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Department of Mathematics
North Carolina State University
Raleigh, North Carolina 27695-8205
U.S.A.
E-mail: pawel@math.ncsu.edu

Department of Mathematics
University of Missouri-Columbia
Columbia, Missouri 65211
U.S.A.
E-mail: stephen@math.missouri.edu

Institute of Mathematics
Warsaw University
Banacha 2
02-097 Warszawa, Poland
E-mail: koles@mimuw.edu.pl

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The boundary Harnack principle for the fractional Laplacian

by

KRZYSZTOF BOGDAN (Wrocław)

Abstract. We study nonnegative functions which are harmonic on a Lipschitz domain with respect to symmetric stable processes. We prove that if two such functions vanish continuously outside the domain near a part of its boundary, then their ratio is bounded near this part of the boundary.

1. Introduction. The boundary Harnack principle (BHP) for nonnegative harmonic functions has important applications in probability theory and potential theory. Among these are approximations to excursion laws for the Brownian motion (see [6]), "3G Theorem" and "Conditional Gauge Theorem" (see [8]). BHP was first proved in [9] for Lipschitz domains by analytic methods (see also [12], [11]). Later, the classical link between harmonic functions and the Brownian motion in $\mathbb{R}^n$ was used to give a probabilistic proof of BHP ([2]). The result and generalizations of BHP to elliptic operators and Schrödinger operators have yielded stimulating interplay between probability theory, harmonic analysis and potential theory (see [7], [3], [8], [6], [1]).

The Brownian motion is a particular (and limiting) instance of the standard rotation invariant α-stable process, $\alpha \in (0, 2]$. The infinitesimal generator $\Delta^{\alpha/2}$ of the latter and the related class of α-harmonic functions have simple homogeneity properties analogous to those of the classical Laplacian and harmonic functions ($\alpha = 2$) in $\mathbb{R}^n$. Also, the potential theory of $\Delta^{\alpha/2}$ ($n > \alpha$) enjoys an explicit formulation in terms of M. Riesz kernels, and is similar to that of the Laplacian in $\mathbb{R}^n$, $n > 2$ ([13]).

The main result of this paper is the following theorem which gives another extension of the classical theory ($\alpha = 2$) to the case $\alpha \in (0, 2)$.

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THEOREM 1. *Let $\alpha \in (0,2)$ and $n \geq 2$. Let D be a Lipschitz domain in $\mathbb{R}^n$ and V be an open set. For every compact set $K \subset V$, there exists a positive constant $C = C(\alpha, D, V, K)$ such that for all nonnegative functions u and v in $\mathbb{R}^n$ which are continuous in V, α-harmonic in $D \cap V$, vanish on $D^c \cap V$, and satisfy $u(x_0) = v(x_0) > 0$ for some $x_0 \in D \cap K$, we have*

$$C^{-1}u(x) \leq v(x) \leq Cu(x), \quad x \in D \cap K. \tag{1.1}$$

Moreover, there exists a constant $\eta = \eta(\alpha, D, V, K) > 0$ such that the function $u(x)/v(x)$ is Hölder continuous of order η in $K \cap D$. In particular, for every $Q \in \partial D \cap V$, $\lim u(x)/v(x)$ exists as $D \ni x \to Q$.

Generally, we follow the approach designed in [7] for elliptic operators (see also [11]). In particular, Lemmas 1, 3, 4, 10–13 and 16 below have their analogues in [12], [11] and [7], with major changes in the proofs. The main obstacle to our development is the non-locality of the integro-differential operator $\Delta^{\alpha/2}$, resulting in non-locality of the definition of α-harmonic function and even of the notion of nonnegativity for such functions. This makes many of the arguments essentially different compared with the case of elliptic operators. In reward we are confronted with new concepts shedding new light on the classical theory. To prove results on the class of α-harmonic functions, we rely on basic properties of the corresponding α-stable process. While a purely analytic approach is possible ([13] provides an analytic introduction to α-harmonic functions), the probabilistic methods are very often more natural and convenient.

2. Preliminaries. For the rest of the paper, let $\alpha \in (0,2)$ and $n \geq 2$. We denote by (X_t, P^x) the standard rotation invariant (or "symmetric") stable process in $\mathbb{R}^n$, with the index of stability α, and the characteristic function

$$E^0 e^{i\xi X_t} = e^{-t|\xi|^\alpha}, \quad \xi \in \mathbb{R}^n,\ t \geq 0 \tag{2.1}$$

(see [5] for an explicit definition). As usual, E^x is the expectation with respect to the distribution P^x of the process starting from $x \in \mathbb{R}^n$. The process X_t has the potential operator

$$U_\alpha f(x) = \mathcal{A}(n,\alpha) \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy$$

(see [5]), and the infinitesimal generator $\Delta^{\alpha/2}$,

$$\Delta^{\alpha/2}u(x) = \mathcal{A}(n,-\alpha) \int_{\mathbb{R}^n} \frac{u(x+y)-u(x)}{|y|^{n+\alpha}}\, dy, \tag{2.2}$$

where $\mathcal{A}(n,\gamma) = \Gamma((n-\gamma)/2)/(2^\gamma \pi^{n/2}|\Gamma(\gamma/2)|)$ (cf. [13]). The justification for the notation $\Delta^{\alpha/2}$ is in the fact that the Fourier transform of the generator and the Fourier transform of the Laplacian $\Delta = \sum_{i=1}^n \partial_i^2$ satisfy the

equation

$$\mathcal{F}(-\Delta^{\alpha/2})(\xi) = |\xi|^\alpha = (\mathcal{F}(-\Delta)(\xi))^{\alpha/2}. \tag{2.3}$$

The proof of (2.3) can be found in [13] (see also [16, IX.11] for another justification). We point out that our notation, and in particular the definition of the Fourier transform:

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^n} e^{i\xi x} f(x)\, dx,$$

are different from those of [13].

For $x \in \mathbb{R}^n$, $r > 0$ and $a > 0$, we set $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$, so that $\operatorname{diam}(B(x,r)) = 2r$; and we write $aB(x,r) = B(x,ar)$. As usual, A^c is the complement of A, and $\operatorname{dist}(A,B) = \inf\{|x-y| : x \in A,\ y \in B\}$ for $A, B \subset \mathbb{R}^n$. For a Borel set $A \subset \mathbb{R}^n$, we define $T(A) = \inf\{t \geq 0 : X_t \in A\}$, the *first entrance time* of A.

DEFINITION 1. Let u be a Borel measurable function on $\mathbb{R}^n$ which is bounded from below (above). We say that u is *α-harmonic* in an open set $D \subset \mathbb{R}^n$ if

$$u(x) = E^x u(X_{T(B^c)}), \quad x \in B, \tag{2.4}$$

for every bounded open set B with the closure $\overline{B}$ contained in D. We say that u is *regular α-harmonic* in D if

$$u(x) = E^x u(X_{T(D^c)}), \quad x \in D. \tag{2.5}$$

By the strong Markov property of X_t, (2.5) implies (2.4), so that regular α-harmonic functions are α-harmonic. The converse is not true, as the example of the Green function (defined below) demonstrates (see also [14]). For u regular α-harmonic in D, we regard $u(x)$, $x \in D^c$, as the "boundary condition" in the sense that, in view of (2.5), it determines (and defines) u completely.

In case $B = B(0,r) \subset \mathbb{R}^n$, $r > 0$, and $|x| < r$, the P^x distribution of $X_{T(B^c)}$ has the density function $P_r(x,\cdot)$ (the Poisson kernel), explicitly given by the formula

$$P_r(x,y) = \begin{cases} c_\alpha^n \left[\dfrac{r^2-|x|^2}{|y|^2-r^2}\right]^{\alpha/2} |x-y|^{-n}, & |y| > r, \\ 0, & |y| \leq r, \end{cases} \tag{2.6}$$

where $c_\alpha^n = \Gamma(n/2)\pi^{-n/2-1}\sin(\pi\alpha/2)$. The proof of (2.6) is given in [13] (see also the *first passage relation* [14, (3.1)], to translate the potential theoretic result of [13] into the probabilistic assertion (2.6)). Consequently, for a function u which is α-harmonic in D, we have

$$u(x) = \int\limits_{|y-\theta|>r} P_r(x-\theta, y-\theta)u(y)\,dy, \qquad x \in B(\theta, r), \tag{2.7}$$

provided $\overline{B(\theta,r)} \subset D$. By (2.6), such a function u is C^∞ on D and satisfies $\int_{\mathbb{R}^n} |u(y)|(1+|y|)^{-n-\alpha}\,dy < \infty$. We note that (2.7) is an analogue of the Poisson integral for the ball. However, unlike Brownian motion, the process X_t has jumps, and the support of $P_r(x,\cdot)$ is the whole of $B(0,r)^c$.

As in the classical case ($\alpha = 2$), Definition 1 is not the only description of α-harmonicity (cf. [13], [4]). For instance, let u be again a Borel measurable function on $\mathbb{R}^n$ which is bounded from below (above) and (say) C^2 in D. Then the Dynkin characteristic operator (see [10]),

$$\mathcal{U}u(x) = \lim_{r\to 0} \frac{E^x u(X_{T(B(x,r))^c}) - u(x)}{E^x T(B(x,r))^c}, \tag{2.8}$$

is well defined and equals $\Delta^{\alpha/2}u(x)$ for $x \in D$. If u is α-harmonic in D, this clearly implies $\Delta^{\alpha/2}u(x) = 0$ for $x \in D$. The converse is also true, and seems to be well known. Thus α-harmonic functions bounded from below (above) are identified with the solutions u of $\Delta^{\alpha/2}u(x) = 0$, $x \in D$, which are bounded from below (above) and C^2 in D.

Lipschitz domains are recognized as an approximate borderline between the class of domains where the classical ($\alpha = 2$) BHP holds in its strongest form, and the class of domains where only partial results may be obtained (see [1, Chapter III]). Also in this paper we restrict ourselves to Lipschitz domains in order to present the methods, rather than to achieve maximal generality.

A (connected) region $D \subset \mathbb{R}^n$ is called a *Lipschitz domain* if it is bounded and for each Q in its boundary ∂D there are: a Lipschitz function $\Gamma_Q : \mathbb{R}^{n-1} \to \mathbb{R}$, an orthonormal coordinate system CS_Q, and a number $R_Q > 0$ such that if $y = (y_1, \ldots, y_n)$ in CS_Q coordinates, then

$$D \cap B(Q, R_Q) = \{y : y_n > \Gamma_Q(y_1, \ldots, y_{n-1})\} \cap B(Q, R_Q).$$

We note that by compactness of ∂D, the radii R_Q are not less than a constant $R_0 > 0$ (the *localization radius* of D), and the Lipschitz constants of the functions Γ_Q are not greater than a constant $\lambda < \infty$ (the *Lipschitz constant* of D).

For the rest of the paper let D be a Lipschitz domain with localization radius R_0 and Lipschitz constant λ. Many estimates below vary according to these (and other) parameters. Such a dependence is always explicitly indicated in the notation $C = C(x,y,z)$, which means that the *constant* C depends *only* on x, y, z. "Constants" are *always* numbers in $(0,\infty)$, so that we can freely multiply and divide them to get other constants.

Below we list a few important properties of Lipschitz domains.

P1 (Outer cone property). *There exists a constant $\eta = \eta(\lambda)$ and a cone $\mathcal{C} = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : 0 < x_n,\ \|(x_1, \ldots, x_{n-1})\| < \eta x_n\}$ such that for every $Q \in \partial D$, there is a cone $\mathcal{C}_Q$ with vertex Q, isometric with $\mathcal{C}$ and satisfying $\mathcal{C}_Q \cap B(Q, R_0) \subset D^c$.*

P2. *There exists a constant $\kappa = \kappa(\lambda) \in (0,1)$ such that for every $r \in (0, R_0)$ and $Q \in \partial D$, there is a point $A \in D \cap B(Q,r)$, denoted by $A_r(Q)$, such that $B(A, \kappa r) \subset D \cap B(Q,r)$.*

P3. *There exists a constant $\mathcal{M} = \mathcal{M}(D) \in \mathbb{N}$ such that for all $x_1, x_2 \in D$, $\varepsilon > 0$ and $k \in \mathbb{N}$ satisfying*

$$\operatorname{dist}(\{x_1, x_2\}, D^c) > \varepsilon, \tag{2.9}$$

$$|x_1 - x_2| < 2^k \varepsilon, \tag{2.10}$$

there is a sequence of balls $B_1, \ldots, B_{\mathcal{M}k} \subset D$ satisfying

$$\tfrac{1}{2} \operatorname{diam} B_j < \operatorname{dist}(B_j, \partial D) < 2 \operatorname{diam} B_j, \qquad j = 1, \ldots, \mathcal{M}k,$$

which join x_1 and x_2 in the sense that x_1 is the center of B_1, x_2 is the center of $B_{\mathcal{M}k}$, and $B_j \cap B_{j+1} \neq \emptyset$, $j = 1, \ldots, \mathcal{M}k - 1$. Also, there is a constant $\mathcal{L} = \mathcal{L}(D)$ such that the above sequence B_j is contained in $D \cap B(Q,r)$, provided

$$x_1, x_2 \in D \cap B(Q, \mathcal{L}r), \qquad Q \in \partial D,\ r > 0. \tag{2.11}$$

Moreover, there is a constant $L = L(\lambda)$ such that the sequence B_j is contained in $D \cap B(Q,r)$, provided

$$x_1, x_2 \in D \cap B(Q, Lr), \qquad Q \in \partial D,\ 0 < r < R_0, \tag{2.12}$$

and, in this case, the constant $\mathcal{M}$ above may be chosen to depend only on λ. This number will be denoted by $M = M(\lambda)$.

Remark 1. In property P1, the existence of the cone $\mathcal{C}_Q$ in local coordinates CS_Q easily follows from the Lipschitz condition on Γ_Q. A calculation yields $\eta = \lambda^{-1}$. Property P2 follows from the existence of inner cones $\mathcal{C}'_Q$ for D (with the same aperture as $\mathcal{C}_Q$), and a calculation gives $\kappa = (2\sqrt{1+\lambda^2})^{-1}$. We observe that the point A need not be unique, so there is a little abuse of language in the notation $A_r(Q)$.

In turn, P3 follows *locally* (i.e. in the part concerning L, M) from a straightforward construction. In coordinates CS_Q, we consider two cones with vertices on ∂D, isometric with, say, $\mathcal{C}' = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : 0 < x_n,\ \|(x_1, \ldots, x_{n-1})\| < x_n/(2\lambda + 2))\}$, with axes parallel to the y_n axis and containing x_1 and x_2, respectively. If x_1 and x_2 are sufficiently close to Q, the cones intersect within $D \cap B(Q,r)$, and the sequence B_j is easily constructed by stipulating that it should be included within the cones. Globally (in the part concerning $\mathcal{M}, \mathcal{L}$), property P3 is a consequence of the local result and of the compactness and *connectedness* of the set $D_\delta = \{x \in D : \operatorname{dist}(x, D^c) \geq \delta\}$ for all $\delta \in (0, \delta_0)$ (some $\delta_0 > 0$). We note that, unlike M, L, the constants $\mathcal{M}, \mathcal{L}$ depend heavily on global geometric properties of D, which makes estimates expressed in terms of $\mathcal{M}, \mathcal{L}$ less suitable for our purposes.

Admittedly, the statement of P3 is quite complicated. In part this is due to the fact that it is tailor-made for the Harnack chain argument for harmonic functions. In the case of α-harmonic functions, $\alpha \in (0,2)$, treated in this paper, we give a convenient replacement for this argument. As a result, our considerations do not depend on P3, nor on the connectedness of D, which is used only in the justification of P3. However, we mention the property for the sake of comparison and because it is possibly more versatile when different processes and the corresponding harmonic functions are considered.

3. Estimates for individual α-harmonic functions.

The following is an analogue of the classical chain Harnack principle (see [11]).

LEMMA 1. *There is a constant $\mathcal{K} = \mathcal{K}(D)$ such that for all $x_1, x_2 \in D$, $Q \in \partial D$, $r > 0$ and $k \in \mathbb{N}$ satisfying* (2.9)–(2.11) *of* P3,

$$\mathcal{K}^{-k} u(x_2) \leq u(x_1) \leq \mathcal{K}^k u(x_2) \tag{3.1}$$

for every function $u \geq 0$, α-harmonic in $D \cap B(Q,r)$. Also, there is a constant $K = K(n, \lambda)$ such that for all $x_1, x_2 \in D$, $Q \in \partial D$, $r > 0$ and $k \in \mathbb{N}$ satisfying (2.9), (2.10), *and* (2.12) *of* P3,

$$K^{-k} u(x_2) \leq u(x_1) \leq K^k u(x_2) \tag{3.2}$$

for every function $u \geq 0$, α-harmonic in $D \cap B(Q,r)$.

Proof. By (2.7) for the ball $2B_j$ with the same center and twice the diameter of B_j, and (2.6), we see that there is a constant $c = c(n)$ such that

$$c^{-1} u(x'') \leq u(x') \leq c u(x''), \qquad x', x'' \in B_j,\ j = 1, \ldots, \mathcal{M}k.$$

This is the usual Harnack principle for the ball (see [4] for an explicit formula for c). It follows by induction that $c^{-\mathcal{M}k} u(x_2) \leq u(x_1) \leq c^{\mathcal{M}k} u(x_2)$, hence we can take $\mathcal{K} = c^{\mathcal{M}}$ in (3.1). Similar arguments prove (3.2). ∎

The following lemma proves to be a convenient replacement for the above chain Harnack principle. However, it should be noted that apart from evident analogies there are essential differences between these two results, as well as between their proofs.

LEMMA 2. *Let $x_1, x_2 \in \mathbb{R}^n$, $r > 0$ and $k \in \mathbb{N}$ satisfy $|x_1 - x_2| < 2^k r$. Let $u \geq 0$ be a function which is α-harmonic in $B(x_1, r) \cup B(x_2, r)$. Then*

$$J^{-1} 2^{-k(n+\alpha)} u(x_2) \leq u(x_1) \leq J 2^{k(n+\alpha)} u(x_2) \tag{3.3}$$

for a constant $J = J(n, \alpha)$.

Proof. We may and do assume that $x_1 = 0$ and $|x_1 - x_2| > \frac{3}{2} r$. As in the proof of Lemma 1, we have

$$u(x) \geq c^{-1} u(x_2), \qquad x \in B(x_2, r/2), \tag{3.4}$$

with a constant $c = c(n)$. By (2.6), for $x \in B(x_2, r/2)$ we obtain

$$\begin{aligned} P_{r/2}(0,x) &\geq c_\alpha^n 2^{-\alpha} r^\alpha |x|^{-(n+\alpha)} \geq c_\alpha^n 2^{-\alpha} r^\alpha (2^{k+1} r)^{-(n+\alpha)} \\ &= c_\alpha^n 2^{-(n+2\alpha)} 2^{-k(n+\alpha)} r^{-n}. \end{aligned}$$

Using (2.7) and remembering that $u \geq 0$, we get

$$\begin{aligned} u(x_1) = u(0) &= \int\limits_{|x|>r/2} P_{r/2}(0,x) u(x)\,dx \geq \int\limits_{B(x_2,r/2)} P_{r/2}(0,x) u(x)\,dx \\ &\geq |B(x_2, r/2)| c_\alpha^n 2^{-(n+2\alpha)} 2^{-k(n+\alpha)} r^{-n} c^{-1} u(x_2), \end{aligned}$$

and, by symmetry, (3.3) follows. ∎

We now come to the main results of the section.

LEMMA 3. *There exist constants $\beta = \beta(n, \alpha, \lambda)$ and $M_1 = M_1(n, \alpha, \lambda)$ such that for all $Q \in \partial D$ and $r \in (0, R_0)$, and functions $u \geq 0$, regular α-harmonic in $D \cap B(Q,r)$ and satisfying $u(x) = 0$ on $D^c \cap B(Q,r)$, we have*

$$u(x) \leq M_1 (|x - Q|/r)^\beta c(u), \qquad x \in D \cap B(Q,r), \tag{3.5}$$

where $c(u) = \sup\{u(y) : y \in D \cap B(Q,r)\}$.

LEMMA 4. *There exists a constant $M_2 = M_2(n, \alpha, \lambda)$ such that for all $Q \in \partial D$ and $r \in (0, R_0/2)$, and functions $u \geq 0$, regular α-harmonic in $D \cap B(Q, 2r)$ and satisfying $u(x) = 0$ on $D^c \cap B(Q, 2r)$, we have*

$$u(x) \leq M_2 u(A_r(Q)), \qquad x \in D \cap B(Q,r). \tag{3.6}$$

The above estimates for (regular) α-harmonic functions correspond to those used in the proofs of the classical BHP in [12] and [2]. Lemma 3 is a scaling-invariant statement about the decay of α-harmonic functions with a locally vanishing "boundary condition", and Lemma 4 is an analogue of the Carleson estimate.

For $|x| < r$ and $|y| > r$, we consider the quotient

$$\frac{P_r(x,y)}{P_r(0,y)} = \left[\frac{r^2 - |x|^2}{r^2}\right]^{\alpha/2} \frac{|x-y|^{-n}}{|y|^{-n}}.$$

We take $r=1$. It is clear that for each $\varepsilon>0$, there is a neighborhood of $0\in\mathbb{R}^n$, say $B_N=B(0,2^{-N})$ for some large $N=N(\varepsilon)\in\mathbb{N}$, which satisfies the following important inequality:

$$(1+\varepsilon)^{-1}\le\frac{P_1(x,y)}{P_1(0,y)}\le 1+\varepsilon,\qquad x\in B_N,\ |y|>1. \tag{3.7}$$

Proof of Lemma 3. It is clearly enough to show that, under the assumptions of the lemma, we have

$$u(x)\le M_1'(|x-Q|/r)^{\beta}c(u),\qquad x\in D\cap B(Q,r/2), \tag{3.8}$$

with a constant $M_1'=M_1'(n,\alpha,\lambda)$. We make the following simplifications. For the domain D, the function $u\ge 0$, regular α-harmonic in $D\cap B(Q,r)$, and a positive number d, the function $u'(x)=u(x/d)\ge 0$ is regular α-harmonic in $d(D\cap B(Q,r))$. This is a simple consequence of the homogeneity of the process:

$$X_{at}\stackrel{D}{=}a^{1/\alpha}X_t,\qquad a>0.$$

On the other hand, the inequality (3.8) holds for (D,R_0,λ,Q,r,u) if and only if it holds for $(dD,dR_0,\lambda,dQ,dr,u')$. By such a *scaling* we may and do restrict our considerations to the case $R_0\ge 1$, $r=1$ (note that the Lipschitz constant for D is not changed by the scaling). Without loosing generality, we also assume that $Q=0$, u is *bounded* on $D\cap B(0,1)$, and, finally, $u(x)\le 1$ for $x\in B(0,1)$, so that $c(u)\le 1$. For every integer k, we set

$$r_k=2^{-k},\qquad B_k=B(0,r_k),$$
$$u_k=\sup\{u(x):x\in B_k\},\qquad \Pi_k=B_k\setminus B_{k+1},$$

and we claim that there are constants $c_1=c_1(n,\alpha,\lambda)$ and $c_2=c_2(n,\alpha,\lambda)<1$ such that

$$u_k\le c_1c_2^k,\qquad k>0. \tag{3.9}$$

Our task now is to prove (3.9), since it is clearly equivalent to (3.8).

For $k>0$ and $x_k\in B_k$, we have

$$u(x_k)\le\int_{B_k^c}P_{r_k}(x_k,y)u(y)\,dy. \tag{3.10}$$

Indeed, by regular α-harmonicity,

$$u(x_k)=E^{x_k}u(X_{T(D\cap B_k)^c})=E^{x_k}\{u(X_{T(D\cap B_k)^c});\ X_{T(D\cap B_k)^c}\notin D^c\cap B_k\},$$

since $X_{T(D\cap B_k)^c}\in D^c\cap B_k$ implies $u(X_{T(D\cap B_k)^c})=0$. The semicolon above means as usual that the integration is over the subsequent set. Now, assuming $X_{T(D\cap B_k)^c}\notin D^c\cap B_k$, we see that the process X_t leaves $D\cap B_k$ and B_k at the same time. Therefore, remembering that $u\ge 0$, we conclude

$$u(x_k)=E^{x_k}\{u(X_{T(B_k^c)});\ X_{T(D\cap B_k)^c}\notin D^c\cap B_k\}\le E^{x_k}u(X_{T(B_k^c)}).$$

We split the integral in (3.10) into two parts

$$I_k(x_k)=\int_{B_0^c}P_{r_k}(x_k,y)u(y)\,dy, \tag{3.11}$$

$$J_k(x_k)=\int_{B_0\setminus B_k}P_{r_k}(x_k,y)u(y)\,dy. \tag{3.12}$$

The estimate for I_k is easy. We set $x_1=2^{k-1}x_k\in B_1$. We have

$$\begin{aligned}I_k(x_k)&=c_\alpha^n\int_{|y|>1}\left[\frac{r_k^2-|x_k|^2}{|y|^2-r_k^2}\right]^{\alpha/2}|x_k-y|^{-n}u(y)\,dy,\\&=c_\alpha^n\int_{|y|>1}\left[\frac{r_1^2-|x_1|^2}{|y|^2-r_1^2}\right]^{\alpha/2}|x_1-y|^{-n}u(y)\\&\quad\times\left[\frac{r_k^2-|x_k|^2}{r_1^2-|x_1|^2}\right]^{\alpha/2}\left[\frac{|y|^2-r_1^2}{|y|^2-r_k^2}\right]^{\alpha/2}\left(\frac{|x_1-y|}{|x_k-y|}\right)^n dy.\end{aligned}$$

Also, we have

$$\begin{aligned}&\left[\frac{r_k^2-|x_k|^2}{r_1^2-|x_1|^2}\right]^{\alpha/2}\left[\frac{|y|^2-r_1^2}{|y|^2-r_k^2}\right]^{\alpha/2}\left(\frac{|x_1-y|}{|x_k-y|}\right)^n\\&=2^{-(k-1)\alpha}\left[\frac{|y|^2-r_1^2}{|y|^2-r_k^2}\right]^{\alpha/2}\left(\frac{|x_1-y|}{|x_k-y|}\right)^n\le 2^{-(k-1)\alpha}\left(\frac{\frac32|y|}{\frac12|y|}\right)^n\le 2^\alpha 3^n 2^{-k\alpha},\end{aligned}$$

therefore $I_k(x_k)\le 2^\alpha 3^n I_1(x_1)2^{-k\alpha}$. This yields

$$\sup\{I_k(x):x\in B_k\}\le 2^\alpha 3^n\sup\{I_1(x):x\in B_1\}2^{-k\alpha},\qquad k>0. \tag{3.13}$$

Let $A=A_1(0)$, of P2, so that $B=B(A,\kappa)\subset D\cap B_0$. Fix $x\in B_1$ and $y\in B_0^c$. Let $\widehat{P}$ be the density function of the P^A distribution of $X_{T(B^c)}$. We have

$$\begin{aligned}P_{r_1}(x,y)/\widehat{P}(y)&=\left[\frac{r_1^2-|x|^2}{|y|^2-r_1^2}\right]^{\alpha/2}\left[\frac{|y-A|^2-\kappa^2}{\kappa^2}\right]^{\alpha/2}\frac{|y-A|^n}{|y-x|^n}\\&\le(2\kappa)^{-\alpha}\frac{|y-A|^\alpha}{(|y|^2-r_1^2)^{\alpha/2}}\cdot\frac{|y-A|^n}{|y-x|^n}\\&\le(2\kappa)^{-\alpha}\frac{(2|y|)^\alpha}{(\frac34|y|^2)^{\alpha/2}}\cdot\frac{(2|y|)^n}{(\frac12|y|)^n}=\kappa^{-\alpha}2^{2n+\alpha}3^{-\alpha/2},\end{aligned}$$

which yields, by (3.11), (2.7), and the assumptions that u is regular α-harmonic on $D\cap B(Q,r)\supset B$ and $c(u)\le 1$, the estimate

$$I_1(x)\le\kappa^{-\alpha}2^{2n+\alpha}3^{-\alpha/2}u(A)\le\kappa^{-\alpha}2^{2n+\alpha}3^{-\alpha/2}.$$

By (3.13), we thus see that there is a constant $c_3 = c_3(n,\lambda)$ such that

$$\sup\{I_k(x) : x \in B_k\} \le c_3 2^{-k\alpha}, \quad k > 0. \tag{3.14}$$

For $k, l > 0$ and $x \in B_k$, we denote by $P_{r_k}(x, \Pi_{k-l})$ the probability of the event that the process X_t (starting from x) leaves B_k by hitting Π_{k-l}. This notation is consistent with (2.6). Similarly, $P_{r_k}(x, \Pi_{k-l} \cap C_0)$ denotes the probability of hitting $\Pi_{k-l} \cap C_0$ for the outer cone C_0 in property P1. The existence of the cone, (3.7) and the homogeneity of P_r:

$$P_r(x,y) = r^{-n} P_1(xr^{-1}, yr^{-1}), \quad |x| < r,\ |y| > r,\ r > 0,$$

clearly imply the existence of a constant $p = p(n,\alpha,\lambda) \in (0,1)$ such that

$$P_{r_k}(x, \Pi_{k-l} \cap C_0) \ge p P_{r_k}(x, \Pi_{k-l}), \quad 1 \le l \le k,\ x \in B_{k+N}, \tag{3.15}$$

for N satisfying (3.7). The constant p is independent of $\varepsilon > 0$ in (3.7) provided $\varepsilon < 1$, which we assume to hold in the sequel. Next we observe that

$$\frac{P_1(0, \Pi_{-(k+1)})}{P_1(0, \Pi_{-k})} \le 2^{-\alpha}, \quad k > 0. \tag{3.16}$$

The verification is straightforward. For $y_k \in \Pi_{-k}$ and $y_{k+1} = 2y_k \in \Pi_{-(k+1)}$, we have

$$\frac{P_1(0, y_{k+1})}{P_1(0, y_k)} = \frac{(|y_{k+1}|^2 - 1)^{-\alpha/2} |y_{k+1}|^{-n}}{(|y_k|^2 - 1)^{-\alpha/2} |y_k|^{-n}} \le 2^{-n} \frac{(|y_k|^2 - 1)^{\alpha/2}}{(|y_{k+1}|^2 - 4)^{\alpha/2}} = 2^{-n} 2^{-\alpha}.$$

Using (3.7) and (3.16), and defining

$$p_k = \sup\{P_1(x, \Pi_{-k}) : x \in B_N\}, \quad k > 0,$$

we obtain

$$p_{k+1}/p_k \le (1+\varepsilon)^2 2^{-\alpha}, \quad k > 0. \tag{3.17}$$

Also by (3.7), $p_k \le (1+\varepsilon) P_1(0, \Pi_{-k})$, $k > 0$, hence

$$\sum_{k=1}^{\infty} p_k \le 1 + \varepsilon. \tag{3.18}$$

Let $x \in B_{k+N}$, $k > 0$. By (3.10), (3.14), scaling and (3.15),

$$u(x) \le c_3 2^{-k\alpha} + (1-p) \sum_{l=1}^{k} p_l u_{k-l}. \tag{3.19}$$

Therefore we have

$$u_{k+N} \le c_3 2^{-k\alpha} + (1-p) \sum_{l=1}^{k} p_l u_{k-l}, \quad k > 0.$$

Let $\{d_k\}_{k=1}^{\infty}$ be the sequence satisfying the conditions

$$d_1 = d_2 = \ldots = d_N = 1,$$

$$d_{k+N} = c_3 2^{-k\alpha} + (1-p) \sum_{l=1}^{k} p_l d_{k-l}, \quad k > 0.$$

Clearly $d_k \ge u_k$, $k > 0$.

We denote by m a natural number whose value will be specified later so as to depend only on n, α, λ. Using the definition of d_k and (3.17), we get

$$\begin{aligned} d_{k+N+m} &= c_3 2^{-(k+m)\alpha} + (1-p) \sum_{l=1}^{m} p_l d_{k+m-l} + (1-p) \sum_{l=m+1}^{k+m} p_l d_{k+m-l} \\ &= c_3 2^{-(k+m)\alpha} + (1-p) \sum_{l=1}^{m} p_l d_{k+m-l} + (1-p) \sum_{l=1}^{k} p_{l+m} d_{k-l} \\ &\le c_3 2^{-(k+m)\alpha} + (1-p) \sum_{l=1}^{m} p_l d_{k+m-l} + [2^{-\alpha}(1+\varepsilon)^2]^m \sum_{l=1}^{k} p_l d_{k-l} \\ &\le (1-p) \sum_{l=1}^{m} p_l d_{k+m-l} + [2^{-\alpha}(1+\varepsilon)^2]^m d_{k+N}, \quad k > 0. \end{aligned}$$

Let $\widetilde{d}_k = \max\{d_i : k \le i < k+N+m\}$. We have by (3.18), for $k > 0$,

$$\begin{aligned} d_{k+N+m} &\le \Big\{(1-p) \sum_{l=1}^{m} p_l + [2^{-\alpha}(1+\varepsilon)^2]^m\Big\} \widetilde{d}_k \\ &\le \{(1-p)(1+\varepsilon) + [2^{-\alpha}(1+\varepsilon)^2]^m\} \widetilde{d}_k = c_4 \widetilde{d}_k, \end{aligned} \tag{3.20}$$

with the constant $c_4 = c_4(n,\alpha,\lambda) = (1-p)(1+\varepsilon) + [2^{-\alpha}(1+\varepsilon)^2]^m \in (0,1)$ depending only on the choice of $\varepsilon = \varepsilon(n,\alpha,\lambda)$, subordinate to the conditions $(1-p)(1+\varepsilon) < 1$ and $2^{-\alpha}(1+\varepsilon)^2 < 1$, and a suitable choice of $m = m(n,\alpha,\lambda)$, which must be made at the moment. In particular, we get

$$d_{k+N+m} \le \widetilde{d}_k, \quad k > 0, \tag{3.21}$$

which yields $\widetilde{d}_{k+1} \le \widetilde{d}_k$, $k > 0$. Thus, by induction, (3.20) and (3.21), we get for every $k > 0$,

$$d_{k+N+m+i} \le c_4 \widetilde{d}_k, \quad i > 0.$$

It follows that

$$\widetilde{d}_{k+N+m} \le c_4 \widetilde{d}_k,$$

and so

$$\widetilde{d}_k \le c_4^{k/(N+m)-1} \widetilde{d}_1.$$

The observations that $\widetilde{d}_1 = \widetilde{d}_1(n, \alpha, \lambda)$, and

$$u_k \le d_k \le \widetilde{d}_k \le c_4^{-1} \widetilde{d}_1 (c_4^{1/(N+m)})^k, \qquad k > 0,$$

finish the proof of (3.8). ■

The emphasis in this section is on the analogies between our results and the theory of nonnegative harmonic functions. Nevertheless, it is noteworthy how much the discontinuity of the paths of X_t complicates the simple idea of the proof of Lemma 3, which is taken from its classical ($\alpha = 2$) counterpart (see [12, Lemma 5.1]).

We note that the estimates in the proof of Lemma 3 above do not guarantee that u is bounded on the whole of $D \cap B(Q, r)$, which is the condition for (3.5) to be meaningful. The boundedness is obtained under the slightly stronger assumptions of Lemma 4.

Proof of Lemma 4. We first show that u is bounded on $D \cap B(Q, r)$. As in the proof of Lemma 3, we may and do restrict our considerations to the case $Q = 0$, $r = 1$, without loosing generality. Let $\sigma \in (1, 2)$, so that

$$\overline{B(0,1)} \subset B(0, \sigma) \subset B(0, 2). \tag{3.22}$$

We define

$$u_\sigma(z) = \int_{|y|>\sigma} P_\sigma(z, y) u(y)\, dy, \qquad z \in B(0, \sigma). \tag{3.23}$$

The same argument as that used to prove (3.10) yields

$$u_\sigma(z) \ge u(z), \qquad z \in D \cap B(0, \sigma).$$

We observe that if $u_\sigma(0)$ is finite, then, by (2.6), u_σ is finite and regular α-harmonic on $B(0, \sigma)$ (with the "boundary condition" equal to u on $B(0, \sigma)^c$). Then, using (3.22), we conclude that u_σ is bounded on $D \cap B(0, 1)$, and so is u. Moreover, we have

$$\sup_{z \in B(0,1)} u(z) \le \sup_{z \in B(0,1)} u_\sigma(z) \le c u_\sigma(0), \tag{3.24}$$

with a constant $c = c(n, \alpha, \sigma)$ (see Lemma 2). Thus, we only need to show that the integral in (3.23) converges for $z = 0$, and for at least one $\sigma \in (1, 2)$. We recall that u satisfies $\int_{|y|>1} u(y)(1 + |y|)^{-(n+\alpha)}\, dy < \infty$. Therefore, $u_\sigma(0) < \infty$ is equivalent to

$$\int_{2>|y|>\sigma} P_\sigma(0, y) u(y)\, dy < \infty,$$

and, consequently, to

$$I(\sigma) = \int_{2>|y|>\sigma} [|y|^2 - \sigma^2]^{-\alpha/2} u(y)\, dy < \infty$$

(see (2.6)). We make an auxiliary calculation:

$$\int_0^t [t^2 - \sigma^2]^{-\alpha/2}\, d\sigma \le t^{-\alpha/2} \int_0^t [t - \sigma]^{-\alpha/2}\, d\sigma = \frac{2}{2-\alpha} t^{1-\alpha}, \qquad t > 0. \tag{3.25}$$

Then, by Fubini's theorem and (3.25), we have

$$\begin{aligned} \int_1^2 I(\sigma)\, d\sigma &= \int_1^2 d\sigma \int_{2>|y|>\sigma} [|y|^2 - \sigma^2]^{-\alpha/2} u(y)\, dy \\ &= \int_{2>|y|>1} u(y)\, dy \int_1^{|y|} [|y|^2 - \sigma^2]^{-\alpha/2}\, d\sigma \\ &\le \frac{2}{2-\alpha} \int_{2>|y|>1} u(y)|y|^{1-\alpha}\, dy \le \frac{4}{2-\alpha} \int_{2>|y|>1} u(y)\, dy, \end{aligned} \tag{3.26}$$

which is finite by the local integrability of u. Clearly, $I(\sigma) < \infty$ σ-a.e. on $(1, 2)$, which proves that u is bounded on $D \cap B(0, 1)$, and even on $D \cap B(0, 2 - \varepsilon)$ for $\varepsilon > 0$ arbitrarily small. The latter fact and Lemma 3 imply that $u(x) \to 0$ as $D \ni x \to S \in \partial D \cap B(0, 2)$. This continuity and Lemma 3 are the only ingredients needed to get estimates analogous to (3.6) by verbatim repetition of standard arguments developed for classical harmonic functions (see e.g. [11, Lemma (4.4)]).

Actually, if we follow this way, we need to use the (local) chain Harnack principle (3.2), with the balls B_j of P3 necessarily included in the domain of α-harmonicity $D \cap B(Q, 2r)$ of u. Still, it is so in the case when u is α-harmonic on the whole of D, assuming we want M_2 to depend only on n, α, λ, and not otherwise on D. The stipulation renders

$$u(x) \le M_2' u(A_{Lr}(Q)), \qquad x \in D \cap B(Q, Lr),$$

with a constant $M_2' = M_2'(n, \lambda)$ and $L = L(\lambda)$ of P3. The estimate is slightly less convenient than (3.6), but sufficient for our method of the proof of BHP for α-harmonic functions (we could also use (3.3), with easier calculations).

However, we present a simpler and more explicit proof making further use of the estimates obtained above. As before, $Q = 0$ and $r = 1$. We take $\sigma \in (4/3, 5/3)$ such that $I(\sigma) \le 3 \frac{4}{2-\alpha} \int_{2>|y|>1} u(y)\, dy$ (cf. (3.26)). Since $\sigma > 4/3$, the constant c in (3.24) may be chosen independent of σ, i.e. $c = c(n, \alpha)$. We have, by (3.23) and (2.6),

$$\begin{aligned} u_\sigma(0) &= \int_{|y|>\sigma} P_\sigma(0, y) u(y)\, dy \le \int_{2>|y|>\sigma} + \int_{|y|>2} \\ &\le c_\alpha^n \Big[I(\sigma) \sigma^\alpha + \int_{|y|>2} \frac{\sigma^\alpha}{[|y|^2 - \sigma^2]^{\alpha/2}} |y|^{-n} u(y)\, dy \Big] \end{aligned} \tag{3.27}$$

$$\leq c_\alpha^n\left[\frac{48}{2-\alpha}\int_{2>|y|>1} u(y)\,dy + \int_{|y|>2}\frac{\sigma^\alpha|y|^{-n}u(y)}{[|y|^2-\sigma^2]^{\alpha/2}}\,dy\right].$$

We take a point $A = A_1(0) \in D\cap B(0,1)$ such that $B(A,\kappa)\subset D\cap B(0,1)$ (see P2). Then we have, by (2.7),

$$(3.28)\qquad u(A) = \int_{B(A,\kappa)^c} P_\kappa(0,y-A)u(y)\,dy$$

$$= c_\alpha^n\int_{|y-A|>\kappa}\frac{\kappa^\alpha}{[|y-A|^2-\kappa^2]^{\alpha/2}}|y-A|^{-n}u(y)\,dy.$$

To compare the integrals in (3.27) and (3.28) we note that

$$\frac{\kappa^\alpha}{[|y-A|^2-\kappa^2]^{\alpha/2}}|y-A|^{-n}\geq c_1,\qquad 1<|y|<2,$$

with a constant $c_1 = c_1(n,\lambda)$. Also, we have

$$c_2\frac{\sigma^\alpha}{[|y|^2-\sigma^2]^{\alpha/2}}|y|^{-n}\leq\frac{\kappa^\alpha}{[|y-A|^2-\kappa^2]^{\alpha/2}}|y-A|^{-n},\qquad |y|>2,$$

for a constant $c_2 = c_2(n,\lambda)$. Indeed, $|y|>2$ and $\sigma<5/3$ imply

$$|y-A|\leq|y|+|A|\leq|y|+1\leq\frac{3}{2}|y|$$

and

$$|y|^2-\sigma^2\geq|y|^2-(5/3)^2\geq|y|^2-\frac{25}{36}|y|^2$$

$$=\frac{11}{36}|y|^2\geq\frac{11}{36}\left(\frac{2}{3}\right)^2|y-A|^2\geq\frac{11}{81}[|y-A|^2-\kappa^2].$$

Now, it is clear that

$$u(A)\geq\min\left\{c_1\frac{2-\alpha}{48},c_2\right\}u_\sigma(0),$$

and, consequently, (3.24) yields

$$\sup_{z\in B(0,1)}u(z)\leq c\left(\min\left\{c_1\frac{2-\alpha}{48},c_2\right\}\right)^{-1}u(A).$$

By a remark above, $c = c(n,\alpha)$, which finishes the proof. For later convenience we note that clearly the following improved version of (3.6) also holds:

$$(3.29)\qquad \sup_{x\in B(0,5r/4)}u(x)\leq c_3u(A_r(Q)),$$

with a constant $c_3 = c_3(n,\alpha,\lambda)$. ∎

The following lemma reverses the inequality in Lemma 3. It is crucial that $\gamma<\alpha$ below, since by (2.7), we clearly have (3.30) with $\gamma=\alpha$.

LEMMA 5. *There exist constants $M_3 = M_3(n,\alpha,\lambda)$ and $\gamma=\gamma(n,\alpha,\lambda)<\alpha$ such that for all $Q\in\partial D$ and $r\in(0,R_0)$, and functions $u\geq 0$, α-harmonic in $D\cap B(Q,r)$, we have*

$$(3.30)\qquad u(A_s(Q))\geq M_3(|A_s(Q)-Q|/r)^\gamma u(A_r(Q)),\qquad s\in(0,r).$$

Proof. We may and do assume that $r=1$ and $u(A_r(Q))=1$. Let $T=2/\kappa$, with the constant $\kappa=\kappa(\lambda)$ of P2. For $k=0,1,\ldots$, we define

$$r_k = T^{-k},\qquad A_k = A_{r_k}(Q)\quad\text{(see P2)},\qquad B_k = B(A_k,r_{k+1}).$$

As in the proof of Lemma 1, we have a constant $c_1 = c_1(n)$ such that

$$(3.31)\qquad \frac{u(x)}{u(y)}\leq c_1,\qquad x,y\in B_k,\ k=0,1,\ldots$$

Also, there is a constant $c_2 = c_2(n,\lambda)$ such that

$$|B_l|\geq c_2^{-1}T^{-nl},\qquad \operatorname{diam}(B_k)\geq c_2^{-1}T^{-k},$$
$$\operatorname{dist}(A_k,y)\leq c_2T^{-l},\qquad k>l\geq 0,\ y\in B_l.$$

Now, by (2.7), (3.31) and (2.6) it follows easily that for $k=0,1,\ldots$, we have

$$u(A_k)\geq c_1^{-1}\sum_{l=0}^{k-1}u(A_l)\int_{B_l}P_{r_{k+1}}(0,y-A_k)\,dy\geq c_3\sum_{l=0}^{k-1}T^{-(k-l)\alpha}u(A_l),$$

with a constant $c_3 = c_3(n,\alpha,\lambda)$. We set $b_m = T^{m\alpha}u(A_m)$, $m=0,1,\ldots$ The above inequality yields

$$(3.32)\qquad b_k\geq c_3\sum_{l=0}^{k-1}b_l.$$

We claim that there are constants $c_4 = c_4(n,\alpha,\lambda)$ and $c_5 = c_5(n,\alpha,\lambda)>1$ such that for $k=0,1,\ldots$, we have

$$(3.33)\qquad b_k\geq c_4c_5^k.$$

This is proved by induction. The values of c_4 and c_5 are specified in the course of the proof. Indeed, by (3.32) and induction we have

$$b_k\geq c_3\sum_{l=0}^{k-1}c_4c_5^l = c_3c_4\frac{c_5^k-1}{c_5-1}.$$

For $c_5>1$ small enough (e.g. $c_5 = 1+c_3/2$) this yields (3.33) for $k\geq k_0$, where $k_0 = k_0(n,\alpha,\lambda)\in\mathbb{N}$ is a constant. By (3.3) we can choose $c_4 = c_4(n,\alpha,\lambda)$ small enough to satisfy (3.33) for $k<k_0$. This proves (3.30) for

$s = T^{-k}$, $k = 0, 1, \ldots$ For the remaining values of s, we use the partial result and (3.3). ■

4. Estimates for α-harmonic measure. Let V be a bounded open set in $\mathbb{R}^n$. For a point $x \in \mathbb{R}^n$, the P^x distribution of $X_{T(V^c)}$ is a probability measure on V^c, called *α-harmonic measure* (in x with respect to V) and denoted by ω_V^x (we drop α from the notation). Unlike Brownian motion, (X_t) has discontinuous paths a.s. and, for $x \in V$, ω_V^x is usually supported on all of V^c. The latter fact follows from (2.6) and the estimate $\omega_V^x \geq \omega_B^x$ on V^c, for every ball $B \subset V$. For $x \in V^c$, since $P^x(X_0 = x) = 1$, we have $\omega_V^x = \varepsilon_x$, the Dirac measure in x.

It is important to notice that $\omega_V^x(A)$, $x \in \mathbb{R}^n$, is a nonnegative function regular α-harmonic in V for every Borel set $A \subset \mathbb{R}^n$. Indeed, we have $\omega_V^x(A) = E^x(\mathbf{1}_A(X_{T(V^c)}))$, $x \in \mathbb{R}^n$ (cf. (2.5)). We observe that the "boundary condition" for $\omega_V^x(A)$ is the indicator function $\mathbf{1}_{A \cap V^c}$.

Our basic tool in estimating the α-harmonic measure is the following lemma.

LEMMA 6. *Let $V \subset \mathbb{R}^n$ be a bounded open set with the outer cone property. Let A_1 and A_2 be Borel sets in V^c. Assume that there is a constant C such that for every ball $B = B(x,r) \in V$ satisfying* $\operatorname{dist}(B, V^c) = \operatorname{diam}(B)$, *we have*

$$\omega_B^x(A_1) \leq C\omega_B^x(A_2). \tag{4.1}$$

Then

$$\omega_V^x(A_1) \leq C\omega_V^x(A_2), \quad x \in V. \tag{4.2}$$

The α-harmonic measure ω_V^x is concentrated on $\operatorname{int} V^c$ *and is absolutely continuous with respect to the Lebesgue measure on V^c. The corresponding density function $f^x(y)$, $x \in V$, $y \in V^c$, is continuous in $(x,y) \in V \times \operatorname{int} V^c$.*

Proof. For $x \in V$, let $r_x = \frac{1}{3}\operatorname{dist}(x, V^c)$ and $B_x = B(x, r_x)$, so that $\operatorname{diam} B_x = \operatorname{dist}(B_x, V^c)$. We note that r_x depends continuously on x. Let A be a Borel subset of V^c. For brevity we write

$$F(x) = \omega_V^x(A) = P^x(X_{T(V^c)} \in A), \quad x \in V.$$

By a remark above, $F(x)$ is continuous in V. The following equality holds:

$$F(x) = P^x(X_{T(B_x^c)} \in A) + E^x(F(X_{T(B_x^c)});\ X_{T(B_x^c)} \in V), \quad x \in V. \tag{4.3}$$

Indeed, $T(V^c) = T(B_x^c) + T(V^c) \circ \theta_{T(B_x^c)}$ (we follow the standard notation of [5], with θ being the translation operator $X_t \circ \theta_s = X_{t+s}$). This implies $X_{T(V^c)} \circ \theta_{T(B_x^c)} = X_{T(V^c) \circ \theta_{T(B_x^c)} + T(B_x^c)} = X_{T(V^c)}$. By the strong Markov property, we have, for $x \in V$,

$$\begin{aligned}
P^x(X_{T(V^c)} \in A) &= E^x P\{X_{T(V^c)} \in A \mid X_{T(B_x^c)}\} \\
&= E^x P\{X_{T(V^c)} \circ \theta_{T(B_x^c)} \in A \mid X_{T(B_x^c)}\} \\
&= E^x P^{X_{T(B_x^c)}}(X_{T(V^c)} \in A) \\
&= E^x\{P^{X_{T(B_x^c)}}(X_{T(V^c)} \in A);\ X_{T(B_x^c)} \in V^c\} \\
&\quad + E^x\{P^{X_{T(B_x^c)}}(X_{T(V^c)} \in A);\ X_{T(B_x^c)} \in V\} \\
&= P^x(X_{T(B_x^c)} \in A) + E^x(F(X_{T(B_x^c)});\ X_{T(B_x^c)} \in V).
\end{aligned}$$

We denote the two terms on the right hand side of (4.3) by $p_0(x, A)$ and $r_0(x, A)$, respectively. We observe that $p_0(x, A)$ is the probability of the event that the process (X_t) jumps directly to A when leaving B_x, and $r_0(x, A)$ is the probability of a complementary event that, leaving B_x, it visits V before jumping to A. The definition of B_x and (2.6) yield that $p_0(\cdot, A)$ is continuous on V, thus $r_0(\cdot, A)$ is continuous on V. Using (4.3) we prove inductively that for $k = 0, 1, \ldots,$ we have

$$F(x) = p_0(x, A) + p_1(x, A) + \ldots + p_k(x, A) + r_k(x, A), \quad x \in V, \tag{4.4}$$

with

$$p_{k+1}(x, A) = E^x(p_k(X_{T(B_x^c)}, A);\ X_{T(B_x^c)} \in V), \tag{4.5}$$

and

$$r_{k+1}(x, A) = E^x(r_k(X_{T(B_x^c)}, A);\ X_{T(B_x^c)} \in V). \tag{4.6}$$

Indeed, it is enough to verify that $r_k = p_{k+1} + r_{k+1}$, $k = 0, 1, 2, \ldots$ We may think of p_k as the probability of the event that the process (X_t) goes to A after precisely k jumps from one ball B_x, $x \in V$, to another. We notice that by the definition of B_x, the outer cone property (cf. P1) and (2.6), there is a constant $p = p(\alpha, \operatorname{diam}(V))$ such that

$$P^x(X_{T(B_x^c)} \in V^c) > p, \quad x \in V. \tag{4.7}$$

We use (4.6) and

$$\sup_{x \in V} r_{k+1}(x, A) \leq (1-p) \sup_{x \in V} r_k(x, A) \leq (1-p)^{k+1} \to 0 \quad \text{as } k \to \infty, \tag{4.8}$$

together with (4.4), to conclude that

$$\omega_V^x(A) = \sum_{k=0}^{\infty} p_k(x, A). \tag{4.9}$$

Let A_1 and A_2 satisfy the hypotheses of the lemma. By (4.1) we have $p_0(x, A_1) \leq Cp_0(x, A_2)$ for $x \in V$. Using this, (4.5) and induction, we get

$$p_k(x, A_1) \leq Cp_k(x, A_2), \quad x \in V,\ k = 0, 1, 2, \ldots \tag{4.10}$$

An application of (4.9) yields (4.2).

If $A_1 \subset V^c$ is a Borel set of zero Lebesgue measure, then (4.1) is satisfied with $C = 0$ (and, say, $A_2 = V^c$), thus $\omega_V^x(A_1) = 0$ for $x \in V$. Therefore ω_V^x is absolutely continuous with respect to Lebesgue measure, with a density function (the Poisson kernel for V) denoted by $f^x(y)$, $x \in V$, $y \in V^c$. Similarly, $p_k(x,\cdot)$ has a density function denoted by $\widehat{p}_k(x,y)$, $x \in V$, $y \in V^c$. We may and do choose $\widehat{p}_0$ continuous by letting the equality $\widehat{p}_0(x,y) = P_{r_x}(0, y-x)$, $x \in V$, $y \in V^c$ (see (2.6)), hold pointwise in y. For $k = 0,1,2,\ldots$, and for every Borel set $A \subset V^c$, by (4.5), and Fubini's theorem, we have

$$\begin{aligned} p_{k+1}(x,A) &= \int_V p_k(z,A) P_{r_x}(0,z-x)\,dz \\ &= \int_A \int_V \widehat{p}_k(z,y) P_{r_x}(0,z-x)\,dz\,dy, \qquad x \in V. \end{aligned}$$

Therefore we may and do assume that (pointwise)

$$\widehat{p}_{k+1}(x,y) = \int_V \widehat{p}_k(z,y) P_{r_x}(0,z-x)\,dz, \qquad x \in V,\ y \in V^c. \tag{4.11}$$

Let K be an arbitrary compact subset of $\operatorname{int} V^c$. We claim that $\widehat{p}_k(x,y)$ is uniformly continuous in $V \times K$ and vanishes as $x \to \partial V$. This is proved by induction. Let $V \ni x_n \to Q \in \overline{V}$ and $K \ni y_n \to S \ni K$. If $Q \in \partial V$ then, for every $\varepsilon > 0$, by (4.11) we have

$$\begin{aligned} \widehat{p}_{k+1}(x_n,y_n) &= \int_V \widehat{p}_k(z,y_n) P_{r_{x_n}}(0,z-x_n)\,dz \\ &= \int_{V\cap B(Q,\varepsilon)} + \int_{V\setminus B(Q,\varepsilon)}. \end{aligned}$$

By induction, the first integral on the right hand side can be made arbitrarily small provided $\varepsilon > 0$ is small enough. Since $r_{x_n} \to 0$, for every $\varepsilon > 0$ fixed, the second integral vanishes as $n \to \infty$. Thus $\widehat{p}_{k+1}(x_n,y_n) \to 0$ as $n \to \infty$.

Now assume that $Q \in V$. We have

$$\begin{aligned} &\widehat{p}_{k+1}(x_n,y_n) - \widehat{p}_{k+1}(Q,S) \\ &\quad = \int_V \widehat{p}_k(z,y_n) P_{r_{x_n}}(0,z-x_n) - \widehat{p}_k(z,S) P_{r_Q}(0,z-Q)\,dz \\ &\quad = \int_V P_{r_{x_n}}(0,z-x_n)[\widehat{p}_k(z,y_n) - \widehat{p}_k(z,S)]\,dz \\ &\qquad + \int_V \widehat{p}_k(z,S)[P_{r_{x_n}}(0,z-x_n) - P_{r_Q}(0,z-Q)]\,dz. \end{aligned}$$

Since, by induction, $\widehat{p}_k(z,y_n) \to \widehat{p}_k(z,S)$, uniformly in $z \in V$, the first integral on the right hand side tends to zero as $n \to \infty$. We also note that $r_{x_n} \to r_Q > 0$, and by (2.6), the measures μ_n, given by the density functions $P_{r_{x_n}}(0,z-x_n)$, $z \in \mathbb{R}^n$, converge weakly to the measure with the density function $P_{r_Q}(0,z-Q)$, $z \in \mathbb{R}^n$. Thus the second integral also vanishes. In consequence, $\widehat{p}_{k+1}(x_n,y_n) - \widehat{p}_{k+1}(Q,S) \to 0$ as $n \to \infty$. The induction is complete.

By (4.9), we clearly have for $x \in V$, and almost all $y \in V^c$,

$$f^x(y) = \sum_{k=0}^{\infty} \widehat{p}_k(x,y). \tag{4.12}$$

Using (4.11) and (4.7) yields

$$\begin{aligned} \sup_{x\in V,\,y\in K} \widehat{p}_{k+1}(x,y) &\le (1-p) \sup_{x\in V,\,y\in K} \widehat{p}_k(x,y) \\ &\le (1-p)^{k+1} \sup_{x\in V,\,y\in K} P_{r_x}(0,y-x). \end{aligned}$$

We conclude that the series in (4.12) converges uniformly on $V \times K$, and (by letting (4.12) hold pointwise) $f^x(y)$ is uniformly continuous on $V \times K$, and vanishes as $x \to \partial V$. In particular, $f^x(y)$ is continuous in $(x,y) \in V \times \operatorname{int} V^c$. ∎

Remark 2. In fact, $f^x(y)$ is C^∞ in $(x,y) \in V \times \operatorname{int} V^c$. The proof requires only a slight modification of the method applied above, namely a C^∞ selection of the radii r_x, $x \in V$, and is left to the interested reader. We do not use this result in our considerations.

Remark 3. Let $y_1, y_2 \in \operatorname{int} V^c$. Under the notation from the proof of Lemma 6, if $\widehat{p}_0(x,y_1) \le C\widehat{p}_0(x,y_2)$ for $x \in V$, then $f^x(y_1) \le Cf^x(y_2)$ for $x \in V$. This is an easy consequence of (4.10), (4.12) and the continuity of the density functions.

The next lemma states that the asymptotics of $f^x(y)$ as $y \to \infty$ is the same as that of the Poisson kernel (2.6) for the ball.

LEMMA 7. *Let $V \subset \mathbb{R}^n$ be a bounded open set with the outer cone property. Let $\lambda_1 > 0$. For every $y \in V^c$ which satisfies*

$$\operatorname{dist}(y,V) \ge \lambda_1 \operatorname{diam}(V), \tag{4.13}$$

we have

$$C_1^{-1}\frac{s(x)}{\operatorname{dist}(y,V)^{n+\alpha}} \le f^x(y) \le C_1 \frac{s(x)}{\operatorname{dist}(y,V)^{n+\alpha}}, \qquad x \in V, \tag{4.14}$$

with $C_1 = C_1(n,\lambda_1)$ and a function $s(x)$, $x \in V$, depending only on V and α. Moreover, $C_1 \to 1$ as $\lambda_1 \to \infty$.

Proof. We use the notation from the proof of Lemma 6. Let $y \in V^c$ satisfy (4.13) and x be an arbitrary point in V. We claim that there is a constant $c = c(n, \lambda_1)$ such that

$$c^{-1} c_\alpha^n \frac{r_x^\alpha}{\operatorname{dist}(y, V)^{n+\alpha}} \le \widehat{p}_0(x, y) \le c c_\alpha^n \frac{r_x^\alpha}{\operatorname{dist}(y, V)^{n+\alpha}}. \tag{4.15}$$

We recall that

$$\widehat{p}_0(x, y) = P_{r_x}(0, y - x) = c_\alpha^n \left[\frac{r_x^2}{|y - x|^2 - r_x^2}\right]^{\alpha/2} |y - x|^{-n}. \tag{4.16}$$

We have

$$|y - x| > r_x + \operatorname{dist}(y, V) \ge r_x + \lambda_1 \operatorname{diam}(V) > (1 + \lambda_1) r_x,$$

and so

$$|y - x| \ge \frac{\lambda_1}{1 + \lambda_1} |y - x| + \frac{1}{1 + \lambda_1} (1 + \lambda_1) r_x = \frac{\lambda_1}{1 + \lambda_1} |y - x| + r_x.$$

Now, we get the estimate

$$\begin{aligned}\widehat{p}_0(x, y) &\le c_\alpha^n \left[\frac{r_x^2}{|y - x|^2 \lambda_1^2/(1 + \lambda_1)^2}\right]^{\alpha/2} |y - x|^{-n} \\ &= c_\alpha^n (1 + 1/\lambda_1)^\alpha r_x^\alpha |y - x|^{-n-\alpha} \le (1 + 1/\lambda_1)^\alpha c_\alpha^n \frac{r_x^\alpha}{\operatorname{dist}(y, V)^{n+\alpha}}.\end{aligned}$$

Using (4.13), we also get

$$|y - x| \le \operatorname{dist}(y, V) + \operatorname{diam}(V) \le \operatorname{dist}(y, V)(1 + 1/\lambda_1),$$

which yields

$$\widehat{p}_0(x, y) \ge c_\alpha^n r_x^\alpha |y - x|^{-n-\alpha} \ge (1 + 1/\lambda_1)^{-n-\alpha} c_\alpha^n \frac{r_x^\alpha}{\operatorname{dist}(y, V)^{n+\alpha}}.$$

As a result, for λ_1 fixed, we obtain (4.15). At the same time it is obvious that we can get rid of the dependence of the constant $c = c(n, \alpha, \lambda_1)$, resulting from our calculations, on $\alpha \in (0, 2)$ by considering the worst case $\alpha \to 2$. Also, we can make $c = c(n, \lambda_1)$ thus obtained as close to 1 as we wish, by taking λ_1 large enough.

We next denote by y_1, y_2 two points y satisfying (4.13). We have

$$c^{-2} \left(\frac{\operatorname{dist}(y_2, V)}{\operatorname{dist}(y_1, V)}\right)^{n+\alpha} \le \frac{\widehat{p}_0(x, y_1)}{\widehat{p}_0(x, y_2)} \le c^2 \left(\frac{\operatorname{dist}(y_2, V)}{\operatorname{dist}(y_1, V)}\right)^{n+\alpha}. \tag{4.17}$$

By this inequality and Remark 3,

$$c^{-2} \left(\frac{\operatorname{dist}(y_2, V)}{\operatorname{dist}(y_1, V)}\right)^{n+\alpha} \le \frac{f^x(y_1)}{f^x(y_2)} \le c^2 \left(\frac{\operatorname{dist}(y_2, V)}{\operatorname{dist}(y_1, V)}\right)^{n+\alpha},$$

or

$$\begin{aligned}c^{-2} f^x(y_2) \operatorname{dist}(y_2, V)^{n+\alpha} &\le f^x(y_1) \operatorname{dist}(y_1, V)^{n+\alpha} \\ &\le c^2 f^x(y_2) \operatorname{dist}(y_2, V)^{n+\alpha}.\end{aligned} \tag{4.18}$$

We now consider a sequence of points y_2 above, denoted by y_2^k, $k = 1, 2, \ldots$ We assume that the corresponding constants $\lambda_1(k) = \operatorname{dist}(y_2^k, V)/\operatorname{diam}(V)$ (cf. (4.13)) satisfy $\lambda_1 \le \lambda_1(k) \to \infty$ as $k \to \infty$. The first consequence of (4.18) is that the sequence

$$f^x(y_2^k) \operatorname{dist}(y_2^k, V)^{n+\alpha}, \quad k = 1, 2, \ldots,$$

is bounded away from zero and infinity. Then we have

$$\begin{aligned}c_1^{-2} f^x(y_2^k) \operatorname{dist}(y_2^k, V)^{n+\alpha} &\le f^x(y_2^m) \operatorname{dist}(y_2^m, V)^{n+\alpha} \\ &\le c_1^2 f^x(y_2^k) \operatorname{dist}(y_2^k, V)^{n+\alpha}, \quad k, m \in \mathbb{N},\end{aligned}$$

with a constant $c_1 = c_1(n, k, m) = c(n, \min(\lambda_1(k), \lambda_1(m)))$. The remark made on the constant c above yields $c_1 \to 1$ as $(k, m) \to \infty$. It is now obvious that

$$s(x) = \lim_k f^x(y_2^k) \operatorname{dist}(y_2^k, V)^{n+\alpha}$$

exists. Therefore, by (4.18) with $y_1 = y$ and $y_2 = y_2^k$, the inequality (4.14) is proved with $C_1 = c^2$. ■

Remark 4. Incidentally, Lemma 7 proves that $s(x) > 0$ and $f^x(y) > 0$ for $x \in V$ and $y \in \operatorname{int} V^c$. An explicit example of the above function $s(x)$ for $V = B(0, r) \subset \mathbb{R}^n$ is the factor $c_\alpha^n (r^2 - x^2)^{\alpha/2}$ in (2.6). For general V, by the proof of Lemma 6, $s(x)$ is (uniformly) continuous on V and vanishes as $V \ni x \to \partial V$.

LEMMA 8. *Let $V \subset \mathbb{R}^n$ be a bounded open set with the outer cone property. Let $\lambda_1 > 0$. For any functions $u, v \ge 0$, regular α-harmonic in V, vanishing on the set $E = \{x : 0 < \operatorname{dist}(x, V) \le \lambda_1 \operatorname{diam}(V)\}$, and such that $u(x_0) = v(x_0)$ for some $x_0 \in V$, we have*

$$C_1^{-4} \le \frac{u(x)}{v(x)} \le C_1^4, \quad x \in V,$$

with the constant $C_1 = C_1(n, \lambda)$ of Lemma 7.

Proof. Let x denote an arbitrary point in V. By (4.14),

$$u(x) = \int_{E^c} u(y)\, \omega_V^x(dy) \le s(x) C_1 \int_{E^c} u(y) (\operatorname{dist}(y, V))^{-n-\alpha}\, dy, \tag{4.19}$$

and

$$v(x) = \int_{E^c} v(y)\, \omega_V^x(dy) \ge s(x) C_1^{-1} \int_{E^c} v(y) (\operatorname{dist}(y, V))^{-n-\alpha}\, dy. \tag{4.20}$$

We denote by I, II the rightmost integrals of (4.19), (4.20), respectively. For u, v not equal to 0 a.s. we have $u(x)/v(x) \le C_1^2 I/II$, and by symmetry, also $u(x)/v(x) \ge C_1^{-2} I/II$. By our assumption and the latter inequality, $u(x_0)/v(x_0) = 1 \ge C_1^{-2} I/II$, hence $u(x)/v(x) \le C_1^4$, and also $u(x)/v(x) \ge C_1^{-4}$. This completes the proof. ■

For points y which are "near" D, we do not have an estimate as precise as (4.14). However, the following lemma states that an important consequence of (4.14) holds also for such points, namely the ratio $f^x(y_1)/f^x(y_2)$, for y_1, y_2 satisfying appropriate additional assumptions, does not essentially depend on x.

LEMMA 9. *Let $V \subset \mathbb{R}^n$ be a bounded open set with the outer cone property. Let $\delta > 0$. There is a constant $C_2 = C_2(n, \delta)$ such that for every pair $y_1, y_2 \in V^c$ satisfying*

$$\operatorname{dist}(\{y_1, y_2\}, V) > \delta|y_1 - y_2|, \tag{4.21}$$

we have

$$C_2^{-1} \le \frac{f^x(y_1)}{f^x(y_2)} \le C_2, \quad x \in V. \tag{4.22}$$

Proof. The method of proof is very similar to that of the proof of Lemma 7 and without further comments we adopt the notation used therein. Let $y_1, y_2 \in V^c$ satisfy (4.21) and x be an arbitrary point in V. We prove (4.22) by means of the inequality

$$C_2^{-1} \le \frac{\widehat{p}_0(x, y_1)}{\widehat{p}_0(x, y_2)} \le C_2, \tag{4.23}$$

with $C_2 = C_2(n, \delta)$ (see Remark 3). Let

$$W = \frac{P_{r_x}(0, y_1 - x)}{P_{r_x}(0, y_2 - x)} = \left[\frac{|y_2 - x|^2 - r_x^2}{|y_1 - x|^2 - r_x^2}\right]^{\alpha/2} \frac{|y_1 - x|^{-n}}{|y_2 - x|^{-n}}.$$

We set $d = \operatorname{dist}(\{y_1, y_2\}, B_x)$. Clearly, by (4.21), we have $|y_2 - y_1| < d/\delta$, and $|y_1 - x| > d$, so that

$$|y_2 - x| \le |y_1 - x| + |y_2 - y_1| \le |y_1 - x| + |y_1 - x|/\delta = (1 + 1/\delta)|y_1 - x|.$$

Also $|y_1 - x| \ge d + r_x$, and

$$|y_2 - x| \le d + |y_2 - y_1| + r_x \le d(1 + 1/\delta) + r_x.$$

Then we have

$$W \le \left[\frac{(d(1+1/\delta) + r_x)^2 - r_x^2}{(d + r_x)^2 - r_x^2}\right]^{\alpha/2} \frac{|x - y_2|^n}{|x - y_1|^n}$$
$$\le \left[\frac{d^2(1+1/\delta)^2 + 2r_x d(1 + 1/\delta)}{d^2 + 2r_x d}\right]^{\alpha/2} \left(1 + \frac{1}{\delta}\right)^n \le \left(1 + \frac{1}{\delta}\right)^{n+\alpha}.$$

By analogy $W \ge (1+1/\delta)^{-n-\alpha}$. This proves (4.23) and consequently (4.22). We see that C_2 may be taken independent of $\alpha \in (0, 2)$, and, naturally, $C_2 \to 1$ as $\delta \to \infty$. ∎

We recall that D denotes a Lipschitz domain in $\mathbb{R}^n$. So far in this section we have been able to avoid property P2, and we have investigated a larger class of domains V satisfying only the outer cone property. Even greater generality is possible, and some clues can be found in the proof of Lemma 17 below. Now we return to our primary focus on Lipschitz domains D.

The α-harmonic measure of the sets $D^c \cap B(Q, r)$, $Q \in \partial D$, $r > 0$, cannot be estimated by means of (4.14) nor (4.22). However, it is possible to obtain bounds for the measure in terms of the Green function G of D, which turn out to be sufficient to complete our proof of BHP.

LEMMA 10. *There exists a constant $\varrho = \varrho(n, \alpha, \lambda) \in (0, 1)$ such that for all $Q \in \partial D$ and $r \in (0, R_0)$,*

$$\omega_D^x(B(Q, r)) \ge 1/2, \quad x \in B(Q, \varrho r). \tag{4.24}$$

Proof. For clarity we first observe that $\omega_D^x(B(Q,r)) = \omega_D^x(D^c \cap B(Q, r))$, $x \in \mathbb{R}^n$. The notation on the left hand side is shorter and therefore will be preferred. We define $v(x) = 1 - \omega_D^x(B(Q, r)) = \omega_D^x(B(Q, r)^c)$, $x \in \mathbb{R}^n$. This is a bounded (by 1) nonnegative function, regular α-harmonic in D, which vanishes on $D^c \cap B(Q, r)$. By Lemma 3 there is a constant $\varrho = \varrho(n, \alpha, \lambda) \in (0, 1)$ such that $v(x) < 1/2$ for $x \in D \cap B(Q, \varrho r)$. Therefore $\omega_D^x(B(Q, r)) = 1 - v(x) > 1/2$ for $x \in D \cap B(Q, \varrho r)$. For $x \in D^c \cap B(Q, \varrho r)$, we even have $\omega_D^x(B(Q, r)) = 1$. ∎

We recall the definition of the Green function G of D. For a measure μ on $\mathbb{R}^n$, let U_α^μ be its *Riesz potential*,

$$U_\alpha^\mu(x) = \mathcal{A}(n, \alpha) \int_{\mathbb{R}^n} \frac{d\mu(y)}{|x - y|^{n-\alpha}}.$$

Let ε_x, $x \in \mathbb{R}^n$, be the Dirac measure. We define

$$G(x, y) = U_\alpha^{\varepsilon_x}(y) - U_\alpha^{\omega_D^x}(y), \quad x, y \in \mathbb{R}^n, \tag{4.25}$$

the (nonnegative) *Green function* of D. It is well known that $G(x, y) \ge 0$ and $G(x, y) = G(y, x)$ for $x, y \in \mathbb{R}^n$, and $G(x, y) = 0$ provided $x \in D^c$ or $y \in D^c$ (we agree to set $G(x, x) = 0$ for $x \in D^c$). It is also well known that the first term on the right hand side of (4.25) is α-harmonic in $x \in \mathbb{R}^n \setminus \{y\}$ (see [13]). The second term is clearly regular α-harmonic in $x \in D$. Therefore, for each $y \in D$ and $r > 0$, $G(x, y)$ is regular α-harmonic in $x \in D \setminus \overline{B(y, r)}$. Certainly, it is *not* regular α-harmonic in $D \setminus \{y\}$.

LEMMA 11. *There exists a constant $C_3 = C_3(n, \alpha, \lambda)$ such that for all $Q \in \partial D$, $r \in (0, R_0)$ and $x \in D \setminus B(A_{\varrho r/2}(Q), \varrho\kappa r/2)$,*

$$r^{n-\alpha} G(x, A_{\varrho r/2}(Q)) \le C_3 \omega_D^x(B(Q, r)), \tag{4.26}$$

with the constant $\varrho = \varrho(n, \alpha, \lambda)$ of Lemma 10.

Proof. We set $A = A_{\varrho r/2}(Q)$ (see P2) and $B = B(A, \varrho\kappa r/2)$. By P2 we have $B \subset B(Q, \varrho r/2)$. Lemma 10 states that $\omega_D^x(B(Q,r)) \geq 1/2$ for $x \in B(Q, \varrho r)$. By (4.25) we clearly have

$$G(x,A) \leq \mathcal{A}(n,\alpha)|x-A|^{\alpha-n}, \quad x \in \mathbb{R}^n, \tag{4.27}$$

and so $r^{n-\alpha}G(x,A) \leq \mathcal{A}(n,\alpha)(\varrho\kappa/2)^{\alpha-n}$ for $x \notin B$. Hence it is obvious that there is a constant $c_1 = c_1(n,\alpha,\lambda)$ such that

$$r^{n-\alpha}G(x,A) \leq c_1\omega_D^x(B(Q,r)), \quad x \in B(Q,\varrho r) \setminus B. \tag{4.28}$$

We define $D_0 = D \setminus \overline{B(Q,\varrho r)}$ and

$$u(x) = cc_1\omega_D^x(B(Q,r)) - r^{n-\alpha}G(x,A), \quad x \in \mathbb{R}^n,$$

where $c = c(n,\alpha,\lambda) > 1$ is another constant to be determined later. We observe that u is a function regular α-harmonic in D_0 (bounded from above), and by (4.28), u is nonnegative on $D_0^c \setminus B$. We observe that for every pair $y_1, y_2 \in B \subset B(Q, \varrho r/2)$, we have

$$|y_1 - y_2| < \varrho r < 2\operatorname{dist}(\{y_1, y_2\}, D_0),$$

so (4.21) is satisfied for y_1, y_2, and the set D_0, with $\delta = 1/2$. We note that while D_0 need not be a Lipschitz domain, the outer cone property holds for D_0, and consequently, by Lemma 9, there is a constant $c_2 = c_2(n)$ such that the density function $f^x(\cdot)$ of $\omega_{D_0}^x(\cdot)$ satisfies the inequality

$$c_2^{-1} \leq f^x(y)/f^x(A) \leq c_2, \quad y \in B,\ x \in D_0. \tag{4.29}$$

We fix $x \in D_0$. We have

$$\begin{aligned} u(x) &= \int_{D_0^c} u(y)f^x(y)\,dy \geq \int_B u(y)f^x(y)\,dy \\ &= cc_1 \int_B \omega_D^y(B(Q,r))f^x(y)\,dy - r^{n-\alpha}\int_B G(y,A)f^x(y)\,dy. \end{aligned}$$

By (4.29) and (4.24), the first term on the right hand side is bounded from below by

$$cc_1c_2^{-1}\frac{1}{2}f^x(A)|B| = cc_3f^x(A)r^n,$$

with a constant $c_3 = c_3(n,\alpha,\lambda)$. Using (4.29), (4.27) and polar coordinates, we estimate the second term from above by

$$\begin{aligned} r^{n-\alpha}&\mathcal{A}(n,\alpha)c_2f^x(A)\int_B |y-A|^{\alpha-n}\,dy \\ &= r^{n-\alpha}\mathcal{A}(n,\alpha)c_2f^x(A)\int_0^{\varrho\kappa r/2} \omega_{n-1}t^{n-1}t^{\alpha-n}\,dt \\ &= \mathcal{A}(n,\alpha)c_2\omega_{n-1}\frac{1}{\alpha}(\varrho\kappa/2)^\alpha f^x(A)r^n, \end{aligned}$$

where ω_{n-1} is the $(n-1)$-dimensional Hausdorff measure of the unit sphere in $\mathbb{R}^n$. Clearly, we can now choose $c = c(n,\alpha,\lambda)$ large enough to provide $u(x) \geq 0$ for $x \in D_0$. This completes the proof. ∎

LEMMA 12. *There exists a constant $C_4 = C_4(n,\alpha,\lambda)$ such that for all $Q \in \partial D$ and $r \in (0, R_0/2)$,*

$$\omega_D^x(B(Q,r)) \leq C_4 r^{n-\alpha}G(x, A_{\varrho r/2}(Q)), \quad x \in D \setminus B(Q,2r), \tag{4.30}$$

with the constant $\varrho = \varrho(n,\alpha,\lambda)$ of Lemma 10.

Proof. Let $x \in D \setminus B(Q,2r)$ and $g(y) = G(x,y)$, $y \in \mathbb{R}^n$. Let $\phi \in C_0^\infty(\mathbb{R}^n)$. From

$$\Delta^{\alpha/2}\phi(x) = \mathcal{A}(n,-\alpha)\int_{\mathbb{R}^n} \frac{\phi(x+y)-\phi(x)}{|y|^{n+\alpha}}\,dy,$$

it follows easily that

$$|\Delta^{\alpha/2}\phi(y)| \leq c_1(1+|y|)^{-n-\alpha}, \quad y \in \mathbb{R}^n, \tag{4.31}$$

for a constant $c_1 = c_1(n,\alpha,\phi)$. We claim that under the assumption $\phi(x) = 0$,

$$\int_{\mathbb{R}^n} g(y)\Delta^{\alpha/2}\phi(y)\,dy = \int_{\mathbb{R}^n} \phi(y)\,d\omega_D^x(y). \tag{4.32}$$

Indeed, in the sense of distributions,

$$\Delta^{\alpha/2}\left(\frac{\mathcal{A}(n,\alpha)}{|x-\cdot|^{n-\alpha}}\right) = -\varepsilon_x, \quad x \in \mathbb{R}^n \tag{4.33}$$

(see [13, Lemma 1.11]). This is used twice in the following identities:

$$\begin{aligned} \int_{\mathbb{R}^n} &g(y)\Delta^{\alpha/2}\phi(y)\,dy \\ &= \mathcal{A}(n,\alpha)\int_{\mathbb{R}^n}\Big\{|x-y|^{\alpha-n} - \int_{\mathbb{R}^n}|y-z|^{\alpha-n}\,d\omega_D^x(z)\Big\}\Delta^{\alpha/2}\phi(y)\,dy \\ &= -\phi(x) - \mathcal{A}(n,\alpha)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|z-y|^{\alpha-n}\Delta^{\alpha/2}\phi(y)\,dy\,d\omega_D^x(z) \\ &= \int_{\mathbb{R}^n}\phi(z)\,d\omega_D^x(z). \end{aligned}$$

Assume next that $\phi \geq 0$, $\phi(y) = 1$ for $y \in B(0,1)$, and $\phi(y) = 0$ for $y \notin B(0,2)$. We define $\phi_r(y) = \phi(\frac{y-Q}{r})$, $y \in \mathbb{R}^n$. By a simple change of variable we obtain

$$(4.34)\qquad \Delta^{\alpha/2}\phi_r(y) = \mathcal{A}(n,-\alpha)\int_{\mathbb{R}^n} \frac{\phi\left(\frac{y+z-Q}{r}\right)-\phi\left(\frac{y-Q}{r}\right)}{|z|^{n+\alpha}}\,dz$$
$$= \mathcal{A}(n,-\alpha)\int_{\mathbb{R}^n} \frac{\phi\left(\frac{y-Q}{r}+z'\right)-\phi\left(\frac{y-Q}{r}\right)}{|z'|^{n+\alpha}r^{n+\alpha}}r^n\,dz'$$
$$= r^{-\alpha}\Delta^{\alpha/2}\phi\left(\frac{y-Q}{r}\right),\qquad y\in\mathbb{R}^n.$$

We observe that for $x\in D\setminus B(Q,2r)$, we have $\phi_r(x)=0$, thus (4.32) applies to ϕ_r. By definition of ϕ, (4.32), (4.34) and (4.31), we get

$$\omega_D^x(B(Q,r)) \le \int_{\mathbb{R}^n}\phi_r(y)\,d\omega_D^x(y) = \int_{\mathbb{R}^n} g(y)\Delta^{\alpha/2}\phi_r(y)\,dy$$
$$\le \int_{\mathbb{R}^n} g(y)|\Delta^{\alpha/2}\phi_r(y)|\,dy$$
$$= r^{-\alpha}\int_{\mathbb{R}^n} g(y)\left|\Delta^{\alpha/2}\phi\left(\frac{y-Q}{r}\right)\right|dy$$
$$\le c_1 r^{-\alpha}\int_{\mathbb{R}^n} g(y)\left(1+\left|\frac{y-Q}{r}\right|\right)^{-n-\alpha}dy$$
$$= c_1 r^{-\alpha}\Big[\int_{B(Q,r)\cap D} + \int_{D\setminus B(Q,r)}\Big] = I + II.$$

By (3.29) in the proof of Lemma 4, we have $g(y)\le c_2 g(A_r(Q))$ for $y\in B(Q,r)$, with a constant $c_2=c_2(n,\alpha,\lambda)$. Indeed, g is nonnegative, regular α-harmonic on $D\cap B(Q,8r/5)$, and vanishes on D^c, so the conditions leading to (3.29) are fulfilled (we use (3.29) rather than (3.6) to simplify the notation in (4.30)). Now we set $A=A_{\varrho r/2}(Q)$, $B=B(A,\varrho\kappa r/2)$, and we claim that there is a constant $c_3=c_3(n,\alpha,\lambda)$ such that

$$(4.35)\qquad g(A_r(Q))\le c_3 g(A).$$

To prove (4.35), we could use the chain Harnack principle of Lemma 1, but this requires the additional assumptions

$$x\in D\setminus B(Q,r/L)\quad\text{and}\quad r\in(0,R_0L),$$

with the constant $L=L(\lambda)$ of P3, and so we get involved into somewhat clumsy notation concerning among other things the range of r.

Therefore we prefer to use (3.3), which gives (4.35) immediately. The moderate price we pay, in this particular case, for using such an argument is that the constant c_3 obtained, depending on α, tends to zero as $\alpha\to 2$, and the estimate (4.35) becomes inaccurate near $\alpha=2$. It follows that

$$I\le r^{-\alpha}c_1c_2c_3 g(A)|B(Q,r)| = r^{n-\alpha}g(A)c_1c_2c_3\omega_{n-1}/n.$$

To estimate II, we denote by $\widehat{P}$ the density function of the α-harmonic measure ω_B^A. For $y\in B^c$, we have by (2.6),

$$\widehat{P}(y) = P_{\kappa\varrho r/2}(0,y-A) = c_\alpha^n\left[\frac{r^2(\kappa\varrho/2)^2}{|y-A|^2-r^2(\kappa\varrho/2)^2}\right]^{\alpha/2}|y-A|^{-n}$$
$$= c_\alpha^n(\kappa\varrho/2)^\alpha r^{-n}\left[\frac{1}{(|y-A|/r)^2-(\kappa\varrho/2)^2}\right]^{\alpha/2}(|y-A|/r)^{-n}$$
$$\ge c_\alpha^n(\kappa\varrho/2)^\alpha r^{-n}(|y-A|/r)^{-n-\alpha}.$$

We will further assume that $|y-Q|\ge r$. We have $|A-Q|<\varrho r/2$, hence $|y-A|\le|y-Q|+\varrho r/2\le|y-Q|(1+\varrho/2)$. It follows that

$$\widehat{P}(y)\ge c_\alpha^n(\kappa\varrho/2)^\alpha(1+\varrho/2)^{-n-\alpha}r^{-n}(|y-Q|/r)^{-n-\alpha}$$
$$\ge c_\alpha^n(\kappa\varrho/2)^\alpha(1+\varrho/2)^{-n-\alpha}r^{-n}(1+|y-Q|/r)^{-n-\alpha}.$$

The conclusion reads

$$II\le r^{n-\alpha}c_1(c_\alpha^n)^{-1}(\kappa\varrho/2)^{-\alpha}(1+\varrho/2)^{n+\alpha}\int_{B^c} g(y)\widehat{P}(y)\,dy.$$

We recall that g is regular α-harmonic on B, and the last integral is equal to $g(A)$, which yields

$$II\le r^{n-\alpha}c_1(c_\alpha^n)^{-1}(\kappa\varrho/2)^{-\alpha}(1+\varrho/2)^{n+\alpha}g(A).$$

This estimate concludes the proof of (4.30). ∎

We summarize the above results as follows.

Corollary 1. *There exists a constant $C_5=C_5(n,\alpha,\lambda)$ such that for all $Q\in\partial D$ and $r\in(0,R_0/2)$,*

$$(4.36)\qquad C_6^{-1}\le\frac{\omega_D^x(B(Q,r))}{r^{n-\alpha}G(x,A_{\varrho r/2}(Q))}\le C_5,\qquad x\in D\setminus B(Q,2r).$$

5. Proof of the boundary Harnack principle. For Lipschitz domains D, we have frequently considered the intersections $D\cap B(Q,r)$ with balls centered at the boundary points of D. This method of *localization* has the disadvantage that the sets obtained need not be Lipschitz domains themselves, and the estimates for α-harmonic measure we have proved above do not apply. However, it is well known that Lipschitz domains can be *localized* near the boundary, in a slightly different way described below, enabling us to overcome this obstacle (see [12]).

P4 (Localization property) *There is a constant $R=R(n,\lambda)\ge 1$ such that for all $Q\in\partial D$ and $r\in(0,R_0)$, there is a Lipschitz domain Ω with the*

Lipschitz constant $\lambda' = \lambda R$ *and localization radius* $R_0' = R_0/R$, *having the property*

$$D \cap B(Q, r/R) \subset \Omega \subset D \cap B(Q, r). \tag{5.1}$$

Remark 5. The set Ω above may be obtained in local coordinates CS_Q as the intersection of D and a cylinder about the y_n axis.

LEMMA 13. *There exists a constant* $C_6 = C_6(n, \alpha, \lambda)$ *such that for all* $Q \in \partial D$ *and* $r \in (0, R_0/2)$, *and functions* $u, v \geq 0$, *regular* α*-harmonic in* $D \cap B(Q, 2r)$, *which vanish on* $D^c \cap B(Q, 2r)$, *we have*

$$C_6^{-1} v(x) \leq u(x) \leq C_6 v(x), \quad x \in D \cap B(Q, r), \tag{5.2}$$

provided $u(A_r(Q)) = v(A_r(Q))$.

Proof. We first prove that there is a constant $C_6' = C_6'(n, \alpha, \lambda)$ such that

$$(C_6')^{-1} v(x) \leq u(x) \leq C_6' v(x), \quad x \in D \cap B(Q, r/(2R)), \tag{5.3}$$

under all the assumptions of the lemma except that $u(A_r(Q)) = v(A_r(Q))$ is replaced by $u(A) = v(A) > 0$ with $A = A_{r/(2R)}(Q)$ (see P2).

Let $\Omega = \Omega(D, Q, r)$ be the set of P4 and let λ' be its Lipschitz constant. Let $c = c(n, \lambda)$ be a constant to be specified below. We define

$$F_1 = \{S \in \partial\Omega : 0 < \mathrm{dist}(S, D^c) < cr\},$$
$$F_2 = \{S \in \partial\Omega : \mathrm{dist}(S, D^c) \geq cr\},$$
$$\Delta = \{x \in D \setminus \Omega : \mathrm{dist}(x, F_1 \cup F_2) < cr/2\}.$$

We observe that, for every point x on the axis π of the inner cone $\mathcal{C}_Q'$, we have $\mathrm{dist}(x, (\mathcal{C}_Q')^c) = |x - Q|/\sqrt{1 + \lambda^2}$ (see Remark 1). It follows that for $S_1 \in \pi \cap \{S \in \partial\Omega : 0 < \mathrm{dist}(S, D^c)\}$, we have

$$\mathrm{dist}(S_1, D^c) \geq \min\{\mathrm{dist}(S_1, (\mathcal{C}_Q')^c), \mathrm{dist}(S_1, B(Q, 2r)^c)\} \geq r/(R\sqrt{1 + \lambda^2}).$$

Now we stipulate that the constant c above satisfy $c \leq 1/(4R\sqrt{1 + \lambda^2})$. It follows that $S_1 \in F_2$. In fact, the ball $B_1 = B(S_1, cr)$ satisfies $\mathrm{dist}(B_1, D^c) \geq cr$. Also, there exist a constant $d = d(n, c) = d(n, \lambda) \in \mathbb{N}$ and a family of balls $B_i = B_i(S_i, cr)$, $i = 1, \ldots, d$, (including B_1) such that $S_i \in F_1 \cup F_2$, $i = 1, \ldots, d$, and

$$\Delta \subset \bigcup_{i=1}^{d} B_i. \tag{5.4}$$

Indeed, such a covering may be easily constructed from a covering of $\mathbb{R}^n$ by balls of diameter $cr/4$, with equally spaced centers, by means of moving each ball B in this covering which intersects Δ so as to position its center at a point $S \in F_1 \cup F_2$ sufficiently close to B, and, subsequently, by expanding the resulting ball B' by a factor of 4, to assure that the new ball $4B' = B(S, cr)$ covers B. Since $|Q - S_i| \geq r/R$ and $c \leq 1/(4R)$, for all $x \in D \cap B(Q, r/(2R))$ we clearly have $x \notin 2B_i$, $i = 1, \ldots, d$. By (4.36), there is a constant $c_1 = c_1(n, \alpha, \lambda') = c_1(n, \alpha, \lambda)$ such that

$$c_1^{-1} \leq \frac{\omega_\Omega^x(B_i)}{(cr)^{n-\alpha} G_\Omega(x, A_i)} \leq c_1, \quad x \in D \cap B(Q, r/(2R)), \tag{5.5}$$

where $A_i = A_{\varrho cr/2}(S_i)$, $i = 1, \ldots, d$, $\varrho = \varrho(n, \alpha, \lambda')$ (cf. Lemma 10). The points $\{A_i\}$ depend on Ω and in particular on λ' via P2.

We fix $x \in B(Q, r/(2R))$. The Green function $G_\Omega(x, y)$ is nonnegative and α-harmonic in $y \in \Omega \setminus \{x\}$. Lemma 2 easily yields

$$c_2^{-1} \leq \frac{G_\Omega(x, A_i)}{G_\Omega(x, A_1)} \leq c_2, \quad i = 1, \ldots, d, \tag{5.6}$$

with a constant $c_2 = c_2(n, \alpha, \lambda)$. Alternatively we could use the chain Harnack argument from the proof of Lemma 1, and the connectedness of $F_1 \cup F_2 = \partial\Omega \setminus \partial D$, for Ω described in Remark 5.

By (5.4)–(5.6), we get

$$\omega_\Omega^x(\Delta) \leq \sum_{i=1}^{d} \omega_\Omega^x(B_i) \leq c_3 \omega_\Omega^x(B_1), \tag{5.7}$$

with a constant $c_3 = c_3(n, \alpha, \lambda)$. To end the proof of (5.3), we define functions u_1, u_2, regular α-harmonic in Ω, by stipulating the following "boundary conditions":

$$u_1(y) = \begin{cases} u(y), & y \in \Delta, \\ 0, & y \in \Omega^c \setminus \Delta, \end{cases} \qquad u_2(y) = \begin{cases} 0, & y \in \Delta, \\ u(y), & y \in \Omega^c \setminus \Delta, \end{cases}$$

so that $u_1, u_2 \geq 0$ and $u_1 + u_2 = u$. In the same way we define v_1, v_2. There is a constant $c_4 = c_4(n, \alpha, \lambda)$ such that (by (3.29) and (3.3))

$$u_1(y) = u(y) \leq c_4 u(A), \quad y \in \Delta \subset B(Q, \tfrac{5}{4} r), \tag{5.8}$$

and (by (3.3))

$$v_1(y) = v(y) \geq c_4^{-1} v(A), \quad y \in B_1 \cap \Delta. \tag{5.9}$$

Therefore by (5.7) and (5.8), we obtain

$$u_1(x) \leq c_3 c_4 u(A) \omega_\Omega^x(B_1),$$

and similarly, by (5.9), we get

$$v_1(x) \geq c_4^{-1} v(A) \omega_\Omega^x(B_1). \tag{5.10}$$

We conclude that

$$u_1(x)/v_1(x) \leq c_3 c_4^2. \tag{5.11}$$

To estimate $u_2(x)/v_2(x)$, we use Lemma 8, for the Lipschitz domain Ω and $\lambda_1 = cr/(2\sqrt{1+\lambda^2}\operatorname{diam}(\Omega)) \geq c/(4\sqrt{1+\lambda^2})$, to get

$$c_5^{-1} \leq \frac{u_2(x)}{v_2(x)} \Big/ \frac{u_2(A)}{v_2(A)} \leq c_5, \tag{5.12}$$

with a constant $c_5 = c_5(n, \alpha, \lambda)$. We notice that $u_2(A) \leq u(A)$. By (3.3) and (2.7), it is not difficult to see that $v_2(A) \geq c_6 v(A)$, with a constant $c_6 = c_6(n, \alpha, \lambda)$. Therefore, by (5.11) and (5.12), there is another constant $c_7 = c_7(n, \alpha, \lambda)$ such that

$$u(x)/v(x) \leq c_7, \quad x \in B(Q, r/(2R)),$$

and, by symmetry,

$$u(x)/v(x) \geq c_7^{-1}, \quad x \in B(Q, r/(2R)).$$

The proof of (5.3) is now complete. The estimate (5.2) is an easy consequence of (5.3) and (3.3). We omit the details. ■

Let dy be a nonempty open subset of $D^c \setminus B(Q, 2r)$. We recall that the α-harmonic measure $\omega_D^x(dy)$ is a regular α-harmonic function in $x \in D$. Hence, by Lemma 13, for any two such sets dy, dy_0, and every $x_0 \in D \cap B(Q, r)$, we have

$$C_6^{-2} \leq \frac{\omega_D^x(dy)}{\omega_D^{x_0}(dy)} \Big/ \frac{\omega_D^x(dy_0)}{\omega_D^{x_0}(dy_0)} \leq C_6^2, \quad x \in D \cap B(Q, r).$$

When dy and dy_0 shrink to $y, y_0 \in \operatorname{int} D^c \setminus \overline{B(Q, 2r)}$, respectively, we get the following result on the density function $f^x(y)$ of $\omega_D^x(dy)$.

Corollary 2. *Let $Q \in \partial D$ and $r \in (0, R_0/2)$. For any two pairs $y, y_0 \in \operatorname{int} D^c \setminus \overline{B(Q, 2r)}$ and $x, x_0 \in D \cap B(Q, r)$, we have*

$$C_6^{-2} \leq \frac{f^x(y)}{f^{x_0}(y)} \Big/ \frac{f^x(y_0)}{f^{x_0}(y_0)} \leq C_6^2, \tag{5.13}$$

with the constant $C_6 = C_6(n, \alpha, \lambda)$ of Lemma 13.

We note that analogous estimates are deductible, under more stringent assumptions, from (4.14) and (4.22). We also observe that for x_0, y_0 fixed, (5.13) yields an approximate factorization of the density function

$$f^x(y) \approx f^x(y_0) f^{x_0}(y)/f^{x_0}(y_0),$$

where the first term on the right hand side depends only on x, the second on y, and the third is a normalizing constant (cf. (2.6)). As a matter of fact, this is equivalent (see the proof of Lemma 8) to the (local) BHP stated in Lemma 13.

The proof of the full statement of Theorem 1 requires some additional preparation. The exponential estimate (5.14) in the following lemma may be regarded as an approximate principle of localization for α-harmonic functions.

Lemma 14. *There exist constants $\xi = \xi(n, \alpha, \lambda) < 1$ and $C_7 = C_7(n, \alpha, \lambda)$ such that for all $Q \in \partial D$ and $r \in (0, R_0/2)$, and functions $u \geq 0$, regular α-harmonic in $D \cap B(Q, 2r)$, which vanish on $D^c \cap B(Q, 2r)$, we have*

$$E^x\{u(X_{T_{(D\cap B_k)^c}});\ X_{T_{(D\cap B_k)^c}} \in B_0^c\} \leq C_7 \xi^k u(x), \quad x \in D \cap B_k, \tag{5.14}$$

for $B_k = B(Q, 2^{-k} r)$, $k = 0, 1, \ldots$

Proof. We may and do assume that $r = 1$ and $\sup_{x \in B_0} u(x) = 1$ (see Lemma 4 and the beginning of the proof of Lemma 3). We define

$$u_k(x) = E^x\{u(X_{T_{(D\cap B_k)^c}});\ X_{T_{(D\cap B_k)^c}} \in B_0^c\}, \quad x \in \mathbb{R}^n,$$

(so that $u_0 = u$) and $A_k = A_{2^{-k}}(Q)$, $k = 0, 1, \ldots$ Clearly, each u_k is nonnegative and regular α-harmonic in $D \cap B_k$. By an argument similar to that justifying (3.10) we get $u_{k+1}(x) \leq u_k(x)$, $x \in \mathbb{R}^n$, and (using also (3.14) from the proof of Lemma 3) there is a constant $c = c(n, \alpha, \lambda)$ such that

$$u_k(A_k) \leq E^{A_k}\{u(X_{T_{B_k^c}});\ X_{T_{B_k^c}} \in B_0^c\} \leq c 2^{-k\alpha}, \quad k = 1, 2, \ldots \tag{5.15}$$

By Lemmas 4 and 5, $u(A_k) \geq M_2^{-1} M_3 (\kappa 2^{-k})^\gamma$, with $\gamma < \alpha$, M_3 of Lemma 5, and M_2 of Lemma 4. Finally, by Lemma 13 and (5.15), for $k = 1, 2, \ldots$ and $x \in D \cap B_k$, we have

$$\frac{u_k(x)}{u(x)} \leq \frac{u_{k-1}(x)}{u(x)} \leq C_6 \frac{u_{k-1}(A_{k-1})}{u(A_{k-1})} \leq C_6 M_2 M_3^{-1} c \kappa^{-\gamma} 2^{-(k-1)(\alpha-\gamma)},$$

which completes the proof. ■

Lemma 15. *Assume that nonnegative numbers $u_0, u_1, \ldots, u_k$; $v_0, v_1, \ldots, v_k$; $a, b_1, \ldots, b_k$; and $\varepsilon_1, \ldots, \varepsilon_k$ satisfy the following conditions:*

$$a \leq b_i, \quad i = 1, \ldots, k, \tag{5.16}$$

$$u_i \leq b_i v_i, \quad v_i \leq \varepsilon_i v_0, \quad i = 1, \ldots, k, \tag{5.17}$$

$$u_0 \leq a v_0. \tag{5.18}$$

Then

$$\sum_{i=0}^k u_i \leq \Big[a + \sum_{i=1}^k (b_i - a)\varepsilon_i\Big] \sum_{i=0}^k v_i. \tag{5.19}$$

Proof. The verification is straightforward:

$$\sum_{i=0}^{k} u_i \le u_0 + \sum_{i=1}^{k} b_i v_i \le a v_0 + \sum_{i=1}^{k} b_i v_i$$
$$= a\sum_{i=0}^{k} v_i + \sum_{i=1}^{k}(b_i - a) v_i \le a\sum_{i=0}^{k} v_i + \sum_{i=1}^{k}(b_i - a)\varepsilon_i v_0$$
$$\le \Big[a + \sum_{i=1}^{k}(b_i - a)\varepsilon_i\Big]\sum_{i=0}^{k} v_i. \ \blacksquare$$

The following lemma may be seen as a culmination of our study, yielding the most precise estimates for the ratio of nonnegative regular α-harmonic functions in Lipschitz domains, with possible application in determining the Martin boundary of Lipschitz domains for α-harmonic functions.

LEMMA 16. *There exist constants $C_8 = C_8(n,\alpha,\lambda)$ and $\nu = \nu(n,\alpha,\lambda)$ such that for all $Q \in \partial D$ and $r \in (0, R_0/2)$, and functions $u, v \ge 0$, regular α-harmonic in $D \cap B(Q,2r)$, which vanish on $D^c \cap B(Q,2r)$ and satisfy $u(A_r(Q)) = v(A_r(Q)) > 0$, the limit $g = \lim_{D\ni x\to Q} u(x)/v(x)$ exists, and we have*

$$\left|\frac{u(x)}{v(x)} - g\right| \le C_9|x - Q|^{\nu}, \quad x \in D \cap B(Q,r). \tag{5.20}$$

The proof uses elements of the corresponding proofs from [2] and [12]. However, the maximum principle exploited there does not hold for α-harmonic functions. Instead we use the approximate localization principle stated in Lemma 14, which naturally complicates the estimates.

Proof of Lemma 16. We may and do assume that $r = 1$ and $u(A_r(Q)) = v(A_r(Q)) = 1$. In what follows, k_0 denotes a positive integer, to be specified in a suitable moment so that $k_0 = k_0(n,\alpha,\lambda)$. Let $c = 2^{k_0}$. For $k = 0,1,\ldots,$ we define

$$r_k = c^{-k}, \quad B_k = B(Q, r_k), \quad D_k = D \cap B_k,$$
$$\Pi_k = D_k \setminus D_{k+1}, \quad \Pi_{-1} = B_0^c,$$

and for $l = -1, 0, 1, \ldots, k-1$,

$$u_k^l(x) = E^x\{u(X_{T(D_k^c)});\ X_{T(D_k^c)} \in \Pi_l\}, \quad x \in \mathbb{R}^n, \tag{5.21}$$
$$v_k^l(x) = E^x\{v(X_{T(D_k^c)});\ X_{T(D_k^c)} \in \Pi_l\}, \quad x \in \mathbb{R}^n. \tag{5.22}$$

Let ε denote a number in $(0,1)$, to be specified below, so that $\varepsilon = \varepsilon(n,\alpha,\lambda)$. By Lemma 14, for $k = 0,1,\ldots$ and $x \in D_k$, we have

$$u_k^l(x) \le C_7(\xi^{k_0})^{k-1-l} u(x), \quad l = -1, 0, 1, \ldots, k-2, \tag{5.23}$$

and

$$\sum_{l=-1}^{k-2} u_k^l(x) \le C_7 \xi^{k_0} u(x). \tag{5.24}$$

Therefore we can *define* $k_0 = k_0(\varepsilon)$ by stipulating, for $k = 1,2,\ldots$ and $l = -1, 0, 1, \ldots, k-2$,

$$u_k^l(x) \le \varepsilon^{k-1-l} u_k^{k-1}(x), \quad x \in D_k, \tag{5.25}$$

and, by symmetry,

$$v_k^l(x) \le \varepsilon^{k-1-l} v_k^{k-1}(x), \quad x \in D_k. \tag{5.26}$$

We claim that there exist constants $c_1 = c_1(n,\alpha,\lambda)$ and $\zeta = \zeta(n,\alpha,\lambda) \in (0,1)$ such that for $l = 0, 1, \ldots,$

$$\sup_{x\in D_l} \frac{u(x)}{v(x)} \le (1 + c_1\zeta^l) \inf_{x\in D_l} \frac{u(x)}{v(x)}. \tag{5.27}$$

Clearly, in view of (5.2), (5.27) is equivalent to (5.20). We prove (5.27) by induction. Let $k = 0,1,\ldots$ Assume that (5.27) holds for $l = 0,1,\ldots,k$. By definitions (5.21) and (5.22) we have

$$u(x) = \sum_{l=-1}^{k} u_{k+1}^l(x), \quad x \in \mathbb{R}^n, \tag{5.28}$$
$$v(x) = \sum_{l=-1}^{k} v_{k+1}^l(x), \quad x \in \mathbb{R}^n, \tag{5.29}$$

each summand in the above sums being a nonnegative regular α-harmonic function in D_{k+1}.

For a function f on a set A we define

$$\mathrm{Osc}_A f = \sup_{x\in A} f(x) - \inf_{x\in A} f(x).$$

Let $g(x) = u_{k+1}^k(x)/v_{k+1}^k(x)$, $x \in D_k$. We claim that

$$\mathrm{Osc}_{D_{k+2}} g \le \delta\, \mathrm{Osc}_{D_k} g, \tag{5.30}$$

with $\delta = \delta(n,\alpha,\lambda) \in (0,1)$. Indeed, let $a_1 = \inf_{x\in D_k} g(x)$ and $a_2 = \sup_{x\in D_k} g(x)$. We note that $0 < a_1 \le a_2 < \infty$ by (5.21), (5.22) and (5.2). If $a_1 = a_2$, then (5.30) is proved with $\delta = 0$; otherwise, let

$$g'(x) = \frac{g(x) - a_1}{a_2 - a_1} = \frac{u_{k+1}^k(x) - a_1 v_{k+1}^k(x)}{v_{k+1}^k(x)(a_2 - a_1)}, \quad x \in D_k,$$

and we see that g' is a quotient of nonnegative (nonzero) functions regular α-harmonic in D_{k+1}. Clearly $\mathrm{Osc}_{D_k} g' = 1$. The functions g and g' satisfy

$$\mathrm{Osc}_{D_{k+2}} g = \mathrm{Osc}_{D_{k+2}} g'\, \mathrm{Osc}_{D_k} g. \tag{5.31}$$

Now, if $\sup_{x\in D_{k+2}} g'(x) \le 1/2$, we are satisfied with the conclusion

$$\mathrm{Osc}_{D_{k+2}} g' \le 1/2. \tag{5.32}$$

Otherwise, by Lemma 13, we have $\inf_{x\in D_{k+2}} g'(x) \ge \frac{1}{2}C_6^{-2}$, with $C_6 = C_6(n,\alpha,\lambda)$ of Lemma 13. Then, obviously,

$$\mathrm{Osc}_{D_{k+2}} g' \le 1 - \tfrac{1}{2}C_6^{-2}. \tag{5.33}$$

By (5.31)–(5.33), we get (5.30) with $\delta = \max\left(\frac{1}{2}, 1-\frac{1}{2}C_6^{-2}\right) = 1-\frac{1}{2}C_6^{-2} \in (0,1)$. Taking into account the inequalities

$$\inf_{x\in D_{k+2}} \frac{u_{k+1}^k(x)}{v_{k+1}^k(x)} \ge \inf_{x\in D_k} \frac{u_{k+1}^k(x)}{v_{k+1}^k(x)} \ge \inf_{x\in D_k} \frac{u(x)}{v(x)}$$

and

$$\sup_{x\in D_k} \frac{u_{k+1}^k(x)}{v_{k+1}^k(x)} \le \sup_{x\in D_k} \frac{u(x)}{v(x)}$$

(see (5.21) and (5.22)), we easily check that (5.30) yields

$$\sup_{x\in D_{k+2}} \frac{u_{k+1}^k(x)}{v_{k+1}^k(x)} \Big/ \inf_{x\in D_{k+2}} \frac{u_{k+1}^k(x)}{v_{k+1}^k(x)} - 1 \le \delta\left(\sup_{x\in D_k} \frac{u(x)}{v(x)} \Big/ \inf_{x\in D_k} \frac{u(x)}{v(x)} - 1\right). \tag{5.34}$$

Using (5.34), and (5.27) for $l = k$, we obtain

$$\sup_{x\in D_{k+2}} \frac{u_{k+1}^k(x)}{v_{k+1}^k(x)} = (1 + c_1\delta d\zeta^k) \inf_{x\in D_{k+2}} \frac{u_{k+1}^k(x)}{v_{k+1}^k(x)}, \tag{5.35}$$

with a suitably chosen number $d \in [0,1]$ (independent of $x \in D_{k+1}$). We note that this is equivalent to

$$\sup_{x\in D_{k+2}} \frac{v_{k+1}^k(x)}{u_{k+1}^k(x)} = (1 + c_1\delta d\zeta^k) \inf_{x\in D_{k+2}} \frac{v_{k+1}^k(x)}{u_{k+1}^k(x)}. \tag{5.36}$$

We next observe that

$$\inf_{x\in D_{k+2}} \frac{u_{k+1}^k(x)}{v_{k+1}^k(x)} \ge \inf_{x\in D_l} \frac{u(x)}{v(x)}, \quad l = 0,1,\ldots,k.$$

Thus (5.27) yields

$$\sup_{x\in D_l} \frac{u(x)}{v(x)} \le (1 + c_1\zeta^l) \inf_{x\in D_{k+2}} \frac{u_{k+1}^k(x)}{v_{k+1}^k(x)}, \quad l = 0,1,\ldots,k. \tag{5.37}$$

We now fix $x \in D_{k+2}$. Using (5.25), we have

$$\frac{u(x)}{v(x)} = \frac{\sum_{l=0}^k u_{k+1}^l(x) + u_{k+1}^{-1}(x)}{\sum_{l=0}^k v_{k+1}^l(x) + v_{k+1}^{-1}(x)} \le (1+\varepsilon^{k+1}) \frac{\sum_{l=0}^k u_{k+1}^l(x)}{\sum_{l=0}^k v_{k+1}^l(x)}.$$

We are in a position to apply Lemma 15, with the corresponding notation

$$u_0 = u_{k+1}^k(x), \quad v_0 = v_{k+1}^k(x),$$
$$a = (1 + c_1\delta d\zeta^k) \inf_{y\in D_{k+2}} \frac{u_{k+1}^k(y)}{v_{k+1}^k(y)},$$
$$u_i = u_{k+1}^{k-i}(x), \quad v_i = v_{k+1}^{k-i}(x),$$
$$b_i = (1 + c_1\zeta^{k-i}) \inf_{y\in D_{k+2}} \frac{u_{k+1}^k(y)}{v_{k+1}^k(y)},$$
$$\varepsilon_i = \varepsilon^i, \quad i = 1,\ldots,k.$$

Indeed, (5.18) of Lemma 15 follows from (5.35), (5.17) follows from (5.37), (5.21), (5.22), and from (5.26). Hence, by (5.28), (5.29) and Lemma 15,

$$\frac{u(x)}{v(x)} \le \tau \inf_{y\in D_{k+2}} \frac{u_{k+1}^k(y)}{v_{k+1}^k(y)}, \tag{5.38}$$

with $\tau = (1+\varepsilon^{k+1})(1 + c_1\delta d\zeta^k + c_1\sum_{i=1}^k \zeta^{k-i}\varepsilon^i)$. Using (5.36) instead of (5.35), we also get

$$\frac{v(x)}{u(x)} \le \tau \inf_{y\in D_{k+2}} \frac{v_{k+1}^k(y)}{u_{k+1}^k(y)}. \tag{5.39}$$

Using (5.38), (5.35) and (5.39) results in

$$\sup_{x\in D_{k+2}} \frac{u(x)}{v(x)} \le \tau \inf_{y\in D_{k+2}} \frac{u_{k+1}^k(y)}{v_{k+1}^k(y)} = \frac{\tau}{1 + c_1\delta d\zeta^k} \sup_{y\in D_{k+2}} \frac{u_{k+1}^k(y)}{v_{k+1}^k(y)}$$
$$\le \frac{\tau^2}{1 + c_1\delta d\zeta^k} \inf_{y\in D_{k+2}} \frac{u(y)}{v(y)}.$$

We note that for $l \le k_1 \in \mathbb{N}$, by a suitable choice of $c_1 = c_1(n,\alpha,\lambda,k_1)$, and by (5.2), we clearly have (5.27) provided (say) $\zeta > 1/2$. The proof will be concluded if we can choose constants $\varepsilon = \varepsilon(n,\alpha,\lambda)$ and $\zeta = \zeta(n,\alpha,\lambda) \in (1/2,1)$ so that for $k > k_1 = k_1(n,\alpha,\lambda) \in \mathbb{N}$, $\tau^2/(1 + c_1\delta d\zeta^k) \le 1 + c_1\zeta^{k+2}$, with the very c_1 chosen for $k \le k_1$. This is possible, elementary, and lengthy, and is left to the reader ($\zeta = \sqrt{(1+\delta)/2}$ and $\varepsilon = (1-\delta)/20$ will do). ■

The following lemma enables us to translate the results on regular α-harmonic functions into those on α-harmonic functions continuously vanishing at the boundary of a domain (cf. [14, Proposition 24.10]).

LEMMA 17. *Let $V \subset \mathbb{R}^n$ be a bounded open set with the outer cone property. Let u be a function bounded from below (above), α-harmonic in D and bounded on V. Then u is regular α-harmonic in V.*

Proof. Let V_n, $n = 1, 2, \ldots$, be an increasing sequence of (bounded) open sets such that $\overline{V}_n \subset V$, and $\bigcup_{n=1}^{\infty} V_n = V$. We claim that

$$\lim_{n\to\infty} P^x\{X_{T(V_n^c)} = X_{T(V^c)}\} = 1, \qquad x \in V. \tag{5.40}$$

This is proved as follows. We note that $\{T(V_n^c)\}$ is an increasing sequence of stopping times. We define $T = \sup\{T(V_n^c) : n = 1, 2, \ldots\}$. This is a stopping time finite a.s. and clearly $T \le T(V^c)$. Since the process (X_t) is *quasi-left-continuous* (see [5]), we have

$$\lim_{n\to\infty} X_{T(V_n^c)} = X_T \quad \text{a.s.} \tag{5.41}$$

By closedness of V_n^c, $n = 1, 2, \ldots$, and by the right-continuity of the paths of (X_t), $V^c = \bigcap_{n=1}^{\infty} V_n^c$ implies that $X_T \in V^c$ a.s., hence $T \ge T(V^c)$ a.s. Thus we have

$$\lim_{n\to\infty} X_{T(V_n^c)} = X_{T(V^c)} \quad \text{a.s.} \tag{5.42}$$

By Lemma 6, we have $X_{T(V^c)} \in \operatorname{int} V^c$ P^x-a.s. In view of (5.42), this is only possible when (5.40) is satisfied.

Without loosing generality, we further assume that u is nonnegative. Fix $x \in V$. Using (2.4) (for n large enough), we have

$$\begin{aligned} u(x) &= E^x u(X_{T(V_n^c)}) \\ &= E^x\{u(X_{T(V^c)});\ X_{T(V_n^c)} \in V^c\} + E^x\{u(X_{T(V_n^c)});\ X_{T(V_n^c)} \in V \setminus V_n\}. \end{aligned}$$

The second term on the right hand side tends to 0 as $n \to \infty$, by (5.40) and the boundedness of u on V. Monotone convergence proves that the first term tends to $E^x u(X_{T(V^c)})$. Therefore $u(x) = E^x u(X_{T(V^c)})$, and the regularity is proved. ■

The reader may have noticed that the crucial assertion (5.40) in the proof of Lemma 17 is implicit in the proof of Lemma 6 (see (4.9)). The proof of (5.40) given above has the advantage that it applies to more general sets V such that ω_V^x is concentrated on $\operatorname{int} V^c$.

Proof of Theorem 1. First we note that, by an application of Lemma 17, functions u and v satisfying the assumptions of the theorem are regular α-harmonic in $D \cap V'$, where V' is an (arbitrary) open (bounded) set such that $K \subset V' \subset \overline{V'} \subset V$. It is also noteworthy that, conversely, if u and v are regular α-harmonic in $D \cap V'$, then their continuous decay at points of $\partial D \cap V'$ is a consequence of the other hypotheses of the theorem and of Lemmas 3 and 4.

Taking this into account, the first part of the theorem follows easily from the compactness of $\overline{D} \cap K$, and Lemmas 2 and 13. In particular, by taking $K = \overline{V'}$, we see that the quotient $q(x) = u(x)/v(x)$ is bounded away from zero and infinity on V'.

We note that the full strength of Lemma 13, that is, the independence of C_6 from r in (5.2), is not used in this argument.

We now sketch the proof of the Hölder continuity of q on $D \cap K$. By Lemma 16, we may and do extend q to $\partial D \cap V$. Let $x, y \in D \cap K$, $d = |x - y|$ and $r = \operatorname{dist}(\{x, y\}, V'^c)$, with V' as above. We claim that

$$|q(x) - q(y)| \le c_1 d/r, \tag{5.43}$$

with $c_1 = c_1(D, V, K, \alpha)$. Indeed, if $d < r/2$, then $x, y \in B(x, \frac{1}{2}r)$. By (2.7) for the ball $B(x, r)$, and by the boundedness of q on V', we get (5.43) (see also the verification of (3.7)). For $d \ge r/2$, (5.43) follows from the boundedness alone.

Next, for a suitable $Q \in \partial D \cap K$, by Lemma 16 we have

$$|q(x) - q(y)| \le |q(x) - q(Q)| + |q(y) - q(Q)| \le c_2(d + r)^{\nu}, \tag{5.44}$$

with $c_2 = c_2(D, V, K, \alpha)$ and ν of Lemma 16, provided $d + r \le \frac{1}{2}\operatorname{dist}(K, V'^c))$. Actually, by the boundedness of q on V', the inequality (5.44) also holds for the remaining values of d, r. The combination of (5.43) and (5.44) easily yields

$$|q(x) - q(y)| \le c_3 d^{\eta}, \qquad d \le 1, \tag{5.45}$$

with constants $c_3 = c_3(D, V, K, \alpha)$ and $\eta = \nu/(1 + \nu)$. The proof of the theorem is complete. ■

Remark 6. The formula (2.6) also holds for $n = 1$ (see [15]). Simple structure of open sets in $\mathbb{R}^1$ and (2.7) immediately extend the validity of BHP to the case $n = 1$.

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Institute of Mathematics
Technical University of Wrocław
50-370 Wrocław, Poland
E-mail: bogdan@im.pwr.wroc.pl

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Isoperimetric problem for uniform enlargement

by

S. G. BOBKOV (Syktyvkar and Bielefeld)

Abstract. We consider an isoperimetric problem for product measures with respect to the uniform enlargement of sets. As an example, we find (asymptotically) extremal sets for the infinite product of the exponential measure.

1. Introduction. Let (X,μ) be a separable topological space equipped with a Borel probability measure. Assume that to each point $x \in X$ there corresponds an open neighborhood $D(x)$ with the following symmetry property: for any $x, y \in X$,

$$\text{if} \quad x \in D(y), \quad \text{then} \quad y \in D(x). \tag{1.1}$$

For every non-empty set $A \subset X$, we define its *enlargement* by

$$\operatorname{enl}(A) = \bigcup_{a \in A} D(a), \tag{1.2}$$

and consider the problem of finding the function

$$R_\mu(p) = \inf_{\mu(A) \ge p} \mu(\operatorname{enl}(A)), \tag{1.3}$$

where the sup is over all Borel sets $A \subset X$ of measure $\mu(A) \ge p$.

Usually the enlargement is built with the help of a metric (or pseudometric) in X, say d, by taking for $D(x)$ the open ball $D(x,h)$ with center x and radius $h > 0$. Then $\operatorname{enl}(A) = A^h$ is the open h-neighbourhood of A, and $R_\mu(p) = R_\mu(p,h)$ depends also on h. Next, in applications of (1.3) to distribution of Lipschitz functions f on X, one fixes p and varies h. For $A = \{x : f(x) \le m\}$, where m is the median of f, we have $A^h \subset \{x : f(x) <$

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