ACTA ARITHMETICA
LXXXV.4 (1998)

# On the average number of unitary factors of finite abelian groups

by

Wenguang Zhai (Jinan) and Xiaodong Cao (Beijing)

**1. Introduction.** Let $t(G)$ denote the number of unitary factors of $G$ and

$$T(x) = \sum t(G),$$

where the summation is taken over all abelian groups of order not exceeding $x$. It was first proved by Cohen [2] that

$$T(x) = c_1 x(\log x + 2\gamma - 1) + c_2 x + E_0(x) \tag{1.1}$$

with $E_0(x) \ll \sqrt{x}$. Krätzel [3] proved that

$$E_0(x) = c_3\sqrt{x} + E_1(x) \quad \text{with} \quad E_1(x) \ll x^{11/29}\log^2 x.$$

Let $\theta$ denote the smallest $\alpha$ such that

$$E_1(x) \ll x^{\alpha+\varepsilon}. \tag{1.2}$$

Then the exponents $\theta \le 31/82, 3/8, 77/208$ were obtained by H. Menzer [7], P. G. Schmidt [8], H. Q. Liu [6], respectively.

The aim of this paper is to further improve the exponent 77/208. We have the following Theorem 1.

Theorem 1. *$\theta \le 9/25$.*

Following Krätzel [3], we only need to study the asymptotic behavior of the divisor function $d(1,1,2;n)$ which is defined by

$$d(1,1,2;n) = \sum_{n_1 n_2 n_3^2 = n} 1.$$

Let $\Delta(1,1,2;x)$ denote the error term of the summation function

$$D(1,1,2;x) = \sum_{n\le x} d(1,1,2;n).$$

1991 *Mathematics Subject Classification*: Primary 11N37.

We then have

THEOREM 2. $\Delta(1,1,2;x)=O(x^{9/25+\varepsilon})$.

Theorem 1 immediately follows from Theorem 2.

*Notations*. $e(t)=\exp(2\pi it)$; $n\sim N$ means $c_1N<n<c_2N$ for some absolute constants $c_1$ and $c_2$; $\varepsilon$ is a sufficiently small number which may be different at each occurrence; $\Delta(t)$ always denotes the error term of the Dirichlet divisor problem; $\psi(t)=t-[t]-1/2$.

**2. A non-symmetric expression of $\Delta(1,1,2;x)$.** In this paper we shall use a non-symmetric expression of $\Delta(1,1,2;x)$ instead of its symmetric expression used in previous papers. This is the following lemma.

BASIC LEMMA. *We have*

$$\Delta(1,1,2;x)=\sum_{m\le x^{1/3}}\Delta\left(\frac{x}{m^2}\right)+O(x^{1/3}\log x).$$

Proof. We only sketch the proof since it is elementary and direct. We begin with

$$\begin{aligned}D(1,1,2;x)&=\sum_{n\le x}d(1,1,2;x)=\sum_{n_1n_2n_3^2\le x}1=\sum_{m^2n\le x}d(n)\\&=\sum_{n\le x^{1/3}}d(n)[\sqrt{x/n}]+\sum_{n\le x^{1/3}}D\left(\frac{x}{m^2}\right)-[x^{1/2}]D(x^{1/3}),\end{aligned}$$

where $D(x)=\sum_{n\le x}d(n)$. We apply the well-known abelian partial summation formula

$$\sum_{n\le u}d(n)f(n)=D(u)f(u)-\int_1^u D(t)f'(t)\,dt$$

to the first sum in the above expression and the Euler–Maclaurin summation formula

$$\sum_{n\le x}f(n)=\int_1^u f(t)\,dt+\frac{f(1)}{2}-\psi(u)f(u)+\int_1^u D(t)f'(t)\,dt$$

to the second sum, and combine

$$D(u)=u\log u+(2\gamma-1)u+\Delta(u)$$

with $\Delta(u)\ll u^{1/3}$ to get

$$D(1,1,2;x)=\text{main terms}+\sum_{m\le x^{1/3}}\Delta\left(\frac{x}{m^2}\right)$$

$$
\begin{aligned}
&- \sum_{n\le x^{1/3}} d(n)\psi(\sqrt{x/n}) + O(x^{1/3})\\
&= \text{main terms} + \sum_{m\le x^{1/3}} \Delta\left(\frac{x}{m^2}\right) + O(x^{1/3}\log x),
\end{aligned}
$$

whence our lemma follows.

## 3. Some preliminary lemmas

LEMMA 1. *Suppose* $0 < c_1\lambda \le |f'(x)| \le c_2\lambda$ *and* $|f''(x)| \sim \lambda N^{-1}$ *for* $N \le n \le cN$. *Then*

$$
\sum_{a<n\le cN} e(f(n)) \ll (\lambda N)^{1/2} + \lambda^{-1}.
$$

LEMMA 2 (see [4]). *Suppose* $f(x)$ *and* $g(x)$ *are algebraic functions in* $[a,b]$ *and*

$$
\begin{gathered}
|f''(x)| \sim R^{-1}, \qquad |f'''(x)| \ll (RU)^{-1},\\
|g(x)| \ll G, \qquad |g'(x)| \ll GU_1^{-1}, \qquad U, U_1 \ge 1.
\end{gathered}
$$

*Then*

$$
\begin{aligned}
\sum_{a<n\le b} g(n)e(f(n)) = {}& \sum_{\alpha<u\le\beta} b_u \frac{g(n_u)}{\sqrt{|f''(n_u)|}} e(f(n_u) - un_u + 1/8)\\
&+ O(G\log(\beta-\alpha+2) + G(b-a+R)(U^{-1}+U_1^{-1}))\\
&+ O\left(G\min\left(\sqrt{R}, \frac{1}{\langle\alpha\rangle}\right) + G\min\left(\sqrt{R}, \frac{1}{\langle\beta\rangle}\right)\right),
\end{aligned}
$$

*where* $[\alpha,\beta]$ *is the image of* $[a,b]$ *under the mapping* $y = f'(x)$, $n_u$ *is the solution of the equation* $f'(x) = u$,

$$
b_u = \begin{cases} 1 & \text{for } \alpha < u < \beta,\\ 1/2 & \text{for } u = \alpha \in \mathbb{Z} \text{ or } u = \beta \in \mathbb{Z}; \end{cases}
$$

*and the function* $\langle t\rangle$ *is defined as follows*:

$$
\langle t\rangle = \begin{cases} \|t\| & \text{if } t \text{ is not an integer},\\ \beta-\alpha & \text{otherwise}, \end{cases}
$$

*where* $\|t\| = \min_{n\in\mathbb{Z}}\{|t-n|\}$.

LEMMA 3 (see [5]). *Let* $H \ge 1$, $X \ge 1$, $Y \ge 1000$; *let* $\alpha, \beta$ *and* $\gamma$ *be real numbers such that* $\alpha\gamma(\gamma-1)(\beta-1) \ne 0$, *and* $A > C(\alpha,\beta,\gamma) > 0$, $f(h,x,y) = Ah^\alpha x^\beta y^\gamma$. *Define*

$$
S(H,X,Y) = \sum_{(h,x,y)\in D} c_1(h,x)c_2(y)e(f(h,x,y)),
$$

*where $D$ is a region contained in the rectangle*

$$\{(h,x,y) \mid h\sim H,\ x\sim X,\ y\sim Y\}$$

*such that for any fixed pair $(h_0,x_0)$, the intersection $D\cap\{(h_0,x_0,y)\mid y\sim Y\}$ has at most $O(1)$ segments. Also, suppose*

$$|c_1(h,x)|\le 1,\qquad |c_2(y)|\le 1,\qquad F=AH^\alpha X^\beta Y^\gamma \gg Y.$$

*Then*

$$\begin{aligned}L^{-3}S(H,X,Y) \ll (HX)^{19/22}Y^{13/22}F^{3/22} + HXY^{5/8}(1+Y^7F^{-4})^{1/16}\\ + (HX)^{29/32}Y^{28/32}F^{-2/32}M^{5/32} + Y(HX)^{3/4}M^{1/4},\end{aligned}$$

*where $L=\log(AHXY+2)$, $M=\max(1,FY^{-2})$.*

LEMMA 4 (see [4]). *Let $M>0,\ N>0,\ u_m>0,\ v_n>0,\ A_m>0,\ B_n>0$ $(1\le m\le M,\ 1\le n\le N)$, and let $Q_1$ and $Q_2$ be given non-negative numbers with $Q_1\le Q_2$. Then there is a $q$ such that $Q_1\le q\le Q_2$ and*

$$\begin{aligned}\sum_{m=1}^{M}A_mq^{u_m}+\sum_{n=1}^{N}B_nq^{-v_n} \ll \sum_{m=1}^{M}\sum_{n=1}^{N}(A_m^{v_n}B_n^{u_m})^{1/(u_m+v_n)}\\ + \sum_{m=1}^{M}A_mQ_1^{u_m}+\sum_{n=1}^{N}B_nQ_2^{-v_n}.\end{aligned}$$

LEMMA 5. *Suppose $X$ and $Y$ are large positive numbers, $A>0$, $\alpha$ and $\beta$ are rational numbers (not non-negative integers). Suppose $D$ is a subdomain of $\{(x,y)\mid x\sim X,\ y\sim Y\}$ embraced by $O(1)$ algebraic curves, and $F=AX^\alpha Y^\beta \gg Y$, $|a(x)|\le 1$, $|b(y)|\le 1$. Then*

$$\begin{aligned}S&=\sum_{(x,y)\in D}a_xb_ye(Ax^\alpha y^\beta)\\ &\ll (XY^{1/2}+F^{4/20}X^{13/20}Y^{15/20}+F^{4/23}X^{15/23}Y^{18/23}+F^{1/6}X^{2/3}Y^{7/9}\\ &\quad + F^{1/5}X^{3/5}Y^{4/5}+F^{1/10}X^{4/5}Y^{7/10})\log^4 F.\end{aligned}$$

Proof. This is Theorem 3 of the old version of our paper [9]. The procedure of the proof is the same as Theorem 2 of [1]. The difference lies in that we use Lemma 4 above three times to choose parameters optimally and in the last step the exponent pair $(1/2,1/2)$ is used.

**4. Proof of Theorem 2.** By our Basic Lemma, we only need to prove that for fixed $1\le M\le x^{1/3}/2$, we have

$$S(M)=\sum_{m\sim M}\Delta\left(\frac{x}{m^2}\right)\ll x^{9/25+\varepsilon}. \tag{4.1}$$

CASE 1: $M \ll x^{1/5}$. By the well-known Voronoï formula, we have

$$S(M) = \sum_{m\sim M} \frac{x^{1/4}}{m^{1/2}} \sum_{n\le x^{7/25}} \frac{d(n)}{n^{3/4}} \cos\left(\frac{4\pi\sqrt{nx}}{m} - \frac{\pi}{4}\right) + O(x^{9/25+\varepsilon}).$$

Hence for some $1 \ll N \ll x^{7/25}$, we have

$$x^{-\varepsilon} S(M) \ll \left| \sum_{m\sim M} \frac{x^{1/4}}{m^{1/2}} \sum_{n\sim N} \frac{d(n)}{n^{3/4}} e\left(\frac{2\sqrt{nx}}{m}\right)\right| + x^{9/25}. \tag{4.2}$$

So it suffices to estimate the sum on the right side of (4.2), denoted by $S(M,N)$.

Let $a_m = M^{1/2}m^{-1/2}$, $b_n = d(n)N^{3/4-\varepsilon}n^{-3/4}$. Then obviously

$$x^{-\varepsilon} S(M,N) \ll x^{1/4}M^{-1/2}N^{-3/4}\left|\sum_{m\sim M}\sum_{n\sim N} a_m b_n e\left(\frac{2\sqrt{nx}}{m}\right)\right|. \tag{4.3}$$

We suppose $x^{1/20} \ll M \ll x^{1/5}$. For $M \ll x^{1/20}$, we have $S(M) \ll x^{0.35}$ by the trivial estimate $\Delta(t) \ll t^{1/3}$.

Let $T(M,N)$ denote the two-dimensional sum on the right side of (4.3). If $N \ge M$, we use Lemma 5 to bound $T(M,N)$ (take $(X,Y) = (N,M)$) and we get

$$\begin{aligned} x^{-\varepsilon} T(M,N) \ll{}& NM^{1/2} + x^{2/20}N^{15/20}M^{11/20} \\ &+ x^{2/23}N^{17/23}M^{14/23} + x^{1/12}N^{3/4}M^{11/18} \\ &+ x^{1/10}N^{7/10}M^{3/5} + x^{1/20}N^{17/20}M^{6/10}. \end{aligned} \tag{4.4}$$

Inserting (4.4) into (4.3) we have

$$\begin{aligned} x^{-\varepsilon} S(M,N) &\ll (Nx)^{1/4} + x^{7/20}M^{1/20} + x^{31/92}N^{-1/92}M^{5/46} \\ &\quad + x^{1/3}M^{1/9} + x^{7/20}N^{-1/20}M^{1/10} \\ &\quad + x^{3/10}N^{1/10}M^{1/10} \\ &\ll x^{9/25}, \end{aligned} \tag{4.5}$$

where $N \ge M$ and $M \ll x^{1/5}$ were used.

If $x^{2/25}M^{1/2} \ll N < M$, we again use Lemma 5 to bound $T(M,N)$ (whence $S(M,N)$), but this time we take $(X,Y) = (M,N)$, and we get

$$\begin{aligned} x^{-\varepsilon} S(M,N) &\ll x^{1/3}N^{1/12} + x^{7/20}N^{2/20}M^{-1/20} \\ &\quad + x^{31/92}N^{11/92}M^{-1/46} + x^{3/10}M^{1/5} \\ &\quad + x^{7/20}N^{3/20}M^{-1/10} + x^{1/4}N^{-1/4}M^{1/2} \\ &\ll x^{9/25}, \end{aligned} \tag{4.6}$$

where $N < M$ and $M \ll x^{1/5}$ were used.

If $N \ll x^{2/25}M^{1/2}$, we use the exponent pair $(1/6, 4/6)$ to estimate the sum over $m$ and estimate the sum over $n$ trivially to get

$$x^{-\varepsilon}S(M,N) \ll x^{9/25}. \tag{4.7}$$

Case 2: $x^{1/5} \ll M \ll x^{13/60}$. By the well-known expression

$$\Delta(t) = -2\sum_{n\le t^{1/2}} \psi\left(\frac{t}{n}\right) + O(1),$$

we have

$$S(M) = -2\sum_{m\sim M}\ \sum_{nm\le x^{1/2}} \psi\left(\frac{x}{nm^2}\right) + O(x^{1/3}). \tag{4.8}$$

So it suffices to bound the sum

$$S_0(M,N;x) = \sum_{(m,n)\in D} \psi\left(\frac{x}{nm^2}\right),$$

where $D = \{(m,n) \mid m\sim M,\ n\sim N,\ nm \le x^{1/2}\}$.

By the well-known finite Fourier expansion of $\psi(t)$ we have

$$\begin{aligned} S_0(M,N;x) &\ll \frac{MN}{J} + \sum_{h\le J} h^{-1}\Big|\sum_{(m,n)\in D} e\left(\frac{hx}{nm^2}\right)\Big| \\ &\ll \frac{MN}{J} + \sum_{H}\sum_{h\sim H} H^{-1}\Big|\sum_{(m,n)\in D} e\left(\frac{hx}{nm^2}\right)\Big|, \end{aligned} \tag{4.9}$$

where $H$ runs through $\{2^j \mid 0 \le j \le \log J/\log 2\}$. So it suffices to bound

$$\Phi(H,M,N) = \sum_{h\sim H} H^{-1}\Big|\sum_{(m,n)\in D} e\left(\frac{hx}{nm^2}\right)\Big|.$$

By Lemma 2 (for details see Liu [4]) we get

$$\begin{aligned} x^{-\varepsilon}\Phi(H,M,N) \ll{}& \frac{N}{H^{3/2}F^{1/2}}\sum_{h\sim H}\Big|\sum_{(m,r)\in D_1} c(m)b(r)e\left(\frac{2\sqrt{rhx}}{m}\right)\Big| \\ &+ (HF)^{1/2} + x^{1/3}, \end{aligned} \tag{4.10}$$

where $F = x/(NM^2)$, $D_1$ is a subdomain of $\{(m,r) \mid m\sim M,\ r\sim HFN^{-1}\}$, $|c(m)|\le 1$, $|b(r)|\le 1$.

Now using Lemma 3 to estimate the sum in (4.10) we get (take $(h,x,y) = (h,r,m)$)

$$\begin{aligned} x^{-\varepsilon}\Phi(H,M,N) \ll{}& H^{8/22}F^{11/22}N^{3/22}M^{13/22} + H^{1/2}F^{1/2}M^{5/8} \\ &+ H^{4/16}F^{4/16}M^{17/16} + H^{8/32}F^{11/32}N^{3/32}M^{28/32} \\ &+ H^{13/32}F^{16/32}N^{3/32}M^{18/32} + F^{1/4}N^{1/4}M \\ &+ H^{1/4}F^{2/4}N^{1/4}M^{2/4} + x^{1/3}. \end{aligned} \tag{4.11}$$

Insert (4.11) into (4.9) and then choose a best $J \in (0, x^{1/2})$. Via Lemma 4 we get

$$\begin{aligned}(4.12)\qquad x^{-\varepsilon}S_0(M,N;x) \ll\ & F^{11/30}N^{11/30}M^{21/30} + F^{8/24}N^{8/24}M^{18/24}\\ & + F^{4/20}N^{4/20}M^{21/20} + F^{11/40}N^{11/40}M^{36/40}\\ & + F^{16/45}N^{16/45}M^{31/45} + F^{2/5}N^{2/5}M^{3/5}\\ & + F^{1/4}N^{1/4}M + x^{1/3}.\end{aligned}$$

Now if we notice that $F = x/(NM^2)$, $MN \ll x^{1/2}$ and $x^{1/5} \ll M \ll x^{13/60}$ we obtain

$$\begin{aligned}(4.13)\qquad x^{-\varepsilon}S_0(M,N;x) &\ll x^{11/30}M^{-1/30} + x^{1/3}M^{1/12} + x^{4/20}M^{13/20}\\ &\quad + x^{11/40}M^{14/40} + x^{16/45}M^{-1/45}\\ &\quad + x^{2/5}M^{-1/5} + x^{1/4}M^{1/2} + x^{1/3}\\ &\ll x^{9/25},\end{aligned}$$

whence (4.1) is true in this case.

Case 3: $x^{13/60} \ll M \ll x^{1/3}$. We use notations of Case 1. Applying Lemma 1 to the sum over $m$ we get

$$x^{-\varepsilon}S(M,N) \ll (xN)^{1/2}M^{-1} + x^{1/3} \ll x^{9/25}$$

if $N \ll M^2x^{-7/25}$.

Now suppose $N \gg M^2x^{-7/25}$. Applying Lemma 2 to the variable $m$ (we omit the routine details which can be found in Liu [4]) we get

$$\begin{aligned}(4.14)\qquad x^{-\varepsilon}S(M,N) \ll\ & \frac{M}{N}\Big|\sum_{n\sim N}\ \sum_{u\sim\sqrt{nx}M^{-2}} c(n)b(u)e(2\sqrt{2}\,u^{1/2}(nx)^{1/4})\Big|\\ & + x^{1/3},\end{aligned}$$

where $c(n) \ll 1$, $b(u) \ll 1$. We apply Lemma 5 to the sum on the right side of (4.14) to get (take $(X,Y) = (N, F/M)$)

$$\begin{aligned}(4.15)\qquad x^{-\varepsilon}S(M,N) &\ll (xN)^{1/4} + x^{19/40}N^{5/40}M^{-14/20}\\ &\quad + x^{11/23}N^{3/23}M^{-17/23} + x^{17/36}N^{5/36}M^{-13/18}\\ &\quad + x^{5/10}N^{1/10}M^{-4/5} + x^{8/20}N^{4/20}M^{-1/2}\\ &\ll x^{43/120} \ll x^{9/25},\end{aligned}$$

where $N \ll x^{7/25}$ and $M \gg x^{13/60}$ were used.

From our discussions we know (4.1) is true in any case and Theorem 2 follows.

**Acknowledgements.** The authors thank Prof. Pan Changdong for his kind encouragement and the referee for his valuable suggestions.

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Department of Mathematics
Shandong Normal University
Jinan, Shandong 250014
P.R. China
E-mail: arith@sdunetnms.sdu.edu.cn

Beijing Institute
of Petrochemical Technology
Daxing, Beijing 102600
P.R. China
E-mail: biptiao@info.iuol.cn.net



*Received on 29.7.1996*
*and in revised form on 15.7.1997* (3030)