition ($\mathcal{E}$-p) for some $p \in \mathbb{Z}_+$, are given. We remark that Theorem 3.4 generalizes Theorem 6 of [LM2], as condition ($\mathcal{E}$-p), in virtue of Theorem 2.4 and Example 2.6, is more restrictive than convergence to zero of $\|T^n\|/n^p$.

Finally, we remark that (3.4.9) is more general than (1.6.2) (which corresponds to the special case $k = p+1$, $j = 0$). Besides, condition (3.4.9) with $k = p$, $j = 0$ shows that, if $\|T^n\|/n^p \to 0$ as $n \to \infty$, then closedness of $\mathcal{R}((I_X - T)^p)$ ensures that 1 is a pole of order less than or equal to p of $R(\cdot, T)$. The following example shows that, under the weaker (than convergence to zero of $\|T^n\|/n^p$) hypotheses $r(T) \le 1$ and $\alpha(I_X - T) \le p$, closedness of $\mathcal{R}((I_X - T)^p)$ does not suffice to ensure that 1 is a pole of $R(\cdot, T)$ (whereas closedness of $\mathcal{R}((I_X - T)^k)$ for some $k > p$ does, in virtue of [La], 2.7).

Example 3.5. Let S be the bounded linear weighted shift operator on the complex Hilbert space l_2 defined by

$$S(x_n)_{n\in\mathbb{N}} = \sum_{n\in\mathbb{N}}\frac{x_n}{n+1}e_{n+1} \quad \text{for every } (x_n)_{n\in\mathbb{N}} \in l_2$$

(where $(e_n)_{n\in\mathbb{N}}$ is the canonical basis of l_2), and let $A \in L(l_2 \times l_2)$ be defined by

$$A(x,y) = (Sx, x) \quad \text{for every } (x,y) \in l_2 \times l_2.$$

Then $\mathcal{R}(A)$ is closed by Theorem 1.1, as $l_2 \times l_2 = \mathcal{R}(A) \oplus (l_2 \times \{0\})$. Furthermore, since $r(S) = 0$ (see [H], Solution 80), S is not nilpotent and

$$A^k(x,y) = (S^k x, S^{k-1}x) \quad \text{for every } (x,y) \in l_2 \times l_2 \text{ and every } k \in \mathbb{Z}_+,$$

it follows that $r(A) = 0$ and A is not nilpotent (see for instance [Z] for another example of a bounded linear nonnilpotent operator having closed range, whose spectral radius is equal to zero). Hence 0 is not a pole of $R(\cdot, A)$ (see [TL], V, 10.6). Notice also that $\alpha(A) = 1$, as $\mathcal{N}(A^k) = \{0\} \times l_2$ for every $k \in \mathbb{Z}_+$. Hence, if we set $T = I_{l_2\times l_2} - A$, it follows that $\sigma(T) = \{1\}$, $\alpha(I_{l_2\times l_2} - T) = 1$, $\mathcal{R}(I_{l_2\times l_2} - T)$ is closed and nevertheless 1 is not a pole of $R(\cdot, T)$. ∎

We remark that the equivalent conditions of Theorem 3.4 imply the following condition:

(3.6) $\|T^n\|/n^p \to 0$ as $n \to \infty$ and $\mathcal{R}((I_X - T)^{p-k}) + \mathcal{N}((I_X - T)^j)$ is closed for every $k \in \{1, \ldots, p\}$ and for all $j \ge k$.

When $p \geq 2$, the equivalent conditions of Theorem 3.4 imply the following condition as well:

(3.7) $(1/n^p)\sum_{k=0}^{n-1} T^k$ converges in $L(X)$ and $\mathcal{R}((I_X-T)^{p-k})+\mathcal{N}((I_X-T)^j)$ is closed for every $k \in \{2,\ldots,p\}$ and for all $j \geq k$.

Nevertheless, neither (3.6) nor (3.7) implies the equivalent conditions of Theorem 3.4: indeed, if T is some bounded linear operator on a complex Banach space such that the sequence $(T^n)_{n\in\mathbb{N}}$ is bounded and 1 is not a pole of $R(\cdot,T)$ (e.g., if T is the bounded linear operator on the complex Banach space $L_2([0,1])$ defined by $(Tx)(t) = tx(t)$ for almost all $t \in [0,1]$ and for all $x \in L_2([0,1])$—see [W], Beispiel—or by $T = (I_{L_2([0,1])} + V)^{-1}$, where V is the Volterra integral operator—see [MZ], p. 1155), then T satisfies both (3.6) for $p = 1$ and (3.7) for $p = 2$, yet satisfies the equivalent conditions of Theorem 3.4 for no $p \in \mathbb{Z}_+$.

Now we are going to show that convergence to zero of $\|T^n\|/n^p$ (respectively, condition $\mathcal{E}(k,p)$) cannot be replaced by condition $\mathcal{S}(1,p)$ in (3.4.2), (3.4.5) and (3.4.8)–(3.4.10) (respectively, (3.4.3), (3.4.4), (3.4.6) and (3.4.7)): indeed, the following is an example of an operator T such that $(1/n)T^n$ converges strongly to zero (namely, in virtue of Proposition 2.2, T satisfies condition $\mathcal{S}(1,1)$), $r(T) = 1$ and 1 is a pole of order 1 of $R(\cdot,T)$ (which, in virtue of Theorems 1.2–1.4 and Lemma 3.1, implies that T satisfies the second part of each of conditions (3.4.2)–(3.4.10) for $p = 1$), and nevertheless $(1/n)\sum_{k=0}^{n-1} T^k$ does not converge in $L(X)$. Notice that, instead, if a bounded linear operator A with $r(A) = 1$ satisfies condition $\mathcal{S}(1,p)$ and in addition the intersection of $\sigma(A)$ and the unit circle is assumed to consist of a finite number of poles of $R(\cdot,A)$, then Theorems 1.7 and 1.8, together with Theorem 1.3, ensure that A satisfies the equivalent conditions of Theorem 3.4. In virtue of Theorem 1.3 and of [La], 2.7, respectively, together with Theorem 1.7, the condition above about the intersection of $\sigma(A)$ and the unit circle is satisfied when, for instance, $\sigma(A) = \{1\}$, A satisfies $\mathcal{S}(1,p)$ and either $\delta(I_X - A) < \infty$ or $\mathcal{R}((I_X - A)^k)$ is closed for some $k > p$ (see [Z], Theorem 6, for the case $p = 1$).

EXAMPLE 3.8. Let W be the bounded linear operator on the complex Hilbert space $L_2([0,1])$ considered in [W], Beispiel. That is,

$$(Wx)(t) = tx(t) \quad \text{for almost all } t \in [0,1].$$

We recall that $\sigma(W) = [0,1]$, $\mathcal{R}(\lambda I_{L_2([0,1])} - W) \neq L_2([0,1])$ for every $\lambda \in [0,1]$ and $\|W^n\| = 1$ for all $n \in \mathbb{N}$ (see [W], Beispiel).

Now let $A \in L(L_2([0,1]) \times L_2([0,1]))$ be defined by

$$A(x,y) = (Wx, y - x) \quad \text{for all } (x,y) \in L_2([0,1]) \times L_2([0,1]).$$

We remark that for all $\lambda \in \mathbb{C}$ and $(x,y) \in L_2([0,1]) \times L_2([0,1])$, we have

$$(\lambda I_{L_2([0,1])\times L_2([0,1])} - A)(x,y) = ((\lambda I_{L_2([0,1])} - W)x, (\lambda - 1)y + x).$$

Since $\mathcal{R}(\lambda I_{L_2([0,1])} - W) \neq L_2([0,1])$ for all $\lambda \in [0,1]$, it follows that

$$\mathcal{R}(\lambda I_{L_2([0,1])\times L_2([0,1])} - A) \neq L_2([0,1]) \times L_2([0,1]) \quad \text{for all } \lambda \in [0,1].$$

Hence $[0,1] \subset \sigma(A)$. Now, for every $\lambda \in \mathbb{C} \setminus [0,1]$, let the operator $S_\lambda \in L(L_2([0,1]) \times L_2([0,1]))$ be defined by

$$S_\lambda(x,y) = \Big((\lambda I_{L_2([0,1])} - W)^{-1}x, \frac{1}{\lambda - 1}(y - (\lambda I_{L_2([0,1])} - W)^{-1}x)\Big)$$

for all $(x,y) \in L_2([0,1]) \times L_2([0,1])$. Then

$$S_\lambda(\lambda I_{L_2([0,1])\times L_2([0,1])} - A) = (\lambda I_{L_2([0,1])\times L_2([0,1])} - A)S_\lambda = I_{L_2([0,1])\times L_2([0,1])},$$

and consequently $\lambda \in \varrho(A)$.

We have thus proved that $\sigma(A) = [0,1]$ (which implies that 1 is not a pole of $R(\cdot,A)$).

We also remark that, since for every $(x,y) \in L_2([0,1]) \times L_2([0,1])$ we have

$$(I_{L_2([0,1])\times L_2([0,1])} - A)(x,y) = ((I_{L_2([0,1])} - W)x, x),$$

we get $L_2([0,1]) \times L_2([0,1]) = \mathcal{R}(I_{L_2([0,1])\times L_2([0,1])} - A) \oplus (L_2([0,1]) \times \{0\})$. Then $\mathcal{R}(I_{L_2([0,1])\times L_2([0,1])} - A)$ is closed by Theorem 1.1.

Now we prove that $(1/n)A^n$ converges strongly to zero. We begin by remarking that for all $n \in \mathbb{Z}_+$ and $(x,y) \in L_2([0,1]) \times L_2([0,1])$ we have

$$A^n(x,y) = \Big(W^n x, y - \sum_{k=0}^{n-1} W^k x\Big).$$

Since $\|W^n\| = 1$ for every $n \in \mathbb{N}$, and consequently $\|W^n\|/n \to 0$ as $n \to \infty$, it suffices to show that $(1/n)(y - \sum_{k=0}^{n-1} W^k x) \to 0$ in $L_2([0,1])$ as $n \to \infty$ for all $(x,y) \in L_2([0,1]) \times L_2([0,1])$, that is, $(1/n)\sum_{k=0}^{n-1} W^k x \to 0$ in $L_2([0,1])$ as $n \to \infty$ for all $x \in L_2([0,1])$. Fix $x \in L_2([0,1])$. We remark that for all $n \in \mathbb{Z}_+$ we have

$$\frac{1}{n}\Big(\sum_{k=0}^{n-1} W^k x\Big)(t) = \frac{1}{n}\sum_{k=0}^{n-1} t^k x(t) = \frac{x(t)(1-t^n)}{n(1-t)} \quad \text{for almost all } t \in [0,1].$$

Thus $(1/n)(\sum_{k=0}^{n-1} W^k x)(t) \to 0$ as $n \to \infty$ for almost all $t \in [0,1]$. Since

$$\Big|\frac{1}{n}\Big(\sum_{k=0}^{n-1} W^k x\Big)(t)\Big| = \frac{|x(t)|}{n}\sum_{k=0}^{n-1} t^k \leq |x(t)|$$

for almost all $t \in [0,1]$ and for all $n \in \mathbb{Z}_+$, from the dominated convergence theorem, it follows that $(1/n)\sum_{k=0}^{n-1} W^k x \to 0$ in $L_2([0,1])$.

We have thus proved that $(1/n)A^n(x,y) \to 0$ in $L_2([0,1]) \times L_2([0,1])$ for all $(x,y) \in L_2([0,1]) \times L_2([0,1])$.

Now let X denote $L_2([0,1]) \times L_2([0,1])$, and let $T \in L(X \times \mathbb{C})$ be defined by

$$T(u,\lambda) = (-Au,\lambda) \quad \text{for all } (u,\lambda) \in X \times \mathbb{C}.$$

Since $\sigma(-A) = -\sigma(A) = [-1,0]$ and the spectrum of the linear operator on $\mathbb{C}$ mapping every $\lambda \in \mathbb{C}$ into λ is the singleton $\{1\}$, it follows that

$$\sigma(T) = [-1,0] \cup \{1\}$$

and consequently $r(T) = 1$.

We remark that for all $(u,\lambda) \in X \times \mathbb{C}$ we have

$$(I_{X\times\mathbb{C}} - T)(u,\lambda) = ((I_X + A)u, 0).$$

Since $-1 \notin \sigma(A)$, it follows that

$$\mathcal{N}(I_{X\times\mathbb{C}} - T) = \{0\} \times \mathbb{C} \quad \text{and} \quad \mathcal{R}(I_{X\times\mathbb{C}} - T) = X \times \{0\},$$

which gives

$$X \times \mathbb{C} = \mathcal{N}(I_{X\times\mathbb{C}} - T) \oplus \mathcal{R}(I_{X\times\mathbb{C}} - T).$$

In virtue of Theorems 1.2 and 1.3, we may conclude that 1 is a pole of order 1 of $R(\cdot,T)$. Furthermore, since $(1/n)A^n$ converges strongly to zero, it follows that

$$\frac{1}{n}T^n(u,\lambda) = \frac{1}{n}((-1)^n A^n u, \lambda) = \left(\frac{(-1)^n}{n}A^n u, \frac{\lambda}{n}\right) \to 0$$

as $n \to \infty$ for all $(u,\lambda) \in X \times \mathbb{C}$. Thus $(1/n)T^n$ also converges strongly to zero, that is, T satisfies $\mathcal{S}(1,1)$.

Now we prove that $(1/n)\sum_{k=0}^{n-1} T^k$ does not converge in $L(X \times \mathbb{C})$. Otherwise, by Theorem 1.5, $(1/n)T^n \to 0$ in $L(X \times \mathbb{C})$, and consequently $(1/n)A^n \to 0$ in $L(X)$. Since $\mathcal{R}(I_X - A)$ is closed, Theorem 1.5 would show that 1 is a pole of $R(\cdot,A)$, contrary to what we have proved above. ■

Now we are going to provide a sufficient condition in order that the equivalent conditions of Theorem 3.4 be satisfied.

PROPOSITION 3.9. *Let $p \in \mathbb{Z}_+$, let X be a complex Banach space and let $T \in L(X)$. If $(1/n^p)\sum_{k=0}^{n-1} T^k$ converges in $L(X)$ and $\mathcal{R}((I_X - T)^{p-1})$ is closed, then T satisfies the equivalent conditions* (3.4.1)–(3.4.10).

Proof. Let $E \in L(X)$ be such that $\|(1/n^p)\sum_{k=0}^{n-1} T^k - E\| \to 0$ as $n \to \infty$. Then by (3.3.2) we have

$$\left\| \frac{1}{n^p}\sum_{j=p}^{n-1}\binom{n}{j+1}(T - I_X)^j - E + \frac{1}{p!}(T - I_X)^{p-1} \right\| \to 0 \quad \text{as } n \to \infty.$$

Furthermore, since $\mathcal{R}((I_X - T)^{p-1})$ is closed, from (3.3.3) we conclude that $\mathcal{R}(E) \subset \mathcal{N}(I_X - T) \cap \mathcal{R}((I_X - T)^{p-1})$ and consequently

$$\mathcal{R}\left(\frac{1}{n^p}\sum_{j=p}^{n-1}\binom{n}{j+1}(T - I_X)^j - E\right) \subset \mathcal{R}((I_X - T)^{p-1}),$$

for every $n > p$. Since $\mathcal{R}((I_X - T)^{p-1})$ is closed, from Lemma 3.2 it follows that, for sufficiently large n, we have

$$\mathcal{R}\left(\frac{1}{n^p}\sum_{j=p}^{n-1}\binom{n}{j+1}(T - I_X)^j - E\right) = \mathcal{R}((I_X - T)^{p-1}).$$

Consequently,

$$\begin{aligned}\mathcal{R}((I_X - T)^p) &= \mathcal{R}\left(\frac{1}{n^p}\sum_{j=p}^{n-1}\binom{n}{j+1}(T - I_X)^{j+1}\right)\\ &= (I_X - T)^{p+1}\left(\mathcal{R}\left(\sum_{j=p}^{n-1}\binom{n}{j+1}(T - I_X)^{j-p}\right)\right)\\ &\subset \mathcal{R}((I_X - T)^{p+1}).\end{aligned}$$

Hence $\mathcal{R}((I_X - T)^{p+1}) = \mathcal{R}((I_X - T)^p)$, that is, $\delta(I_X - T) \le p$. Since convergence to zero of $\|T^n\|/n^p$ is a consequence of convergence of $(1/n^p)\sum_{k=0}^{n-1} T^k$ in $L(X)$ (see (3.3.1)), it follows that T satisfies (3.4.5), and therefore all the equivalent conditions of Theorem 3.4. ■

We conclude with the following example, showing that the summand $\mathcal{N}(I_X - T)$ cannot be removed from condition (3.4.1), as closedness of $\mathcal{R}((I_X - T)^{p-1})$ is not a necessary condition for 1 to be a pole of order p of the resolvent of an operator T such that $\|T^n\|/n^p \to 0$ as $n \to \infty$.

EXAMPLE 3.10. Let B be some bounded linear operator on a complex Banach space X, with zero square, and such that $\mathcal{R}(B)$ is not closed (that is, $\mathcal{R}(B)$ is a nonclosed subspace of $\mathcal{N}(B)$; such an operator B can be constructed on any infinite-dimensional Banach space X: it suffices to choose two sequences $(x_n)_{n\in\mathbb{N}}$ and $(x_n^*)_{n\in\mathbb{N}}$ of elements of X and of its dual, respectively, such that $\|x_n\| = 1$ for all $n \in \mathbb{N}$ and $x_j^*(x_k) = \delta_{jk}$, where δ_{jk} denotes the Kronecker symbol, for all $j,k \in \mathbb{N}$, and then set

$$Bx = \sum_{n\in\mathbb{N}} \lambda_n x_{2n}^*(x) x_{2n+1} \quad \text{for all } x \in X,$$

where $(\lambda_n)_{n\in\mathbb{N}}$ is some sequence of nonzero scalars such that $\sum_{n\in\mathbb{N}}|\lambda_n| \times \|x_{2n}^*\| < \infty$). If we set $T = I_X - B$, then 1 is a pole of order 2 of T and $\sigma(T) = \{1\}$. From Theorem 1.8 it follows that $\|T^n\|/n^2 \to 0$ as $n \to \infty$. Then T satisfies (3.4.1)–(3.4.10) for $p = 2$. Nevertheless, $\mathcal{R}(I_X - T)$ is not closed, as it coincides with $\mathcal{R}(B)$, whereas $\mathcal{R}(I_X-T)+\mathcal{N}(I_X-T)=\mathcal{N}(B)$, which is closed. ∎

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