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Quasi-multipliers of the algebra of approximable operators and its duals

by

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Abstract. Let A be the Banach algebra $K_0(X)$ of approximable operators on an arbitrary Banach space X. For the spaces of all bilinear continuous quasi-multipliers of A resp. its dual A^* resp. its bidual A^{**}, concrete representations as spaces of operators are given.

1. Introduction. Let X be an arbitrary Banach space (we do not assume any kind of approximation property for X) and denote by A the algebra $K_0(X)$ of approximable operators on X (i.e. of all operators which are uniform limits of continuous linear operators from X to X having finite rank), equipped with the usual operator norm. A can be considered as A-A-bimodule in the natural way; therefore, the first resp. second Banach duals A^* resp. A^{**} of A become A-A-bimodules by the first resp. second adjoints of the actions of A on A.

In this article, we shall give representations of the quasi-multiplier spaces of A, A^* and A^{**}, respectively. The result for A itself is known already for at least 17 years ([G1, 3.24 and 3.26]): $QM(A)$ is isometrically isomorphic to $L(X^*)$ where $g \in L(X^*)$ corresponds to the quasi-multiplier ϕ_g determined by $\iota_X \circ \phi_g(a,b) := a^{**} \circ g^* \circ b^{**} \circ \iota_X$. (The notation is explained in detail in the following section.) For the special case where X^* satisfies the bounded approximation property, this result was restated and proved recently in [AR, Corollary 4.3].

A^*, being isometrically isomorphic to $I(X^*)$, can be considered either as an A-A-bimodule in the natural way or, with multiplication defined by composition of operators, as a Banach algebra in its own right. Adopting the first point of view (as is done in Section 3), $QM(A^*)$ is given as $QM_A(A^*) := B_A^A(A, A; A^*)$ while in the second case (treated in Section 4), $QM_{A^*}(A^*) = B_{A^*}^{A^*}(A^*, A^*; A^*)$ (the subscripts to QM are meant to specify which of the two variants is intended).

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To our knowledge, the first case has not yet been considered in the literature. Concerning the second case, Theorem 4.5 of [AR] presents a norm decreasing natural linear embedding of $L(X^{**})$ into $QM_{A^*}(A^*)$. Below, we will describe $QM_A(A^*)$ as a subspace of $L(X^*)$ and show that the map envisaged in [AR] is in fact an isometric isomorphism between $L(X^{**})$ and $QM_{A^*}(A^*)$.

Concerning $QM(A^{**})$, Theorem 4.8 of [AR] presents a norm decreasing linear mapping of $L(X^*)$ onto $QM(A^{**})$, provided that X has the bounded approximation property. In Section 5, we will show—without any assumption on X—that this map is even an isometric isomorphism.

The basic idea in this paper consists in the observation that a space of quasi-multipliers on some Banach bi-module V can be considered as "iterated" multiplier space, i.e. as the space of right multipliers on the space of left multipliers on V or vice versa: $QM(V) = M_r(M_l(V)) = M_l(M_r(V))$. This method was used extensively in [G1]. It allows us to compute spaces of quasi-multipliers on the basis of results on (one-sided) multiplier spaces. Moreover, this way of proceeding does not require the existence of any kind of approximate identity in the underlying Banach algebras. Contrary to this, the authors of [AR] base a good deal of their arguments in Section 4 on (one-sided) identities in the bidual A^{**} of $A = K_0(X)$; essentially, the existence of the latter is equivalent to the existence of certain kinds of approximate identities in $A = K_0(X)$, which, in turn, is equivalent to X resp. X^* possessing the corresponding kind of bounded approximation property (see [CLM, II.3.10] and [G2, Theorem 4]). Therefore, the occurrence of the bounded approximation property in Corollary 4.3 and Theorem 4.8 of [AR] is due to the choice of methods employed there; it is not inherent to the subject matter itself.

2. Notation and terminology. For any Banach space X, let X^* denote its Banach dual and ι_X the canonical embedding of X into X^{**}. Instead of $x'(x)$ ($x' \in X^*$, $x \in X$) we also write $\langle x, x'\rangle$. For $x' \in X^*$, $x \in X$, $x \otimes x'$ is defined as the bounded linear operator (of rank one) on X acting according to $y \mapsto \langle y, x'\rangle x$ ($y \in X$). Concerning the definition of the various kinds of approximation properties, we refer to [CLM, p. 75ff.].

For any three Banach spaces X, Y, Z, $L(X,Y)$ is defined as the Banach space of bounded linear operators from X to Y and $B(X,Y;Z)$ as the Banach space of bounded bilinear operators from $X \times Y$ to Z, both equipped with the supremum norm. $L(X,X)$ is abbreviated to $L(X)$. Further, $F(X)$ will denote the normed subspace of $L(X)$ consisting of the operators of finite rank and $K_0(X)$ the closure of $F(X)$ in $L(X)$, that is, the Banach space of approximable operators on X. Finally, $I(X)$ denotes the space of integral operators from X to X (in the sense of [CLM, II.2.9 and V.3.1]), which is a Banach space with respect to the integral norm $f \mapsto \|f\|_I$; we have $\|f\| \le \|f\|_I$.

For $f \in L(X,Y)$, the adjoint operator $f^* \in L(Y^*, X^*)$ is defined by $\langle x, f^*(y')\rangle := \langle f(x), y'\rangle$ ($x \in X$, $y' \in Y$). If X is a Banach space and $f \in L(X^{**})$ then we denote by $f^\flat$ the operator in $L(X^*)$ defined by

$$f^\flat := (\iota_X)^* \circ f^* \circ \iota_{X^*} = \iota^* \circ f^* \circ \iota.$$

We have $g^{*\flat} = g$ for $g \in L(X^*)$.

For Banach spaces X, Y, Z, we have natural isometric isomorphisms

$$B(X,Y;Z) \cong L(X, L(Y,Z)) \cong L(Y, L(X,Z))$$

where the relation between corresponding elements $\beta, \widetilde{\beta}, \overline{\beta}$ of the respective spaces is given by

$$\beta(x,y) = (\widetilde{\beta}(x))(y) = (\overline{\beta}(y))(x) \quad (x \in X,\ y \in Y).$$

Let A, B be Banach algebras. A Banach space V is called a *left* [*right*] *Banach A-module* if it is a left [right] A-module in the algebraic sense and $\|av\| \le \|a\|\cdot\|v\|$ [resp. $\|va\| \le \|v\|\cdot\|a\|$]. V is called a *Banach A-B-bimodule* if it is a left Banach A-module and a right Banach B-module and $(av)b = a(vb)$ ($a \in A$, $b \in B$, $v \in V$).

The Banach dual of a Banach module becomes a Banach module itself (with "left" and "right" interchanged) by the adjoints of the action of the Banach algebra on the given module. For example,

$$\langle v, v'a\rangle := \langle av, v'\rangle \quad (a \in A,\ v \in V,\ v' \in V').$$

If V, W are left Banach A-modules then $H_A(V,W)$ denotes the (closed) subspace of $L(V,W)$ consisting of all left A-module homomorphisms, that is, of all bounded linear maps T from V into W satisfying $T(av) = aT(v)$ ($a \in A$, $v \in V$). For right B-modules V, W, $H^B(V,W)$ is defined in an analogous way as the (Banach) space of right B-module homomorphisms from V into W. If both V and W are A-B-bimodules then the space of A-B-bimodule homomorphisms is defined by $H_A^B(V,W) := H_A(V,W) \cap H^B(V,W)$. If also Z is a Banach A-B-bimodule then we define $B_A^B(V,W;Z)$ as the (closed) subspace of $B(X,Y;Z)$ consisting of all β satisfying the relations

$$\beta(av,w) = a\beta(v,w), \quad \beta(v,wb) = \beta(v,w)b \quad (a \in A,\ b \in B,\ v \in V,\ w \in W).$$

Moreover, $H_A(A,V)$ and $H^B(B,V)$ are A-B-bimodules in a natural way for any A-B-bimodule V ([G1, p. 46]). From the corresponding isomorphisms in the case of (bi)linear operators, we get isometric isomorphisms

$$B_A^B(A,B;V) \cong H_A(A, H^B(B,V)) \cong H^B(B, H_A(A,V)).$$

Finally, we give the definitions of multiplier spaces of Banach modules: If V is a left Banach A-module [right Banach B-module] then we define the space of right [left] multipliers on V as $M_r(V) := H_A(A,V)$ [resp. $M_l(V) :=$

$H^B(B;V)$]. Of course, if V is a Banach A-B-bimodule, both $M_r(V)$ and $M_l(V)$ make sense. Moreover, for any Banach A-B-bimodule V, we define the space of quasi-multipliers on V as $QM(V) := B^B_A(A,B;V)$. That is, $m : A \times B \to V$ is a quasi-multiplier if m is bounded, bilinear and satisfies $m(a_1a_2, b) = a_1 m(a_2, b)$ and $m(a, b_1b_2) = m(a, b_1)b_2$ for all $a, a_1, a_2 \in A$ and $b, b_1, b_2 \in B$.

3. Quasi-multipliers of the A-A-bimodule A^*. According to Section 2, $QM_A(A^*)$ is given by $B^A_A(A, A; A^*) \cong H^A(A, H_A(A, A^*))$ or $B^A_A(A, A; A^*) \cong H_A(A, H^A(A, A^*))$. No matter which of the two variants we choose we are led to determine spaces of the form $H^A(A, W)$ as well as $H_A(A, W)$ where $A \subseteq L(X)$ and $W \subseteq L(X^*)$: in fact, $A^* = I(X^*) \subseteq L(X^*)$ and—as we will see soon—also $H_A(A, A^*)$ and $H^A(A, A^*)$ are isomorphic to subspaces of $L(X^*)$. Unfortunately, Theorems 3.4 and 3.18 of [G1] (giving concrete representations for $H_A(A, V)$ resp. $H^B(B, V)$) are not applicable since they presuppose $A, B, V \subseteq L(Z)$ all for the same Banach space Z. However, replacing $A = K_0(X)$ by the "reverse" algebra $A_r := \{a^* \mid a \in K_0(X)\} \subseteq L(X^*)$, each left [right] A-module becomes a right [left] A_r-module by setting $v \cdot a^* := a \cdot v$ [resp. $a^* \cdot v := v \cdot a$]. Now [G1, 3.18] works perfectly well for $H^{A_r}(A_r, W) \cong H_A(A, W)$, yet [G1, 3.4] still cannot be applied to $H_{A_r}(A_r, W) \cong H^A(A, W)$: A_r does not satisfy condition [G1, 3.1(*)] required for B in [G1, 3.4]. Thus, we have to develop a variant of [G1, 3.4] to deal with the latter case.

THEOREM 3.1. *Let X be a Banach space, and B a subalgebra of $L(X)$ which is a Banach algebra with respect to some norm $\| \ \|_B$. Let V be a linear subspace of $L(X^*)$ which is a Banach space with respect to some norm $\| \ \|_V$ such that $B_r \circ V \subseteq V$ (with $B_r = \{b^* \mid b \in B\}$) and $\|b^* \circ v\|_V \le \|b\|_B \|v\|_V$ is satisfied for all $b \in B$, $v \in V$.*

Assume that there exist positive constants C_1, C_2, C_3 satisfying

$$\|v\| \le C_1 \|v\|_V \qquad (v \in V),$$
$$\|b\| \le C_2 \|b\|_B \qquad (b \in B),$$
$$\|x \otimes x'\|_B \le C_3 \|x\| \|x'\| \qquad (x \in X,\ x' \in X^*,\ x \otimes x' \in B).$$

Finally, let B satisfy the following condition:

$$(**) \qquad \forall x \in X\ \exists x' \in X^* :\ x \otimes x' \in B \text{ and } \langle x, x' \rangle = 1.$$

Then V becomes a right B-module resp. a left B_r-module by

$$v \cdot b := b^* \cdot v := b^* \circ v$$

and we have a linear isomorphism

$$H^B(B, V) \cong H_{B_r}(B_r, V) \cong \{d \in L(X^*) \mid b^* \circ d \in V\}$$

where $d \in L(X^*)$ corresponds to the multiplier $T_d : b \mapsto b^* \circ d$. Moreover, $\|d\| \le C_1C_3 \|T_d\|$.

If, in addition, $\| \ \|_V$ is equivalent to the operator norm $\| \ \|$ on V then $d \mapsto T_d$ is even a topological isomorphism; more precisely: if $\|v\|_V \le C_4 \|v\|$ for all $v \in V$ then also $\|T_d\| \le C_2C_4 \|d\|$.

Proof. Let $d \in L(X^*)$ such that $b^* \circ d \in V$ for all $b \in B$. Then $T_d : b \mapsto d \cdot b := b^* \circ d$ is a right module homomorphism from B into V having closed graph, hence $T_d \in H^B(B, V)$. Moreover, $T_d \ne 0$ if $d \ne 0$ by (**).

It remains to show that each $T \in H_{B_r}(B_r, V)$ [$\cong H^B(B, V)$] is of the form T_d as described above. Let $T \in H_{B_r}(B_r, V)$. Let $0 \ne x \in X$ be given. By (**), choose $x' \in X^*$ such that $x \otimes x' \in B$ (i.e. $(x \otimes x')^* = x' \otimes \iota x \in B_r$) and $\langle x, x' \rangle = 1$. Let $y' \ne 0$ be *any* element of X^* such that $y' \otimes \iota x \in B_r$. Then we have

$$\begin{aligned} T(y' \otimes \iota x) &= T((y' \otimes \iota x) \circ (x' \otimes \iota x)) = (y' \otimes \iota x) \circ T(x' \otimes \iota x) \\ &= y' \otimes [T(x' \otimes \iota x)]^*(\iota x). \end{aligned}$$

Since x' does not occur on the left side, $[T(x' \otimes \iota x)]^*(\iota x)$ is independent of x' provided $x \otimes x' \in B$ and $\langle x, x' \rangle = 1$. Define $c : X \to X^{**}$ by $c(x) := [T(x' \otimes \iota x)]^*(\iota x)$ for $x \ne 0$ and $c(0) := 0$. It is easy to show that c is linear (compare [G1, p. 115]). By definition, we have $T(y' \otimes \iota x) = y' \otimes c(x)$ for any $x \otimes y' \in B$. Moreover, c is bounded:

$$\begin{aligned} \|y'\| \cdot \|c(x)\| &= \|y' \otimes c(x)\| \le C_1 \|y' \otimes c(x)\|_V \le C_1 \|T(y' \otimes \iota x)\| \\ &\le C_1 \|T\| \cdot \|x \otimes y'\|_B \le C_1C_3 \|T\| \cdot \|x\| \cdot \|y'\|, \end{aligned}$$

so we see that $\|c\| \le C_1C_3 \|T\|$.

Finally, for $0 \ne b \in B$, $0 \ne x \in X$, choose $0 \ne x' \in X^*$ such that $x \otimes x' \in B$. Then

$$\begin{aligned} x' \otimes [T(b^*)]^*(\iota x) &= (x' \otimes \iota x) \circ T(b^*) = T((x' \otimes \iota x) \circ b^*) \\ &= T(x' \otimes b^{**}(\iota x)) = T(x' \otimes \iota b(x)) = x' \otimes c(b(x)). \end{aligned}$$

Therefore, $[T(b^*)]^* \circ \iota_X = c \circ b$. By taking adjoints and composing with ι_{X^*} from the right, we get

$$T(b^*) = \iota_X{}^* \circ \iota_{X^*} \circ T(b^*) = \iota_X{}^* \circ T(b^*)^{**} \circ \iota_{X^*} = b^* \circ c^* \circ \iota_{X^*}$$

Thus, $d := c^* \circ \iota_{X^*}$ satisfies $T = T_d$ and $\|d\| = \|c\| \le C_1C_3 \|T_d\|$.

Assuming $\|v\|_V \le C_4 \|v\|$ for all $v \in V$ we conclude

$$\|T_d(b)\|_V \le C_4 \|b^* \circ d\| \le C_4 \|b\| \cdot \|d\| \le C_2C_4 \|b\|_B \|d\|,$$

which shows that $\|T_d\| \le C_2C_4 \|d\|$. ∎

Now we are prepared to determine $QM_A(A^*)$.

THEOREM 3.2. *Let X be a Banach space, $A = K_0(X)$. Then $QM_A(A^*)$ is linearly isomorphic to the space*

$$Q := \{k \in L(X^*) \mid b^* \circ k \circ a^* \in I(X^*) \text{ for all } a, b \in K_0(X)\}$$

where $k \in Q$ gives rise to the quasi-multiplier $\vartheta(k) : (a,b) \mapsto b^* \circ k \circ a^*$ $(a, b \in K_0(X))$. Moreover, $\|k\| \le \|\vartheta(k)\|$ for $k \in Q$ and $\|\vartheta(k)\| \le \|k\|_I$ for $k \in I(X^*)$.

If, in addition, X has the bounded [metric] approximation property then $QM_A(A^*)$ is topologically [isometrically] isomorphic to $I(X^*)$ via ϑ^{-1}.

Proof. Applying [G1, 3.18], we get

$$\begin{aligned} H_A(A, A^*) &= H_A(A, I(X^*)) \cong H^{A_r}(A_r, I(X^*)) \\ &\cong \{d \in L(X^*) \mid d \circ a^* \in I(X^*) \text{ for all } a \in K_0(X)\} =: V. \end{aligned}$$

Moreover, $\|d\| \le \|T_d\|$ where $T_d : a \mapsto d \circ a^*$.

In the second step, we want to apply Theorem 3.1 to $H^A(A, V)$. Since $\|d\| \le \|T_d\|$ for $d \in V$, it follows that A and V (the latter equipped with the norm of $H_A(A, A^*)$) satisfy the requirements of Theorem 3.1. We obtain

$$\begin{aligned} QM_A(A^*) &= H^A(A, V) \cong H_{A_r}(A_r, V) \\ &\cong \{k \in L(X^*) \mid b^* \circ k \in V \text{ for all } b \in K_0(X)\} \\ &= \{k \in L(X^*) \mid b^* \circ k \circ a^* \in I(X^*) \text{ for all } a, b \in K_0(X)\} =: Q. \end{aligned}$$

Again, $\|k\|$ is dominated by the norm of $(a,b) \mapsto b^* \circ k \circ a^*$ for $k \in Q$. Only the last assertion remains to be shown, $\|\vartheta(k)\| \le \|k\|_I$ being an immediate consequence of the properties of the integral norm. If X has the bounded [metric] approximation property then A has a bounded left approximate identity [of bound 1]. By [CLM, III.3.11], it follows that $A \mathbin{\widehat{\otimes}_A} A = A$ and hence, by [CLM, III.3.9.2],

$$H_A(A, H^A(A, A^*)) = H_A(A, (A \mathbin{\widehat{\otimes}_A} A)^*) = H_A(A, A^*) = (A \mathbin{\widehat{\otimes}} A)^* = A^*. \ \blacksquare$$

Of course, we could as well have used Theorem 3.1 for the first step of the proof and [G1, 3.18] for the second:

$$\begin{aligned} H^A(A, A^*) &= H^A(A, I(X^*)) \cong H_{A_r}(A_r, I(X^*)) \\ &\cong \{d \in L(X^*) \mid b^* \circ d \in I(X^*) \text{ for all } b \in K_0(X)\} =: W; \\ QM_A(A^*) &= H_A(A, W) \cong H^{A_r}(A_r, W) \\ &\cong \{k \in L(X^*) \mid k \circ a^* \in W \text{ for all } a \in K_0(X)\} \\ &= \{k \in L(X^*) \mid b^* \circ k \circ a^* \in I(X^*) \text{ for all } a, b \in K_0(X)\} = Q. \end{aligned}$$

Since $I(X^*)$ is an ideal in $L(X^*)$, $Q \cong QM_A(A^*)$ is at least as large as $I(X^*)$ itself is. The question remains open if Q can contain $I(X^*)$ properly; of course, X would necessarily fail having the bounded approximation property in such a case.

4. Quasi-multipliers of the Banach algebra A^*. In this section, we will determine a concrete representation of the space $QM_{A^*}(A^*) = B^{A^*}_{A^*}(A^*, A^*; A^*) = H_{A^*}(A^*, H^{A^*}(A^*, A^*))$.

THEOREM 4.1. *Let X be a Banach space, $A = K_0(X)$. Then $QM_{A^*}(A^*)$ is isometrically isomorphic to $L(X^{**})$ where $h \in L(X^{**})$ induces on A^* the quasi-multiplier*

$$\theta(h) : (f, g) \mapsto (h \circ f^*)^\flat \circ g \quad (f, g \in A^*).$$

Proof. In the first step, we are going to represent $H^{A^*}(A^*, A^*)$ as a space of operators by showing that $H^{A^*}(A^*, A^*) = H^{A_r}(A^*, A^*) \cong H_A(A^*, A^*)$. For the latter space, a concrete representation is readily available: by [G2, Theorem 1.A], it is isometrically isomorphic to $L(X^*)$ where $g \in L(X^*)$ corresponds to the multiplier $S_g : f \mapsto g \circ f$.

So let us start by taking $S \in H^{A^*}(A^*, A^*)$. The algebra $F(X)$ of continuous linear operators on X having finite rank is dense in $K_0(X)$ by the very definition of $K_0(X)$. Since $F(X)_r := \{a^* \mid a \in F(X)\}$ is a subalgebra of A^*, it follows that

$$S(a \cdot f) = S(f \circ a^*) = S(f) \circ a^* = a \cdot S(f)$$

for every $a \in F(X)$, $f \in I(X^*)$. By continuity with respect to the operator norm on A and the integral norm on $I(X^*)$, the equation above is valid even for $a \in K_0(X)$, hence $S \in H_A(A^*, A^*) \cong H^{A_r}(A^*, A^*)$. Conversely, if $S \in H_A(A^*, A^*)$, then S is of the form $S_g : f \mapsto g \circ f$ by [G2, Theorem 1.A] (for $g \in L(X^*)$). Obviously, $S_g(f_1 \circ f_2) = S_g(f_1) \circ f_2$ for $f_1, f_2 \in A^*$, and so $S \in H^{A^*}(A^*, A^*)$.

In the second step, we apply [G1, 3.4] to derive a representation of $H_{A^*}(A^*, L(X^*))$ as a space of operators on X^{**}. Routine calculations show that the canonical left module action of $f \in A^*$ on $g \in L(X^*)$ $(\cong H^{A^*}(A^*, A^*))$ is given simply by $f \cdot g = f \circ g$ and that all the conditions required in [G1, 3.4] are satisfied. We obtain

$$\begin{aligned} &H_{A^*}(A^*, L(X^*)) \\ &\quad \cong \{h \in L(X^{**}) \mid \forall f \in I(X^*) : \operatorname{im}(f^{**} \circ h^* \circ \iota_{X^*}) \subseteq \iota_{X^*}(X^*)\}. \end{aligned}$$

Here, $\operatorname{im}(F)$ denotes the image of F for any map F. Now every $f \in A^* = I(X^*)$ is weakly compact ([CLM, V.3.6]). Consequently, f^{**} is of the form $\iota_{X^*} \circ f_0$ for some $f_0 \in L(X^{**}, X)$ and thus

$$H_{A^*}(A^*, L(X^*)) \cong L(X^{**}).$$

By this isomorphism, $h \in L(X^{**})$ corresponds to the multiplier $T_h \in H_{A^*}(A^*, L(X^*))$ defined by

$$\begin{aligned} T_h(f) &= f_0 \circ h^* \circ \iota_{X^*} = \iota_X{}^* \circ \iota_{X^*} \circ f_0 \circ h^* \circ \iota_{X^*} \\ &= \iota_X{}^* \circ f^{**} \circ h^* \circ \iota_{X^*} = \iota_X{}^* \circ (h \circ f^*)^* \circ \iota_{X^*} \\ &= (h \circ f^*)^\flat \quad (f \in I(X^*),\ f_0 \text{ as above}). \end{aligned}$$

Concerning the norm of T_h, it also follows from [G1, 3.4] that $\|T_h\| = \|h\|$.

Due to the form of the natural isomorphism $B_{A^*}^{A^*}(A^*, A^*; A^*) \cong H_{A^*}(A^*, H^{A^*}(A^*, A^*))$, $h \in L(X^{**})$ corresponds to the following multiplier $\theta(h)$ of the Banach algebra A^*:

$$\theta(h) : (f, g) \mapsto S_{T_h(f)}(g) = T_h(f) \circ g = (h \circ f^*)^\flat \circ g \qquad (f, g \in A^* = I(X^*)). \ ■$$

Theorem 4.5 of [AR] states that θ is a norm decreasing injective linear map from $L(X^{**})$ into $QM_{A^*}(A^*)$.

5. Quasi-multipliers of A^{**}. Based on the techniques developed so far, it is easy to determine $QM(A^{**})$.

THEOREM 5.1. *Let X be a Banach space, $A = K_0(X)$. Then $QM(A^{**})$ is isometrically isomorphic to $L(X^*)$ where $g \in L(X^*)$ induces the quasi-multiplier $\omega(g)$ defined by*

$$\langle f, \omega(g)(a, b)\rangle = \langle b, g \circ a^* \circ f\rangle \quad (a, b \in A,\ f \in A^*).$$

Proof. Using once again the isometric isomorphism $g \mapsto S_g$ between $L(X^*)$ and $H_A(A^*, A^*)$ ([G2, Theorem 1.A]) we deduce that

$$\begin{aligned} &(1) & QM(A^{**}) &= B_A^A(A, A; A^{**}) \\ &(2) & &\cong H_A(A, H^A(A, A^{**})) \\ &(3) & &\cong H_A(A, H_A(A^*, A^*)) \\ &(4) & &\cong H_A(A, L(X^*)) \\ &(5) & &\cong H^{A_r}(A_r, L(X^*)) \\ &(6) & &\cong L(X^*). \end{aligned}$$

The last step from (5) to (6) was performed on the basis of 3.18 of [G1]. All the isomorphisms occurring above are isometric.

To determine the explicit form of the composed isomorphism, we denote by $m \leftrightarrow \widetilde{m} \leftrightarrow \widehat{m}$ the natural correspondence

$$B_A^A(A, A; A^{**}) \cong H_A(A, H^A(A, A^{**})) \cong H_A(A, H_A(A^*, A^*)).$$

It is given by

$$\langle f, m(a, b)\rangle = \langle f, \widetilde{m}(a)(b)\rangle = \langle b, \widehat{m}(a)(f)\rangle \quad (a, b \in A,\ f \in A^*).$$

Now let us start with $g \in L(X^*)$ and work bottom-up from (6) to (1). By [G1, 3.18], g corresponds to $(a^* \mapsto g \circ a^*) \in H^{A_r}(A_r, L(X^*))$ in line (5) resp. $(a \mapsto g \circ a^*) \in H_A(A, L(X^*))$ in line (4). Passing from (4) to (3) involves

the isomorphism $h \mapsto S_h = (f \mapsto h \circ f)$. Therefore, in line (3) we arrive at the element $\widehat{m}_g$ acting according to

$$\widehat{m}_g : a \mapsto g \circ a^* \mapsto S_{g \circ a^*} = (f \mapsto (g \circ a^*) \circ f)$$

or

$$\widehat{m}_g(a)(f) = S_{g \circ a^*}(f) = g \circ a^* \circ f.$$

Finally, we get the following explicit formula for the quasi-multiplier m_g corresponding to $g \in L(X^*)$:

$$\langle f, m_g(a, b)\rangle = \langle b, g \circ a^* \circ f\rangle \quad (a, b \in A,\ f \in A^*). \ ■$$

There is no explicit formula for m_g involving only a, b, g since, in general, A^{**} is not a subspace of some $L(Y)$ with Y a dual of X of any order—as was the case for A and A^*.

In Section 4 of [AR], the authors define a map

$$\lambda : M_l(A^{**}) = H^A(A, A^{**}) \to QM(A^{**})$$

by $\lambda(S)(a, b) := aS(b)$ (A^{**} is viewed as an A-A-bimodule here). It is immediate that modulo the isometric isomorphism $H^A(A, A^{**}) \cong H_A(A^*, A^*) \cong L(X^*)$ established in Theorem 1.A of [G2], λ is the same map as ω. Theorem 4.8 of [AR] states that λ is norm decreasing and surjective, provided X^* has the bounded approximation property. Theorem 5.1 above shows that λ is even an isometry, and this fact is true for *any* Banach space X. The remark following Theorem 4.8 in [AR] says that λ is a topological isomorphism if, for example, $I(X^*) = N(X^*)$ (the latter denoting the space of nuclear operators on X^*) and that λ is an isometric isomorphism if, in addition, X^* has the metric approximation property. Theorem 5.1 above shows that these special conditions on X are not necessary for λ to be a topological resp. isometric isomorphism. We should like to point out at this place that there is an obvious misprint in the remark of [AR] quoted above: In the last sentence, $L(X^{**})$ should be replaced by $L(X^*)$ twice.

As we already mentioned in the introduction, the respective proofs of Theorem 4.2, Corollary 4.3 and Theorem 4.8 of [AR] are based on the relations between the multiplier spaces of a bimodule and its bidual. This is exactly the place where the bounded approximation property comes in: It is true that those relations form a valuable tool for studying multiplier spaces (compare Chapter 4 of [G1], especially 4.14 and 4.17) but most of the respective theorems require some kind of approximate identity in the algebras involved, and this amounts to the approximation property for X or X^* if $K_0(X)$ is the relevant Banach algebra. In the present paper, we have avoided using the above-mentioned relations between biduals and multiplier spaces. Thereby Theorems 4.1 and 5.1 and the first part of Theorem 3.2 are valid in full generality concerning X.

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