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Analogs of $\Delta(z)$ for triangular Shimura curves

by

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We construct analogs of the classical Δ-function for quotients of the upper half plane $\mathcal{H}$ by certain arithmetic triangle groups Γ coming from quaternion division algebras B. We also establish a relative integrality result concerning modular functions of the form $\Delta(\alpha z)/\Delta(z)$ for α in B^+. We give two explicit examples at the end.

1. Introduction. By a *triangular Shimura curve*, we mean the canonical model X_Γ of $\Gamma\backslash\mathcal{H}$, the quotient of the upper half plane $\mathcal{H}$ by a cocompact arithmetic triangle subgroup Γ of $SL_2(\mathbb{R})$. To be concise, let F be a totally real algebraic number field of degree d, and B a quaternion algebra over F, with $B\otimes_{\mathbb{Q}}\mathbb{R}\cong M_2(\mathbb{R})\oplus\mathbb{H}^{d-1}$, where $\mathbb{H}$ is the Hamilton quaternion algebra. Let O be an order of B, and $\Gamma(O)=\{\gamma\in O: \gamma O=O, N_{B/F}(\gamma)$ is totally positive$\}$. A Fuchsian group Γ of the first kind is called *arithmetic* if it is commensurable with $\Gamma(O)$ for some B and O. It is *triangular* if it can be generated by 3 elliptic or parabolic elements γ_1, γ_2 and γ_3 such that $\gamma_1\gamma_2\gamma_3=\pm I$. Its fundamental domain is the union of two copies of hyperbolic triangles. Γ is cocompact unless $F=\mathbb{Q}$ and $B=M_2(F)$, in which case Γ is commensurable with $SL_2(\mathbb{Z})$.

A well-known example is given by $\Gamma(2):=\{\gamma\in SL_2(\mathbb{Z}):\gamma\equiv 1 \pmod 2\}$, whose fundamental domain has all of its vertices at infinity ("cusp"), namely at 0, 1, ∞. Even the modular group $SL_2(\mathbb{Z})$ belongs to this class, as its fundamental domain has vertices i, $\varrho=e^{2\pi i/3}$ and ∞. The function fields of $\Gamma(2)\backslash\mathcal{H}$ and $SL_2(\mathbb{Z})\backslash\mathcal{H}$ are generated by the classical elliptic modular functions $\lambda(z)$ and $j(z)$, respectively. Moreover, there is a distinguished modular form $\Delta(z)=q\prod_{n\ge1}(1-q^n)^{24}$, $q=e^{2\pi iz}$ for $SL_2(\mathbb{Z})$, which spans the space of cusp forms of weight 12 for $SL_2(\mathbb{Z})$. By a well-known theorem, one knows that, for any $N\ge1$, the modular function $\Delta(Nz)/\Delta(z)$ is, when suitably normalized, integral over $\mathbb{Z}[j]$ (see [Lang]). This fact leads to many interesting results in number theory and geometry.

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In this paper, we find analogs of $\Delta(z)$ and prove such an integrality result for arithmetic triangular groups Γ which are cocompact. We also give two explicit examples in the last section.

A complete (finite) list of cocompact arithmetic triangle groups Γ, given by congruence conditions, is available ([Ta], §3 & §5; [Sh1], p. 82). Furthermore, one knows by Shimura that the algebraic curve $\Gamma\backslash\mathcal{H}$ and the three vertices are defined over an explicit extension M_Γ of F. For each such Γ, we first find weights k such that $\mathcal{S}_k(\Gamma)$ is one-dimensional, generated by an F_Γ-rational modular function with a unique zero at one of the vertices of Φ; we call this function $\Delta_{B,\Gamma}$ (or Δ_B for short). We also present some explicit analogs $j_B(z)$ of the classical j-function. The fact that they exist is a consequence of Shimura's theory of canonical models ([Sh1]) (see Theorem 3 in §4). We determine an explicit expression of j_B as a quotient with denominator Δ_B in the main cases of interest. We also use the Δ analogs to find algebra generators of appropriate spaces of modular forms. A typical result is as following:

THEOREM A. *Let Γ be the group of signature* $(2,3,7)$, *coming from the quaternion algebra over* $\mathbb{Q}(2\pi/7)$ *which ramifies only at two infinite places. There are Δ forms Δ_{12}, Δ_{16} and Δ_{30} of weight* 12, 16, 30 *respectively*, *which have a simple zero at the three respective vertices. Moreover*, *they generate the algebra of all modular forms of even integral weight. Finally*, $\Delta_{12}^7/\Delta_{42}^3$ *is a generator of the function field.*

We also have the following theorem concerning the integrality:

THEOREM B. *Let (Γ, B, k) be as above. Then $\zeta(\alpha,z)^k\Delta_B(\alpha z)/\Delta_B(z)$ is integral over $M[j_B]$ for all $\alpha\in B^+$, where $\zeta(\alpha,z)$ is an automorphy factor.*

By Shimura's theory of canonical models ([Sh1]), we know that the value of any arithmetically defined modular function relative to a congruence subgroup Γ at any CM point z lies in a class field of a totally imaginary quadratic extension K_z of F. This in particular applies to our functions $\zeta(\alpha,z)^k\Delta_B(\alpha z)/\Delta_B(z)$. One may view our result as an integral refinement in a special case. Since for $F=\mathbb{Q}$, it gives abelian extensions of complex quadratic fields, we are mainly interested in the case $F\neq\mathbb{Q}$.

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2. Notations

- F: totally real number field with $[F:\mathbb{Q}]=d$,
- B: quaternion algebra over F with $B\otimes_{\mathbb{Q}}\mathbb{R}\cong M_2(\mathbb{R})\oplus\mathbb{H}^{d-1}$,

- ξ: the composite map

$$\alpha \in B \xrightarrow{i} B \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R}) \oplus \mathbb{H} \oplus \ldots \oplus \mathbb{H} \xrightarrow{\mathrm{Pr}_1} M_2(\mathbb{R}) \ni \xi(\alpha),$$

- $B^+ = \{b \in B : N_{B/F}(b)$ is totally positive$\}$,
- O: a maximal order of B,
- τ: a two-sided integral O ideal of B,
- $\Gamma = \Gamma(O,\tau) = \{\gamma \in B^+ : \gamma$ is a unit of O and $\gamma - 1 \in \tau\}$ (we also use Γ to denote the image of Γ under ξ),
- F_Γ: the ray class field of F corresponding to $(\tau \cap O_F)\varpi_0$ where ϖ_0 is the product of all archimedean primes of F,
- (X_Γ, ϕ): the Shimura canonical model defined over F_Γ,
- $\mathcal{M}(\Gamma)$: the space of meromorphic modular functions for Γ,
- $\mathcal{M}(\Gamma)_0 = \{f \in \mathcal{M}(\Gamma) : f$ is rational over $F_\Gamma\}$,
- $\mathcal{S}_k(\Gamma)$: the space of holomorphic cusp forms of weight k for Γ; since Γ has no cusps, all holomorphic forms are cusp forms,
- $\mathcal{S}_k(\Gamma)_0 = \{f \in \mathcal{S}_k(\Gamma) : f$ is rational over $F_\Gamma\}$.

3. Analogs of $\Delta(z)$. In this section, we determine the class of (Γ, k) such that $\mathcal{S}_k(\Gamma)$ is one-dimensional and can be generated by a modular form which is nonvanishing outside one elliptic point.

If $\Gamma = \langle \gamma_1, \gamma_2, \gamma_3 \rangle$ with $\gamma_1^{e_1} = \gamma_2^{e_2} = \gamma_3^{e_3} = \gamma_1\gamma_2\gamma_3 = 1$ as an automorphism of $\mathcal{H}$, then (e_1, e_2, e_3) is called the *signature* of Γ. Let $P_{e_1}, P_{e_2}, P_{e_3}$ be fixed points of $\gamma_1, \gamma_2, \gamma_3$ respectively.

THEOREM 1. *For the following Γ and k, $\mathcal{S}_k(\Gamma)$ is one-dimensional, generated by an F_Γ-rational modular form $\Delta_B = \Delta_B(\Gamma, k)$, which is an eigenform of Hecke operators. Moreover, Δ_B is nonzero everywhere except at a unique elliptic point.*

Signature of Γ	*k*	*Divisor of Δ_B*
(2, 3, 8)	12, 16, 32	$2P_8$, P_3, $2P_3$
(2, 4, 5)	8, 16, 24, 32	P_5, $2P_5$, $3P_5$, $4P_5$
(2, 3, 10)	12, 20	$4P_{10}$, $2P_3$
(2, 5, 6)	12	$4P_5$
(2, 3, 7)	12, 24, 28, 36, 42, 48, 56, 60, 72	P_7, $2P_7$, P_3, $3P_7$, P_2, $4P_7$, $2P_3$, $5P_7$, $6P_7$
(2, 3, 9)	18	P_2
(2, 3, 11)	12, 24	$5P_{11}$, $10P_{11}$

Proof. Applying the Riemann–Roch theorem to $\Gamma\backslash\mathcal{H}$, we have the following

LEMMA 1.

$$\dim \mathcal{S}_2(\Gamma) = 0; \tag{1}$$

and for even $k > 2$,

$$\dim \mathcal{S}_k(\Gamma) = \left[\frac{(e_1-1)k}{2e_1}\right] + \left[\frac{(e_2-1)k}{2e_2}\right] + \left[\frac{(e_3-1)k}{2e_3}\right] - k + 1. \tag{2}$$

Furthermore, if $k > 2$ *and* $f \in \mathcal{S}_k(\Gamma)$, *then*

$$\sum_{P \neq P_1, P_2, P_3} O_P(f) + \frac{O_{P_1}(f)}{e_1} + \frac{O_{P_2}(f)}{e_2} + \frac{O_{P_3}(f)}{e_3} = \left(1 - \frac{1}{e_1} - \frac{1}{e_2} - \frac{1}{e_3}\right)\frac{k}{2} \tag{3}$$

where $O_P(f)$ *is the order of* f *at* P.

Proof (of Lemma 1). For the proof of (1) and (2), see ([Sh2], §2.6).

For any $P \in \mathcal{H}$, denote by $\overline{P}$ the image of P under the projection $\mathcal{H} \to \Gamma\backslash\mathcal{H}$. Let G_P be the isotropy group of P and $e(P)$ the order of G_P; then $e(P_i) = e_i$ for $i = 1, 2, 3$ and $e(P) = 1$ for other P. Choose a local parameter z_P such that G_P operates on z_P by multiplication by eth roots of unity; then $t = (z_P)^e$ is a local parameter of $\overline{P}$ in $\Gamma\backslash\mathcal{H}$.

Let $f \in \mathcal{S}_k(\Gamma)$. Then $\omega = f(dz)^{k/2}$ is invariant under Γ; hence represents a holomorphic differential form of $\Gamma\backslash\mathcal{H}$. Let $O_{\overline{P}}(\omega)$ be the order of ω at $\overline{P}$ in $\Gamma\backslash\mathcal{H}$. We have

$$\begin{aligned}\omega = f(t)(dt)^{k/2} &= ut^{(O_{\overline{P}}(\omega))}(dt)^{k/2} = u(z_P)^{e(O_{\overline{P}}(\omega))}(e(z_P)^{e-1}dz_P)^{k/2}\\ &= ue^{k/2}(z_P)^{e(O_{\overline{P}}(\omega))+k(e-1)/2}(dz_P)^{k/2},\end{aligned}$$

where u is locally holomorphic and nonzero around P. Therefore

$$O_P(f) = eO_{\overline{P}}(\omega) + k(e-1)/2. \tag{4}$$

As we know, on an algebraic curve of genus g, the sum of the orders of a differential form of degree 1 is equal to $2g-2$. Here $g = 0$. Hence

$$\sum_{P \neq P_1, P_2, P_3} O_{\overline{P}}(\omega) + O_{\overline{P}_1}(\omega) + O_{\overline{P}_2}(\omega) + O_{\overline{P}_3}(\omega) = -k. \tag{5}$$

(3) results from (4) and (5).

Next we compute $\dim \mathcal{S}_k(\Gamma)$ and possible divisors for those Γ listed in Shimura's table. Theorem 1 gives a complete list of those Γ's and k's for which one knows explicitly from the above formula that $\Delta_B(\Gamma, k)$ is zero at only one elliptic point.

Since X_Γ is defined over F_Γ, and $\mathcal{S}_k(\Gamma) \cong H^0(X_{\Gamma/\mathbb{C}}, \underline{\omega}_k)$ where $\underline{\omega}_k$ is the sheaf of modular forms of weight k which is also rational over F_Γ, this cohomology group evidently admits an F_Γ structure $\mathcal{S}_k(\Gamma)_0$. We choose Δ_B to come from $\mathcal{S}_k(\Gamma)_0$. It is obviously a Hecke eigenform as $\dim \mathcal{S}_k(\Gamma) = 1$. ∎

In many cases, one can even prove that the modular forms nonvanishing outside the elliptic points generate the graded algebra

$$\mathcal{S}(\Gamma) = \bigcup_{k=0,\, k \text{ even}}^{\infty} \mathcal{S}_k(\Gamma).$$

Here are two examples.

THEOREM 2. (1) *Let Γ be the group with signature $(2,3,8)$. Let Δ_{12}, Δ_{16} be the generators of $\mathcal{S}_{12}(\Gamma)$, $\mathcal{S}_{16}(\Gamma)$ respectively, from Theorem* 1. *By* (2) *and* (3) *in Lemma* 1, *$\mathcal{S}_{30}(\Gamma)$ is one-dimensional and generated by a modular form Δ_{30}, whose divisor is $P_2 + P_8$. Then $\mathcal{S}(\Gamma) = \mathbb{C}[\Delta_{12}, \Delta_{16}, \Delta_{30}]$.*

(2) *Let Γ be the group with signature $(2,3,7)$. Let Δ_{12}, Δ_{28}, Δ_{42} be the generators of $\mathcal{S}_{12}(\Gamma)$, $\mathcal{S}_{28}(\Gamma)$, $\mathcal{S}_{42}(\Gamma)$ respectively, from Theorem* 1. *Then $\mathcal{S}(\Gamma) = \mathbb{C}[\Delta_{12}, \Delta_{28}, \Delta_{42}]$.*

P r o o f. We only prove (1). The proof of (2) is similar.

We use induction on k. For $k \le 30$, we can construct the following table from (2) and (3) in Lemma 1:

k	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30
$\dim \mathcal{S}_k(\Gamma)$	0	0	0	0	0	1	0	1	0	0	0	1	0	1	1
generator						Δ_{12}		Δ_{16}				Δ_{12}^2		$\Delta_{12}\Delta_{16}$	Δ_{30}

One sees that all $\mathcal{S}_k(\Gamma)$ for $k \le 30$ can be generated by Δ_{12}, Δ_{16} and Δ_{30}.

For $k > 30$, consider the map $\mathcal{S}_{k-16}(\Gamma) \to \mathcal{S}_k(\Gamma)$ with $f \in \mathcal{S}_{k-16}(\Gamma) \mapsto f\Delta_{16} \in \mathcal{S}_k(\Gamma)$. It is an isomorphism if and only if $\dim \mathcal{S}_k(\Gamma) = \dim \mathcal{S}_{k-16}(\Gamma)$.

By (2) in Lemma 1,

$$\begin{aligned}
\dim \mathcal{S}_k(\Gamma) &= \left[\frac{k}{4}\right] + \left[\frac{k}{3}\right] + \left[\frac{7k}{16}\right] - k + 1 \\
&= \left[\frac{k-16}{4}\right] + 4 + \left[\frac{k}{3}\right] + \left[\frac{7(k-16)}{16}\right] + 7 - (k-16) - 16 + 1 \\
&= \dim \mathcal{S}_{k-16}(\Gamma) + \left[\frac{k}{3}\right] - \left[\frac{k-16}{3}\right] - 5.
\end{aligned}$$

Since

$$\left[\frac{k}{3}\right] = \begin{cases} \left[\dfrac{k-16}{3}\right] + 6 & \text{if } 3 \mid k, \\ \left[\dfrac{k-16}{3}\right] + 5 & \text{otherwise,} \end{cases}$$

we can reduce k by 16 if $k \not\equiv 0 \pmod 3$. If $k \equiv 0 \pmod 3$, then $k \equiv 0 \pmod 6$. As $k > 30$, $k = 12l$ or $k = 30 + 12(l-2)$ for some $l > 2$. So

$\Delta_{12}^l \in \mathcal{S}_k(\Gamma)$ or $\Delta_{30}\Delta_{12}^{l-2} \in \mathcal{S}_k(\Gamma)$. Both are nonzero at P_3 since Δ_{12} and Δ_{30} are. Given any $f \in \mathcal{S}_k(\Gamma)$, we can choose suitable $c \in \mathbb{C}$ such that $g = f - c\Delta_{12}^l$ or $f - c\Delta_{30}\Delta_{12}^{l-2}$ vanishes at P_3. Then $g = h\Delta_{16}$ for some $h \in \mathcal{S}_{k-16}(\Gamma)$. Again k is reduced by 16. ∎

4. Analogs of $j(z)$. Let P, Q and R be the three elliptic points of Γ. In this section, we modify the Shimura canonical model to get a new parametrization j_B with a simple zero at P and a simple pole at Q, and such that it is integral at R.

For any CM point z, let K_z be the associated totally imaginary quadratic extension of F which can be F-linearly embedded into B. By Shimura's Main Theorem 1 (see [Sh1, p. 73]), $F_\Gamma(\phi(z)) = M_z$ is a finite abelian unramified extension of K_z. If the class number of K_z is 1, then $M_z = K_z$. Since P, Q, R are the fixed points of γ_1, γ_2, γ_3, they are CM points (see [Sh1, p. 66]). Let $M_\Gamma = M_P M_Q M_R$.

PROPOSITION 1. *There exists a modular function $j_B = j_B(\Gamma, k)$, rational over M_Γ, such that $\mathcal{M}(\Gamma)_0 \otimes_{F_\Gamma} M_\Gamma = M_\Gamma(j_B)$, $\operatorname{div}(j_B) = P - Q$, and $j_B(R)$ is integral* (*in M_Γ*).

Proof. As (X_Γ, ϕ) is the Shimura canonical model, ϕ gives a birational isomorphism of $\Gamma\backslash\mathcal{H}$ to $X_\Gamma(\mathbb{C})$. Therefore ϕ has a simple zero X and a simple pole Y which are both F_Γ-rational. From the above argument, our j_B can be obtained, up to a nonzero scalar in M_Γ, from ϕ via an automorphism of $\mathbb{P}^1$ over M_Γ which sends X, Y to P, Q respectively. Consequently, j_B is rational over M_Γ. For any CM point z, $j_B(z)$ will take values in $M_z M_\Gamma$. In particular, $j_B(R) \in M_\Gamma$. Now, we normalize j_B so that $j_B(R)$ is integral. ∎

REMARKS. 1. This property of j_B is an analog of the classical property of the j-function, namely: $j(\infty) = \infty$, $j(i) = 0$ and $j(\varrho) = 1728 \in \mathbb{Z}$.

2. Some explicit examples have been developed in the last section, where the class numbers of the relevant CM fields are 1, so $M_\Gamma = K_P K_Q K_R$, the compositum of the fields attached to P, Q, R.

3. If the three elliptic elements of Γ have distinct orders, then j_B is rational over F_Γ ([Sh1], (3.18.3)).

Furthermore, we can write out our j_B explicitly, up to a scalar, in terms of our Δ_B's in the following two cases.

THEOREM 3. (1) *Let Γ be the group with signature* $(2,3,8)$. *Then $\Delta_{12}^4/\Delta_{16}^3$ is a generator of $\mathcal{M}(\Gamma)$ whose divisor is supported at vertices.*

(2) *Let Γ be the group with signature* $(2,3,7)$. *Then $\Delta_{12}^7/\Delta_{42}^3$ is a generator of $\mathcal{M}(\Gamma)$ whose divisor is supported at vertices.*

The proof follows by computing the divisors: For (1), $\operatorname{div}(\Delta_{12}^4/\Delta_{16}^3) = 4(2P_8/8) - 3(P_3/3) = P_8 - P_3$. It has only a simple zero and a simple pole at vertices. (2) is similar.

5. The relative integrality result

THEOREM 4. *Fix any Γ and k in the table of Theorem 1. Let Δ_B be as in Section 3 with zeros only at Q and j_B as in Section 4. For $\alpha \in B^+$, set*

$$\phi_\alpha(z) = \left(\frac{\det(\xi(\alpha))}{j(\xi(\alpha), z)^2}\right)^k \frac{\Delta_B(\alpha z)}{\Delta_B(z)}.$$

Then ϕ_α is a modular function for $\Gamma_\alpha = \Gamma \cap \alpha^{-1}\Gamma\alpha$. Moreover, ϕ_α is integral over $M_\Gamma[j_B]$.

Proof. Let

$$\Delta_B|_\alpha = \left(\frac{\det(\xi(\alpha))}{j(\xi(\alpha), z)^2}\right)^k \Delta_B(\alpha z).$$

Straightforward computation gives

$$(\Delta_B|_\alpha)|_{\alpha^{-1}\gamma\alpha}(z) = \Delta_B|_\alpha(z).$$

Therefore $\phi_\alpha = \Delta_B|_\alpha/\Delta_B$ is invariant under Γ_α. Also it is easy to verify that $\phi_{\gamma\alpha}(z) = \phi_\alpha(z)$ and $\phi_{\alpha\gamma}(z) = \phi_\alpha(\gamma z) = \phi_\alpha|_\gamma(z)$.

Now, let $\Gamma\alpha\Gamma = \bigcup_{i=1}^r \Gamma\alpha_i$ be a disjoint union of right cosets, and ψ be any elementary symmetric function of $\{\phi_{\alpha_i} : i = 1, \ldots, r\}$. Then ϕ_{α_i} depends only on the right coset where α_i lies and $\{\phi_{\alpha_i}|_\gamma : i = 1, \ldots, r\}$ is just a permutation of $\{\phi_{\alpha_i} : i = 1, \ldots, r\}$ for any $\gamma \in \Gamma$. So $\psi|_\gamma = \psi$. Consequently, $\psi \in \mathbb{C}(j_B)$.

Assume $\psi = f(j_B)/g(j_B)$ where f, g are relatively prime polynomials; then ψ has a pole at any point z such that $j_B(z)$ is a root of g. Since ϕ_{α_i} (hence ψ) has poles only at points Γ-equivalent to Q and $j_B(Q) = \infty$, g must be a constant, i.e. $\psi \in \mathbb{C}[j_B]$.

Since Δ_B is F_Γ rational, the map $\widetilde{\alpha} : \mathcal{S}_k(\Gamma)_0 \to \mathcal{S}_k(\alpha^{-1}\Gamma\alpha)_0$ with $\Delta_B \mapsto \Delta_B|_\alpha$ is defined over F_Γ from the theory of canonical models. Therefore $\psi \in M_\Gamma[j_B]$. Hence ϕ_α is a root of the monic polynomial $\prod_{i=1}^r (x - \phi_{\alpha_i}) \in M_\Gamma[j_B][x]$. ∎

COROLLARY. *For each α as above, there exists a nonzero $\beta_\alpha \in O_{M_\Gamma}$ such that for any CM point z, $\beta_\alpha\phi_\alpha(z)$ is an algebraic integer whenever $j_B(z)$ is. In particular, $\beta_\alpha\phi_\alpha(z)$ is integral in $M_\Gamma K(z)^{\mathrm{ab}}$.*

Proof. Assume ϕ_α is a root of the polynomial

$$\beta_\alpha x^n + a_{n-1}(j_B)x^{n-1} + \ldots + a_i(j_B)x^i + \ldots + a_0(j_B)$$

with $0 \neq \beta_\alpha \in O_{M_\Gamma}, a_i(j_B) \in O_{M_\Gamma}[j_B]$ for $i = 0, \ldots, n-1$; then one can check that $\beta_\alpha \phi_\alpha$ is a root of the polynomial

$$x^n + a_{n-1}(j_B)x^{n-1} + \ldots + \beta_\alpha^{n-1-i} a_i(j_B)x^i + \ldots + \beta_\alpha^{n-1} a_0(j_B).$$

Evaluate the polynomial at a CM point z. If $j_B(z)$ is an algebraic integer, then it is a monic polynomial with integral coefficients. Therefore $\beta_\alpha \phi_\alpha(z)$ is an algebraic integer. ∎

6. Examples. In this section, we will give two examples of arithmetic triangular groups and find the relationship between the standard parametrizations.

Proposition 2. *Let $F = \mathbb{Q}(\sqrt{2})$ and $B = F + Fi + Fj + Fk$ where $i^2 = -3$, $j^2 = \sqrt{2}$ and $k = ij = -ji$. Let*

$$x = \frac{1+i}{2}, \quad y = \frac{\sqrt{2}-1}{2} + \frac{(\sqrt{2}-1)i}{6} + \frac{j}{2} + \frac{k}{2}, \quad z = \frac{j}{2} + \frac{k}{2},$$
$$O = \mathbb{Z}[\sqrt{2}][1, x, y, z].$$

Then O is a maximal order of B.

Proof. We first prove that every element in O is an integer by showing the integrality of its reduced trace and reduced norm ([Ji], Chap. 5). The fact that O is a ring follows from the

Multiplication table

	1	x	y	z
1	1	x	y	z
x	x	$x-1$	$(\sqrt{2}-1)x - y + z$	$(\sqrt{2}-1) + (\sqrt{2}-1)x - 3y + 2z$
y	y	$-(\sqrt{2}-1) + 2y - z$	$(\sqrt{2}-1)y + (\sqrt{2}-1)$	$1 - (\sqrt{2}-1)x - (\sqrt{2}-1)y + (\sqrt{2}-1)z$
z	z	$-(\sqrt{2}-1) - (\sqrt{2}-1)x + 3y - z$	$(\sqrt{2}-1) + (\sqrt{2}-1)x + (\sqrt{2}-1)y$	2

Finally, the maximality of O can be verified by showing that the reduced discriminant of O, $\mathrm{disc}(O)$, is $\sqrt{2}$, the only finite prime of F which ramifies in B ([Vi], Corollary 5.3, p. 94). ∎

We consider the two groups $\Gamma^* = \{\gamma \in B^+ : \gamma O = O\gamma\}$ and $\Gamma = \{\gamma \in B^+ : N_{B/F}(\gamma) = 1\}$. We first give them an explicit description.

Let $K = \mathbb{Q}(\sqrt[4]{2})$, a real quadratic extension of F. Fix an embedding

$$B \hookrightarrow M_2(K) \hookrightarrow M_2(\mathbb{R})$$

with

$$i \mapsto \begin{pmatrix} 0 & 1 \\ -3 & 0 \end{pmatrix}, \quad j \mapsto \begin{pmatrix} \sqrt[4]{2} & 0 \\ 0 & -\sqrt[4]{2} \end{pmatrix}, \quad k \mapsto \begin{pmatrix} 0 & -\sqrt[4]{2} \\ -3\sqrt[4]{2} & 0 \end{pmatrix}.$$

Then

$$x \mapsto \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}, \quad y \mapsto \begin{pmatrix} \frac{\sqrt{2}-1+\sqrt[4]{2}}{2} & \frac{\sqrt{2}-1-\sqrt[4]{2}}{6} \\ -\frac{\sqrt{2}-1+\sqrt[4]{2}}{2} & \frac{\sqrt{2}-1-\sqrt[4]{2}}{2} \end{pmatrix},$$

$$z \mapsto \begin{pmatrix} \frac{\sqrt[4]{2}}{2} & -\frac{\sqrt[4]{2}}{2} \\ -\frac{3\sqrt[4]{2}}{2} & -\frac{\sqrt[4]{2}}{2} \end{pmatrix}.$$

Identifying B with its image in $M_2(K)$, let

$$\eta_1 = x = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}, \quad \eta_2 = 1 + x + y = \begin{pmatrix} \frac{2+\sqrt{2}+\sqrt[4]{2}}{2} & \frac{2+\sqrt{2}-\sqrt[4]{2}}{6} \\ -\frac{2+\sqrt{2}+\sqrt[4]{2}}{2} & \frac{2+\sqrt{2}-\sqrt[4]{2}}{2} \end{pmatrix},$$

$$\eta_3 = \eta_1\eta_2 = \begin{pmatrix} 0 & \frac{2+\sqrt{2}-\sqrt[4]{2}}{3} \\ -(2+\sqrt{2}+\sqrt[4]{2}) & 0 \end{pmatrix},$$

$$\gamma_1 = \eta_1 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}, \quad \gamma_2 = \frac{1}{2+\sqrt{2}}\,\eta_2^2 = \begin{pmatrix} \frac{\sqrt{2}+\sqrt[4]{2}}{2} & \frac{2+\sqrt{2}-\sqrt[4]{2}}{6} \\ -\frac{2+\sqrt{2}+\sqrt[4]{2}}{2} & \frac{\sqrt{2}-\sqrt[4]{2}}{2} \end{pmatrix},$$

$$\gamma_3 = \gamma_1\gamma_2 = \begin{pmatrix} -\frac{1}{2} & \frac{2\sqrt{2}+1-2\sqrt[4]{2}}{6} \\ -\frac{2\sqrt{2}+1+2\sqrt[4]{2}}{2} & -\frac{1}{2} \end{pmatrix}.$$

Then as an element of the group $\operatorname{Aut}\mathcal{H}$ of all analytic automorphisms on $\mathcal{H}$, $\eta_1^3 = \eta_2^8 = \eta_3^2 = \eta_1\eta_2\eta_3 = 1$ and $\gamma_1^3 = \gamma_2^4 = \gamma_3^3 = \gamma_1\gamma_2\gamma_3 = 1$. It is easy to check that $\Gamma^* = \langle \eta_1, \eta_2, \eta_3 \rangle$, $\Gamma = \langle \gamma_1, \gamma_2, \gamma_3 \rangle$, and $[\Gamma^* : \Gamma] = 2$ with $\Gamma^* = \Gamma \cup \Gamma\eta_2 = \Gamma \cup \Gamma\eta_3$. Fundamental domains of Γ^* and Γ are shown in Figure 1, where Q_1, Q_2, Q_3 and P_1, P_2, P_3 denote the fixed points of η_1, η_2, η_3 and $\gamma_1, \gamma_2, \gamma_3$ respectively.

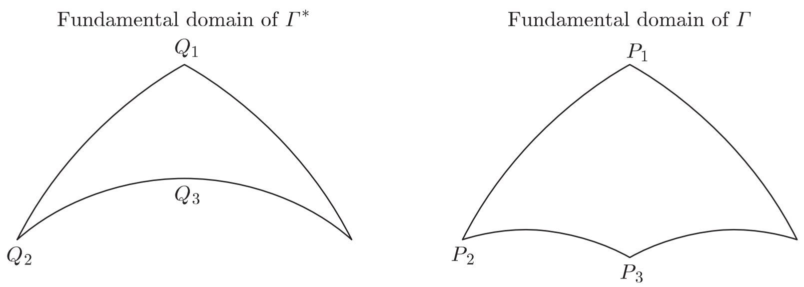

Fig. 1

From Section 4 there are parametrizations for Γ^* and Γ with only a simple zero at one elliptic point and a simple pole at another one. We normalize them as follows:

- j_B : the modular function of Γ^* with
$$\operatorname{div}(j_B) = (Q_1) - (Q_2) \quad \text{and} \quad j_B(Q_3) = 1.$$
- λ_B: the modular function of Γ with
$$\operatorname{div}(\lambda_B) = (P_1) - (P_3) \quad \text{and} \quad \lambda_B(P_2) = 1.$$

PROPOSITION 3. *Let F, j_B and λ_B be as above. Then j_B is rational over F and λ is rational over $M_\Gamma = \mathbb{Q}(\sqrt{2}, \sqrt{3}, i)$.*

Proof. Since the generators η_1, η_2 and η_3 have different orders and the class number of F is 1, it follows from Remark 3 in Section 4 that j_B is rational over F.

To find M_Γ, notice that the characteristic polynomials for $\gamma_1, \gamma_2, \gamma_3$ are
$$P_{\gamma_1}(x) = x^2 - x + 1, \qquad P_{\gamma_2}(x) = x^2 - \sqrt{2}x + 1, \qquad P_{\gamma_3}(x) = x^2 = x + 1.$$
Therefore
$$K_{P_1} = F(\sqrt{3}i) = \mathbb{Q}(\sqrt{2}, \sqrt{3}i), \qquad K_{P_2} = F(\sqrt{2}i) = \mathbb{Q}(\sqrt{2}, i), \qquad K_{P_3} = K_{P_1}.$$
Using the software tool "Pari", one knows the class numbers of K_{P_1}, K_{P_2} are both 1. By Remark 2 in Section 4, we have $M_\Gamma = K_{P_1}K_{P_2} = F(\sqrt{3}i) = \mathbb{Q}(\sqrt{2}, \sqrt{3}, i)$. ∎

In the classical case, for the canonical level 2 modular function
$$\lambda : \Gamma(2)\backslash\mathcal{H}^* \to \mathbb{P}^1(\mathbb{C}),$$
the map from the λ-line to the j-line is given by
$$j(\lambda) = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}.$$
We also have this kind of result for Γ^* and Γ.

THEOREM 5. *Let j_B, λ_B be as above. Then $j_B = -4\lambda_B/(1 - \lambda_B)^2$.*

Proof. It is easy to check
$$Q_1 = P_1 = \eta_3 P_3 = \frac{\sqrt{3}}{3}i, \qquad Q_2 = P_2 = \frac{-\sqrt[4]{2} + \sqrt{2}i}{2 + \sqrt{2} + \sqrt[4]{2}},$$
$$Q_3 = \frac{\sqrt{2 + \sqrt{2}}i}{2 + \sqrt{2} + \sqrt[4]{2}}, \qquad P_3 = \eta_3 P_1 = \frac{\sqrt{3}i}{2\sqrt{2} + 1 + 2\sqrt[4]{2}}.$$
Denote by $[A]^*$ ($[A]$) the Γ^*-equivalent (Γ-equivalent) class represented by A. Let P_r be the natural projection
$$\Gamma\backslash\mathcal{H} \xrightarrow{P_r} \Gamma^*\backslash\mathcal{H}.$$

One sees $P_r^{-1}\{[Q_1]^*\} = \{[P_1],[P_3]\}$, $P_r^{-1}\{[Q_2]^*\} = \{[Q_2]\}$, $P_r^{-1}\{[Q_3]^*\} = \{[Q_3]\}$.

Noticing that $\lambda_B|_{\eta_3} \in \mathcal{M}(\Gamma)$ and

$$\begin{aligned}
\lambda_B|_{\eta_3}(P_1) &= \lambda_B(\eta_3 P_1) = \lambda_B(P_3) = \infty,\\
\lambda_B|_{\eta_3}(P_3) &= \lambda_B(\eta_3 P_3) = \lambda_B(P_1) = 0,\\
\lambda_B|_{\eta_3}(P_2) &= \lambda_B|_{\eta_2}(P_2) = \lambda_B(\eta_2 P_2) = \lambda_B(P_2) = 1,
\end{aligned}$$

we have $\lambda_B|_{\eta_3} = 1/\lambda_B$.

Now look at $1/(1-\lambda_B) \in \mathcal{M}(\Gamma)$. We have

$$\frac{1}{1-\lambda_B}(P_1) = 1, \quad \frac{1}{1-\lambda_B}(P_3) = 0, \quad \frac{1}{1-\lambda_B}(P_2) = \infty,$$
$$\left(\frac{1}{1-\lambda_B}\right)\bigg|_{\eta_3}(P_1) = \frac{1}{1-\lambda_B}(\eta_3 P_1) = \frac{1}{1-\lambda_B}(P_3) = 0,$$
$$\left(\frac{1}{1-\lambda_B}\right)\bigg|_{\eta_3}(P_2) = \left(\frac{1}{1-\lambda_B}\right)\bigg|_{\eta_2}(P_2) = \frac{1}{1-\lambda_B}(\eta_2 P_2) = \frac{1}{1-\lambda_B}(P_2) = \infty.$$

Hence as a modular function of Γ

$$\operatorname{div}\left(\frac{1}{1-\lambda_B}\left(\frac{1}{1-\lambda_B}\right)\bigg|_{\eta_3}\right) = ([P_1]) + ([P_3]) - 2([P_2]).$$

Viewing it as a modular function of Γ^*,

$$\operatorname{div}\left(\frac{1}{1-\lambda_B}\left(\frac{1}{1-\lambda_B}\right)\bigg|_{\eta_3}\right) = ([Q_1]^*) - ([Q_2]^*) = \operatorname{div}(j_B),$$

therefore up to a scalar multiplication, it can be identified with j_B as a modular function of Γ^*.

Since

$$\left(\frac{1}{1-\lambda_B}\right)\bigg|_{\eta_3} = \frac{1}{1-\lambda_B|_{\eta_3}} = \frac{1}{1-1/\lambda_B} = \frac{\lambda_B}{\lambda_B - 1},$$

we have

$$-\frac{\lambda_B}{(1-\lambda_B)^2} = C j_B$$

for some nonzero constant C.

Observe that $\lambda_B^2(Q_3) = \lambda_B(Q_3)\lambda_B(\eta_3 Q_3) = \lambda_B(Q_3)\lambda_B|_{\eta_3}(Q_3) = 1$, so $\lambda_B(Q_3) = \pm 1$.

Since $\lambda_B(P_2) = 1$, P_2 and Q_3 are not Γ-equivalent, $\lambda_B(Q_3) = -1$. Combining this with the fact that $j_B(Q_3) = 1$, we conclude $C = 1/4$. So $-\lambda_B/(1-\lambda_B)^2 = j_B/4$, i.e. $j_B = -4\lambda_B/(1-\lambda_B)^2$. ∎

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