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Application of complex analysis to second order equations of mixed type

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Abstract. This paper deals with an application of complex analysis to second order equations of mixed type. We mainly discuss the discontinuous Poincaré boundary value problem for a second order linear equation of mixed (elliptic-hyperbolic) type, i.e. the generalized Lavrent'ev–Bitsadze equation with weak conditions, using the methods of complex analysis. We first give a representation of solutions for the above boundary value problem, and then give solvability conditions via the Fredholm theorem for integral equations. In [1], [2], the Dirichlet problem (Tricomi problem) for the mixed equation of second order $u_{xx} + \operatorname{sgn} y\, u_{yy} = 0$ was investigated. In [3], the Tricomi problem for the generalized Lavrent'ev–Bitsadze equation $u_{xx} + \operatorname{sgn} y\ u_{yy} + Au_x + Bu_y + Cu = 0$, i.e. $u_{\xi\eta} + au_\xi + bu_\eta + cu = 0$ with the conditions: $a \ge 0$, $a_\xi + ab - c \ge 0$, $c \ge 0$ was discussed in the hyperbolic domain. In the present paper, we remove the above assumption of [3] and obtain a solvability result for the discontinuous Poincaré problem, which includes the corresponding results in [1]–[3] as special cases.

I. Formulation of the discontinuous Poincaré problem for mixed equations of second order. Let D be a simply connected bounded domain D in the complex plane $\mathbb{C}$ with boundary $\partial D = \Gamma \cup L$, where Γ $(\subset \{y > 0\}) \in C^2_\mu$ $(0 < \mu < 1)$ with end points $z^1 = 0$, $z^2 = 2$ and $L = L_1 \cup L_2$, $L_1 = \{x = -y,\ 0 \le x \le 1\}$, $L_2 = \{x = y + 2,\ 1 \le x \le 2\}$, and define $D_1 = D \cap \{y > 0\}$, $D_2 = D \cap \{y < 0\}$ and $z^0 = 1 - i$. Using a conformal mapping, we may assume that $\Gamma = \{|z - 1| = 1,\ y \ge 0\}$.

We consider the second order linear equation of mixed type

$$u_{xx} + \operatorname{sgn} y\, u_{yy} = Au_x + Bu_y + \varepsilon Cu + E \quad \text{in } D, \tag{1.1}$$

where A, B, C, E are functions of z $(\in D)$ and ε is a real parameter. Its complex form is the following equation of second order:

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$$(1.2)\qquad \begin{aligned} u_{z\bar z} &= \operatorname{Re}[A_1(z)u_z] + \varepsilon A_2(z)u + A_3(z), \quad z \in D_1,\\ u_{\overline{z}\overline{z^*}} &= \operatorname{Re}[A_1(z)u_z] + \varepsilon A_2(z)u + A_3(z), \quad z \in \overline{D}_2, \end{aligned}$$

where

$$(1.3)\qquad \begin{aligned} &z = x+iy,\ u_z = \tfrac12[u_x - iu_y],\ u_{\bar z} = \tfrac12[u_x + iu_y],\ u_{z\bar z} = \tfrac14[u_{xx}+u_{yy}],\\ &u_{z^*} = \tfrac12[u_x + iu_y] = u_{\bar z}, \quad u_{\overline{z}\overline{z^*}} = \tfrac12[(u_{\bar z})_x - i(\overline{u_{\bar z}})_y] = \tfrac14[u_{xx} - u_{yy}],\\ &A_1 = \begin{cases} (A+iB)/2 & \text{in } D_1,\\ (A-iB)/2 & \text{in } D_2, \end{cases} \quad A_2 = C/4, \quad A_3 = E/4 \quad \text{in } D. \end{aligned}$$

Suppose that the equation (1.2) satisfies the following conditions:

Condition C. $A_j(z)$ $(j = 1, 2, 3)$ are measurable in $z \in D_1$ and continuous on $\overline{D}_2$ and their L_p and α-Hölder norms satisfy

$$(1.4)\qquad \begin{aligned} L_p[A_j, \overline{D}_1] \le k_0, \quad j = 1, 2, \quad L_p[A_3, \overline{D}_1] \le k_1,\\ C_\alpha[A_j, \overline{D}_2] \le k_0, \quad j = 1, 2, \quad C_\alpha[A_3, \overline{D}_2] \le k_1, \end{aligned}$$

where $p\ (> 2), \alpha\ (0 < \alpha < 1), k_0, k_1$ are nonnegative constants. ■

In order to introduce the discontinuous Poincaré boundary value problem for the equation (1.2), let functions $a(z), b(z)$ have discontinuities of the first kind at $m+2$ distinct points $z_0 = 2, z_1, \ldots, z_m, z_{m+1} = 0 \in \Gamma$, where $Z = \{z_0, z_1, \ldots, z_{m+1}\}$ is arranged according to the positive direction of Γ and m is a positive integer, and let $c(z) = O(|z-z_j|^{-\beta_j})$ in the neighborhood of z_j $(j = 0, 1, \ldots, m+1)$ on Γ, where β_j $(j = 0, 1, \ldots, m+1)$ are small positive numbers. Define $\lambda(z) = a(x)+ib(x)$ and suppose $|a(x)|+|b(x)| \ne 0$; there is no harm in assuming that $|\lambda(z)| = 1$ for $z \in \Gamma^* = \Gamma \setminus Z$. Suppose that $\lambda(z)$, $c(z)$ satisfy the conditions

$$(1.5)\qquad \lambda(z) \in C_\alpha(\Gamma_j), \quad |z - z_j|^{\beta_j} c(z) \in C_\alpha(\Gamma_j), \quad j = 0, 1, \ldots, m+1,$$

where Γ_j is the open arc from z_j to z_{j+1} on Γ, with $z_{m+2} = 2$, and α $(0 < \alpha < 1)$ is a constant.

Problem P. Find a continuously differentiable solution $u(z)$ of (1.2) in $D^* = \overline{D}\setminus \widetilde{Z}$ ($\widetilde{Z} = \{x \pm y = 2,\ y \le 0\} \cup \{z_1, \ldots, z_{m+1}\}$ or $\widetilde{Z} = \{z_0, \ldots, z_m\} \cup \{x \pm y = 0, y \le 0\}$), which is continuous in $\overline{D}$ and satisfies the boundary conditions

$$(1.6)\qquad \begin{aligned} \frac12 \frac{\partial u}{\partial \nu} + \varepsilon\sigma(z)u &= \operatorname{Re}[\overline{\lambda(z)}u_z] + \varepsilon\sigma(z)u\\ &= r(z) + Y(z)h(z), \quad z \in \Gamma, \quad u(0) = c_0, \end{aligned}$$

$$(1.7)\qquad \frac12 \frac{\partial u}{\partial \nu} = \operatorname{Re}[\overline{\lambda(z)}u_{\bar z}] = r(z), \quad z \in L_1 \text{ or } L_2, \quad \operatorname{Im}[\overline{\lambda(z)}u_{\bar z}]|_{z=z^0} = b_0,$$

where ν is a vector at every point on $\Gamma \cup L_j$ $(j = 1$ or $2)$, c_0, b_0 are real constants, $\lambda(z) = a(x)+ib(x) = \cos(\nu,x)-i\cos(\nu,y)$ for $z \in \Gamma$, $\lambda(z) = a(x)+ ib(x) = \cos(\nu,x) + i\cos(\nu,y)$ for $z \in L_j$ $(j = 1$ or $2)$, and $\lambda(z)$, $r(z)$, c_0, b_0 satisfy the conditions

$$\begin{gathered} C_\alpha[\lambda(z),\Gamma] \le k_0,\ C_\alpha[\sigma(z),\Gamma] \le k_0,\ C_\alpha[r(z),\Gamma] \le k_2,\ |c_0|,|b_0| \le k_2, \\ C_\alpha[\lambda(z),L_j] \le k_0,\ C_\alpha[\sigma(z),L_j] \le k_0,\ C_\alpha[r(z),L_j] \le k_2,\ j = 1 \text{ or } 2, \\ \max_{z\in L_1} \frac{1}{|a(x)-b(x)|} \le k_0 \quad \text{or} \quad \max_{z\in L_2} \frac{1}{|a(x)+b(x)|} \le k_0, \end{gathered} \tag{1.8}$$

where α $(1/2 < \alpha < 1)$, k_0 and k_2 are nonnegative constants. Moreover, the functions $Y(z), h(z)$ are as follows:

$$\begin{gathered} Y(z) = \eta \prod_{j=0}^{m+1} |z-z_j|^{\gamma_j} |z-z_*|^l, \qquad z \in \Gamma^*, \\ h(z) = \begin{cases} 0,\ z \in \Gamma, & \text{if } K \ge -1/2, \\ h_j\eta_j(z),\ z \in \Gamma^j, & \text{if } K < -1/2, \end{cases} \end{gathered} \tag{1.9}$$

where Γ^j $(j = 0,1,\dots,m)$ are arcs on $\Gamma^* = \Gamma \setminus Z$ and $\Gamma^j \cap \Gamma^k = \emptyset$, $j \ne k$, $h_j \in J$ $(J = \emptyset$ if $K \ge -1/2$; $J = \{1,\dots,2K'-1\}$ if $K < -1/2$; $K' = [|K|+1/2])$ are unknown real constants to be determined; $h_1 = 0$, $l = 1$ if $2K$ is odd, z_* $(\notin Z) \in \Gamma^*$ is any fixed point, and $l = 0$ if $2K$ is even; Γ^j $(j = 1,\dots,2K'-1)$ are non-degenerate, mutually disjoint arcs on Γ, and $\Gamma^j \cap Z = \emptyset$ for $j = 1,\dots,2K'-1$; $\eta_j(z)$ is a positive continuous function on the interior of Γ^j such that $\eta_j(z) = 0$ on $\overline{\Gamma \setminus \Gamma^j}$ and

$$C_\alpha[\eta_j(z),\Gamma] \le k_0, \qquad j = 1,\dots,2K'-1; \tag{1.10}$$

and $\eta = 1$ or -1 on Γ_j $(0 \le j \le m+1$, $\Gamma_{m+1} = (0,2))$ as in [4], [6].

The above discontinuous Poincaré boundary value problem for (1.2) is called *Problem* P. Problem P for (1.2) with $A_3(z) = 0$ for $z \in \overline{D}$, $r(z) = 0$ for $z \in \Gamma \cup L_j$ $(j = 1$ or $2)$ and $c_0 = b_0 = 0$ will be called *Problem* P_0.

Denote by $\lambda(z_j - 0)$ and $\lambda(z_j + 0)$ the left and right limits of $\lambda(z)$ as $z \to z_j$ $(j = 0,1,\dots,m+1)$ on $\Gamma \cup L_0$, and

$$\begin{gathered} e^{i\phi_j} = \frac{\lambda(z_j-0)}{\lambda(z_j+0)}, \qquad \gamma_j = \frac{1}{\pi i}\ln\left(\frac{\lambda(z_j-0)}{\lambda(z_j+0)}\right) = \frac{\phi_j}{\pi} - K_j, \\ K_j = \left[\frac{\phi_j}{\pi}\right] + J_j, \qquad J_j = 0 \text{ or } 1, \qquad j = 0,1,\dots,m+1; \end{gathered} \tag{1.11}$$

here $z_{m+1} = 0$, $z_0 = 2$, $\lambda(z) = e^{i\pi/4}$ on $L_0 = (0,2)$ and $\lambda(z_0-0) = \lambda(z_{m+1}+0) = \exp(i\pi/4)$, or $\lambda(z) = e^{-i\pi/4}$ on L_0 and $\lambda(z_0-0) = \lambda(z_{m+1}+0) = \exp(-i\pi/4)$; and $0 \le \gamma_j < 1$ when $J_j = 0$ and $-1 < J_j < 0$ when $J_j = 1$,

$0 \le j \le m+1$. The quantity

$$K = \frac{1}{2}(K_0 + K_2 + \ldots + K_{m+1}) = \sum_{j=0}^{m+1} \left(\frac{\phi_j}{2\pi} - \frac{\gamma_j}{2} \right) \tag{1.12}$$

is called the *index* of Problem P and Problem P_0. If $\lambda(z)$ is continuous on $\Gamma \cup L_0$, then $K = \Delta_{\Gamma \cup L_0} \Gamma \arg \lambda(z)/(2\pi)$ is a unique integer. If $\lambda(z)$ is not continuous on $\Gamma \cup L_0$, we may choose $J_j = 0$ or 1, hence the index K is not unique.

Let $\beta_j + \gamma_j < 1$ for $j = 0, 1, \ldots, m+1$. We can require that the solution $u(z)$ satisfies the condition $u_z = O(|z - z_j|^{-\delta_j})$ in the neighborhood of z_j $(j = 0, 1, \ldots, m+1)$ on D^*, where

$$\begin{aligned} \tau_j &= \begin{cases} \beta_j + \tau & \text{for } \gamma_j \ge 0, \text{ and } \gamma_j < 0,\ \beta_j > |\gamma_j|, \\ |\gamma_j| + \tau & \text{for } \gamma_j < 0,\ \beta_j \le |\gamma_j|, \end{cases} \\ \delta_j &= \begin{cases} 2\tau_j, & j = 0,\ m+1, \\ \tau_j, & j = 1, \ldots, m, \end{cases} \end{aligned} \tag{1.13}$$

and τ, δ $(< \tau)$ are small positive numbers. To ensure that the solution $u(z)$ of Problem P is continuously differentiable in D^*, we need to choose $\gamma_1 > 0$ or $\gamma_2 > 0$ respectively.

II. The representation of solutions for the oblique derivative problem for (1.2). Now we give the representation theorems for solutions of the equation (1.2)

THEOREM 2.1. *Let the equation* (1.2) *satisfy Condition* C *in* D_1 *and* $\varepsilon = 0$, $A_2(z) \ge 0$ *in* D_1, *and* $u(z)$ *be a continuous solution of Problem* P *for* (1.2) *in* D_1. *Then* $u(z)$ *can be expressed as*

$$\begin{aligned} &u(z) = U(z)\Psi(z) + \psi(z) \quad \text{in } D_1, \\ &U(z) = 2\,\mathrm{Re} \int_0^z w(z)\,dz + c_0, \quad w(z) = \Phi(z) e^{\phi(z)} \quad \text{in } D_1, \end{aligned} \tag{2.1}$$

where $\psi(z)$, $\Psi(z)$ *are solutions of the equation* (1.2) *in* $\overline{D}_1$ *and of*

$$u_{z\bar z} - \mathrm{Re}[A_1 u_z] - A_2 u = 0 \quad \text{in } D_1, \tag{2.2}$$

respectively and satisfy the boundary conditions

$$\psi(z) = 0, \quad \Psi(z) = 1 \quad \text{on } \Gamma \cup L_0. \tag{2.3}$$

Furthermore, $\psi(z)$, $\Psi(z)$ *satisfy the estimates*

$$C^1_\beta[\psi, \overline{D}_1] \le M_1, \quad \|\psi\|_{W^2_{p_0}(D_1)} \le M_1, \tag{2.4}$$

$$C^1_\beta[\Psi, \overline{D}_1] \le M_2, \quad \|\Psi\|_{W^2_{p_0}(D_1)} \le M_2, \quad \Psi(z) \ge M_3 > 0, \quad z \in \overline{D}_1, \tag{2.5}$$

where β $(0<\beta\le\alpha)$, p_0 $(2<p_0\le p)$, $M_j=M_j(p_0,k,\alpha,D_1)$ $(j=1,2,3)$ *are nonnegative constants, and* $k=(k_0,k_1,k_2)$. *Moreover*, $U(z)$ *is a solution of the equation*

$$U_{z\overline{z}}-\mathrm{Re}[AU_z]=0,\qquad A=-(\ln\Psi)_{\overline{z}}+A_1\quad \text{in } D_1, \tag{2.6}$$

with $\mathrm{Im}[\phi(z)]=0$ *for* $z\in L_0=(0,2)$ *and*

$$C_\beta[\phi,\overline{D}_1]+L_p[\phi_{\overline{z}},\overline{D}_1]\le M_4, \tag{2.7}$$

where β $(0<\beta\le\alpha)$ *and* $M_4=M_4(p,\alpha,k_0,D)$ *are nonnegative constants. Furthermore*, $\Phi(z)$ *is analytic in* D_1, *and* $w(z)$ *satisfies the boundary conditions*

$$\begin{aligned}&\mathrm{Re}[\overline{\lambda(z)}w(z)]=r(z)-\mathrm{Re}[\overline{\lambda(z)}(\psi_z+\Psi_zU(z))]\quad \text{on } \Gamma,\\&\mathrm{Re}[\overline{\lambda(x)}w(x)]=s(x)-\mathrm{Re}[\overline{\lambda(x)}(\psi_z(x)+\Psi_z(x)U(x))]\quad \text{on } L_0,\end{aligned} \tag{2.8}$$

$$\lambda(x)=\begin{cases}1+i \text{ or}\\ 1-i,\end{cases}\qquad x\in L_0=(0,2),\qquad C_\beta[s(x),L_0]\le k_3, \tag{2.9}$$

and the estimate

$$C_\delta[u(z),\overline{D}_1]+C_\delta[w(z)X(z),\overline{D}_1]\le M_5(k_1+k_2+k_3); \tag{2.10}$$

here k_3 *is a nonnegative constant*, $s(x)$ *is given in* (2.19) *below*, $X(z)=\prod_{j=0}^{m+1}|z-z_j|^{\delta_j}$, δ_j $(j=0,1,\dots,m+1)$ *are as stated in* (1.13), *and* $M_5=M_5(p,\delta,k_0,D_1)$ *is a nonnegative constant.*

Proof. According to the method of proof in Chapter 3 of [6], the equations (1.2) and (2.2) have solutions $\psi(z)$, $\Psi(z)$ respectively which satisfy the boundary condition (2.3) and the estimates (2.4) and (2.5). Setting

$$U(z)=[u(z)-\psi(z)]/\Psi(z), \tag{2.11}$$

it can be derived that $U(z)$ is a solution of (2.6) and can be expressed by the second formula in (2.1), where $\phi(z)$ satisfies the estimate (2.7), $\Phi(z)$ is an analytic function in D_1, and $u(z)$, $w(z)=U_z$ satisfy the boundary conditions (2.8), (2.9) and the estimate (2.10). If $s(x)$ in (2.9) is a known function, then by the result in Chapter 4 of [6], the boundary value problem (2.8), (2.9) has a unique solution $w(z)$ as stated in (2.1).

THEOREM 2.2. *Suppose that the equation* (1.2) *satisfies Condition* C *and* $\varepsilon=1$, $A_2\ge 0$ *in* D_1. *Then any solution of Problem* P *for* (1.2) *can be expressed as*

$$u(z)=2\,\mathrm{Re}\int_0^z w(z)\,dz+c_0,\qquad w(z)=w_0(z)+W(z), \tag{2.12}$$

where $w_0(z)$ *is a solution of the following Problem* A:

$$\left\{\begin{matrix} w_{\bar z} \\ \overline{w_{\overline{z^*}}} \end{matrix}\right\} = 0 \quad in \left\{\begin{matrix} D_1 \\ \overline{D_2} \end{matrix}\right\}, \tag{2.13}$$

with the boundary conditions (1.6), (1.7) ($\sigma(z) = 0$, $w_0(z) = u_{0z}$), *and* $W(z)$ *has the form*

$$\begin{aligned} &W(z) = w(z) - w_0(z), \qquad W(z) = \widetilde{\Phi}(z)e^{\widetilde{\phi}(z)} + \widetilde{\psi}(z), \\ &\widetilde{\phi}(z) = \widetilde{\phi}_0(z) + Tg = \widetilde{\phi}_0(z) - \frac{1}{\pi}\iint_{D_1} \frac{g(\zeta)}{\zeta - z}\, d\sigma_\zeta, \ \widetilde{\psi}(z) = Tf \quad in\ D_1, \\ &\overline{W(z)} = \Phi(z) + \Psi(z), \quad \Psi(z) = \int_2^\nu g_1(z)\, d\nu\, e_1 + \int_0^\mu g_2(z)\, d\mu\, e_2, \quad z \in D_2; \end{aligned} \tag{2.14}$$

here $e_1 = (1+i)/2$, $e_2 = (1-i)/2$, $\mu = x + y$, $\nu = x - y$, *and*

$$g(z) = \begin{cases} A_1/2 + \overline{A}_1\overline{w}/(2w), & w(z) \neq 0, \\ 0, & w(z) = 0, \end{cases}$$

$$\begin{aligned} &f(z) = \operatorname{Re}[A_1\widetilde{\phi}_z] + A_2u + A_3 \qquad in\ D_1, \\ &g_1(z) = A\xi + B\eta + Cu + D, \quad g_2(z) = A\xi + B\eta + Cu + D \quad in\ \overline{D}_2, \\ &A = (\operatorname{Re} A_1 + \operatorname{Im} A_1)/2, \\ &B = (\operatorname{Re} A_1 - \operatorname{Im} A_1)/2, \quad C = A_2, \quad D = A_3, \end{aligned} \tag{2.15}$$

where $\xi = \operatorname{Re} w + \operatorname{Im} w$, $\eta = \operatorname{Re} w - \operatorname{Im} w$; *moreover*, $\widetilde{\phi}_0(z)$ *is an analytic function in* D_1 *such that* $\operatorname{Im}[\widetilde{\phi}(x)] = 0$ *on* $L_0 = (0, 2)$, *and* $\widetilde{\Phi}(z), \Phi(z)$ *are solutions of* (2.13) *in* D_1, D_2 *respectively satisfying the boundary conditions*

$$\begin{aligned} &\operatorname{Re}[\overline{\lambda(z)}e^{\widetilde{\phi}(z)}\widetilde{\Phi}(z)] = r(z) - \sigma(z)u(z) - \operatorname{Re}[\overline{\lambda(z)}\widetilde{\psi}(z)], \quad z \in \Gamma, \\ &\operatorname{Re}[\overline{\lambda(x)}(\widetilde{\Phi}(x)e^{\widetilde{\phi}(x)} + \widetilde{\psi}(x))] = s(x), \qquad x \in L_0, \\ &\operatorname{Re}[\overline{\lambda(x)}\Phi(x)] = -\operatorname{Re}[\overline{\lambda(x)}\Psi(x)], \qquad z \in L_0, \\ &\operatorname{Re}[\overline{\lambda(z)}\Phi(z)] = -\operatorname{Re}[\overline{\lambda(z)}\Psi(z)], \qquad z \in L_1 \ or\ L_2, \\ &\operatorname{Im}[\overline{\lambda(z^0)}\Phi(z^0)] = -\operatorname{Im}[\overline{\lambda(z^0)}\Psi(z^0)]. \end{aligned} \tag{2.16}$$

Moreover, *the solution* $w_0(z)$ *of Problem* A *for* (2.13) *satisfies*

$$C_\delta[u_0(z), \overline{D}] + C_\delta[w_0(z)X(z), \overline{D}_1] + C[w_0^\pm(z)\widetilde{X}^\pm(z), \overline{D}_2] \leq M_6(k_1 + k_2) \tag{2.17}$$

(*see* [4]), *where* $X(z), \delta$ *are as stated in* (2.10), $M_6 = M_6(\delta, k_0, D)$ *is a* *nonnegative constant*,

$$w_0^{\pm}(z) = \operatorname{Re} w_0(z) \mp \operatorname{Im} w_0(z), \qquad \widetilde{X}^{\pm}(z) = \prod_{j=1}^{2} |x \pm y - t_j|^{\delta_j}$$

and

$$u_0(z) = 2\operatorname{Re}\int_0^z w_0(z)dz + c_0. \tag{2.18}$$

Proof. Let $u(z)$ be a solution of Problem P for (1.2), and $w(z) = u_z$, $u(z)$ be substituted in place of w, u in (2.15). Thus the functions $g(z)$, $f(z)$, $g_1(z)$, $g_2(z)$, and $\widetilde{\psi}(z), \widetilde{\phi}(z)$ in $\overline{D}_1$ and $\Psi(z)$ in $\overline{D}_2$ in (2.14), (2.15) can be determined. Moreover, we can find the solution $\widetilde{\Phi}(z)$ in D_1 and $\Phi(z)$ in $\overline{D}_2$ of (2.13) with the boundary conditions (2.16), where

$$s(x) = \begin{cases} r(x/2)/[a(x/2) - b(x/2)] \text{ or} \\ r(x/2+1)/[a(x/2+1) + b(x/2+1)], \end{cases} \qquad x \in L_0. \tag{2.19}$$

Thus

$$w(z) = \begin{cases} \widetilde{\Phi}(z)^{\widetilde{\phi}(z)} + \widetilde{\psi}(z) & \text{in } D_1, \\ w_0(z) + W(z) = w_0(z) + \overline{\Phi(z)} + \overline{\Psi(z)} & \text{in } D_2, \end{cases}$$

is the solution of Problem A for the complex equation

$$\begin{Bmatrix} w_{\bar z} \\ \overline{w}_{\overline{z^*}} \end{Bmatrix} = \operatorname{Re}[A_1 w] + A_2 u + A_3 \quad \text{in } \begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix}, \tag{2.20}$$

which can be expressed by the last formula in (2.12), and $u(z)$ is a solution of Problem P for (1.2) given by the first formula in (2.12).

III. The solvability conditions for the discontinuous Poincaré problem for (1.2). Set $w = u_z$ and consider the equivalent boundary value problem (Problem Q) for the mixed complex equation

$$\begin{aligned} & w_{\bar z} - \operatorname{Re}[A_1(z)w] = \varepsilon A_2(z)u + A_3(z), \qquad z \in D_1, \\ & \overline{w}_{\overline{z^*}} - \operatorname{Re}[A_1(z)w] = A_3(z), \qquad z \in D_2, \\ & u(z) = 2\operatorname{Re}\int_0^z w(z)dz + c_0, \end{aligned} \tag{3.1}$$

with the boundary conditions

$$\begin{aligned} & \operatorname{Re}[\overline{\lambda(z)}w] = r(z) - \varepsilon\sigma(z)u + Y(z)h(z), \qquad z \in \Gamma, \\ & \operatorname{Re}[\overline{\lambda(z)}u_{\bar z}] = r(z), \quad z \in L_j \ (j = 1 \text{ or } 2), \quad \operatorname{Im}[\overline{\lambda(z)}u_{\bar z}]|_{z=z^0} = b_0, \end{aligned} \tag{3.2}$$

where c_0, b_0 are real constants. By the result of [4], we can find the general solution of the following Problem Q_1 for the mixed complex equation:

$$(3.3)\qquad \begin{aligned} &w_{\bar z} - \mathrm{Re}[A_1(z)w] = A_3(z), \quad z \in D_1,\\ &\overline{w_{\overline{z^*}}} - \mathrm{Re}[A_1(z)w] = A_3(z), \quad z \in D_2, \end{aligned}$$

with the boundary conditions

$$(3.4)\qquad \begin{aligned} &\mathrm{Re}[\overline{\lambda(z)}w(z)] = r(z) + Y(z)h(z), \quad z \in \Gamma,\\ &\mathrm{Re}[\overline{\lambda(z)}w(z)] = r(z), \quad z \in L_j\ (j = 1 \text{ or } 2), \quad \mathrm{Im}[\overline{\lambda(z)}w(z)]|_{z=z^0} = b_0, \end{aligned}$$

which can be expressed as

$$(3.5)\qquad \widetilde{w}(z) = w_0(z) + \sum_{k=1}^{2K+1} c_k w_{k(z)}$$

where $w_0(z)$ is a special solution of Problem Q_1 and $w_k(z)$ $(k=1,\dots,2K+1)$ is the complete system of linearly independent solutions for the homogeneous problem corresponding to Problem Q_1. Moreover, denote by H_2u the solution of the following Problem Q_2 for the complex equation:

$$(3.6)\qquad \begin{aligned} &w_{\bar z} - \mathrm{Re}[A_1(z)w] = A_2(z)u, \quad z \in D_1,\\ &\overline{w_{\overline{z^*}}} - \mathrm{Re}[A_1(z)w] = A_2(z)u, \quad z \in D_2, \end{aligned}$$

with the boundary conditions

$$(3.7)\qquad \begin{aligned} &\mathrm{Re}[\overline{\lambda(z)}w(z)] = -\sigma(z)u + Y(z)h(z), \quad z \in \Gamma,\\ &\mathrm{Re}[\overline{\lambda(z)}w(z)] = 0, \quad z \in L_j\ \ (j = 1 \text{ or } 2), \quad \mathrm{Im}[\overline{\lambda(z)}w(z)]|_{z=z^0} = 0, \end{aligned}$$

and the point conditions

$$(3.8)\qquad \mathrm{Im}[\overline{\lambda(a_j)}w(a_j)] = 0, \quad j \in J = \begin{cases} \{1,\dots,2K+1\}, & K \ge 0,\\ \emptyset, & K < 0, \end{cases}$$

where $a_j \in \Gamma \setminus Z$ are distinct points. It is easy to see that H_2 is a bounded operator mapping a function $u(z) \in \widetilde{C}^1(\overline{D})$ (i.e. $C(u,\overline{D}) + C(X(z)u_z, \overline{D}_1) + C(\widetilde{X}(z)u_z^{\pm}, \overline{D}_2) < \infty$) to $w(z) \in \widetilde{C}_\delta(\overline{D})$ (i.e. $C_\delta(u,\overline{D}) + C_\delta(X(z)w(z), \overline{D}_1) + C_\delta(\widetilde{X}(z)w^{\pm}(z), \overline{D}_2) < \infty$); here $X(z), \widetilde{X}(z)$ are as stated in Theorem 2.2. Furthermore, set

$$(3.9)\qquad u(z) = H_1 w + c_0 = 2\,\mathrm{Re}\int_0^z w(z)\,dz + c_0$$

where c_0 is an arbitrary real constant. It is clear that H_1 is a bounded operator mapping $w(z) \in \widetilde{C}_\delta(\overline{D})$ to $u(z) \in \widetilde{C}^1(\overline{D})$. By Theorem 2.2, the function $w(z)$ can be expressed as an integral. From (3.9) and $w(z) = \widetilde{w}(z) + \varepsilon H_2 u$, we can obtain a nonhomogeneous integral equation $(K \ge 0)$:

$$(3.10)\qquad u - \varepsilon H_1 H_2 u = H_1 w(z) + c_0 + \sum_{k=1}^{2K+1} c_k H_1 w_k(z).$$

Since H_1H_2 is a completely continuous operator in $\widetilde{C}^1(\overline{D})$, we can use the Fredholm theorem for the integral equation (3.10). Let

$$\varepsilon_j \ (j=1,2,\ldots), \quad 0<|\varepsilon_1|\le|\varepsilon_2|\le\ldots, \tag{3.11}$$

be the discrete eigenvalues for the homogeneous integral equation

$$u-\varepsilon H_1H_2u=0. \tag{3.12}$$

Note that Problem Q for the complex equation (1.2) with $\varepsilon=0$ is solvable, hence $|\varepsilon_1|>0$.

We first discuss the case of $K\ge 0$. If $\varepsilon\ne\varepsilon_j$ $(j=1,2,\ldots)$, then the nonhomogeneous integral equation (3.10) has a solution $u(z)$ and the general solution of Problem Q involves $2K+2$ arbitrary real constants. If ε is an eigenvalue of rank q as in (3.11), then applying the Fredholm theorem, we obtain solvability conditions for the nonhomogeneous integral equation (3.10), which are a system of q algebraic equations for $2K+2$ arbitrary real constants. Letting s be the rank of the corresponding coefficient matrix and $s\le\min(q,2K+2)$, we can determine s equalities in the q algebraic equations, hence Problem Q for (1.2) has $q-s$ solvability conditions. When these conditions hold, then the general solution of Problem Q involves $2K+2+q-s$ arbitrary real constants.

The case of $K<0$ can be similarly discussed. Thus we can state the following theorem.

Theorem 3.1. *Suppose that the linear mixed equation* (1.2) *satisfies Condition* C. *Suppose* $\varepsilon\ne\varepsilon_j$ $(j=1,2,\ldots)$, *where* ε_j $(j=1,2,\ldots)$ *are the eigenvalues of the homogeneous integral equation* (3.12).

(1) *If* $K\ge 0$, *then Problem* P *for* (1.2) *is solvable*, *and the general solution involves* $2K+2$ *arbitrary real constants.*

(2) *If* $K<0$, *then Problem* P *for* (1.2) *has* $-2K-1-s$ *solvability conditions and* $s\le 1$.

Suppose now that ε *is an eigenvalue of the homogeneous integral equation* (3.12) *with rank* q.

(3) *If* $K\ge 0$, *then Problem* P *for* (1.2) *has* $q-s$ *solvability conditions and* $s\le q$.

(4) *If* $K<0$, *then Problem* P *for* (1.2) *has* $-2K-1+q-s$ *solvability conditions and* $s\le\min(-2K-1+q,1+q)$.

Note that the Dirichlet problem (Problem D) for (1.2) with the boundary conditions

$$u(z)=\phi(z) \quad \text{on } \Gamma\cup L_j \ (j=1 \text{ or } 2) \tag{3.13}$$

is a special case of Problem P with index $K=-1/2$. In fact, set $w=u_z$ in D. Then Problem D for the mixed equation (1.2) is equivalent to Problem A for the mixed equation (3.1) with the boundary condition (3.2) and the relation

$$u(z) = 2\,\mathrm{Re}\int_0^z w(z)\,dz + c_0, \qquad z \in D, \tag{3.14}$$

where

$$\begin{aligned}
&\lambda(z) = a + ib = \begin{cases} \overline{i(z-1)}, \ \theta = \arg(z-1) & \text{on } \Gamma, \\ \dfrac{1-i}{\sqrt{2}} \quad \text{on } L_1, \ \text{or} \quad \dfrac{1+i}{\sqrt{2}} & \text{on } L_2, \end{cases} \\
&r(z) = \begin{cases} \phi_\theta & \text{on } \Gamma, \\ \phi_x/\sqrt{2} & \text{on } L_1 \text{ or } \phi_x/\sqrt{2} \text{ on } L_2, \end{cases} \\
&b_0 = \mathrm{Im}\left[\frac{1+i}{\sqrt{2}} u_{\bar z}(z^0)\right] = \frac{\phi_x - \phi_x}{\sqrt{2}}\Big|_{z=z^0} = 0, \ \text{or} \\
&b_0 = \mathrm{Im}\left[\frac{1-i}{\sqrt{2}} u_{\bar z}(z^0)\right] = 0, \\
&c_0 = \phi(0);
\end{aligned} \tag{3.15}$$

here $a = 1/\sqrt{2} \neq b = -1/\sqrt{2}$ on L_1 or $a = 1/\sqrt{2} \neq -b = -1/\sqrt{2}$ on L_2.

The index $K = -1/2$ of Problem D on ∂D_1 can be derived as follows: According to (3.13), the boundary conditions of Problem D in D_1 are

$$\begin{aligned}
&\mathrm{Re}[i(z-1)w(z)] = r(z) = \phi'_\theta & & \text{on } \Gamma, \\
&\mathrm{Re}\left[\frac{1-i}{\sqrt{2}} w(x)\right] = \frac{\phi'(x/2)}{\sqrt{2}}, & & x \in [0,2], \ \text{or} \\
&\mathrm{Re}\left[\frac{1+i}{\sqrt{2}} w(x)\right] = s(x) = \frac{\phi'(x/2+1)}{\sqrt{2}}, & & x \in [0,2].
\end{aligned}$$

It is clear that the possible discontinuity points of $\lambda(z)$ on ∂D_1 are $t_1 = 2$, $t_2 = 0$, and

$$\begin{aligned}
&\lambda(t_1+0) = \frac{3\pi}{2}, \qquad \lambda(t_2-0) = \frac{\pi}{2}, \\
&\lambda(t_1-0) = \lambda(t_2+0) = e^{\pi i/4}, \quad \text{or} \quad \lambda(t_1-0) = \lambda(t_2+0) = e^{7\pi i/4}, \\
&\frac{\lambda(t_1-0)}{\lambda(t_1+0)} = e^{-5\pi i/4} = e^{i\phi_1}, \qquad \frac{\lambda(t_2-0)}{\lambda(t_2+0)} = e^{\pi i/4} = e^{i\phi_2} \ \text{or} \\
&\frac{\lambda(t_1-0)}{\lambda(t_1+0)} = e^{\pi i/4} = e^{i\phi_1}, \qquad \frac{\lambda(t_2-0)}{\lambda(t_2+0)} = e^{-5\pi i/4} = e^{i\phi_2}.
\end{aligned}$$

In order to insure the uniqueness of solutions of Problem D, we choose

$$-1 < \gamma_1 = \frac{\phi_1}{\pi} - K_1 = -\frac{5}{4} - (-1) = -\frac{1}{4} < 0, \ 0 \le \gamma_2 = \frac{\phi_2}{\pi} - K_2 = \frac{1}{4} < 1, \ \text{or}$$

$$0 \le \gamma_1 = \frac{\phi_1}{\pi} - K_1 = \frac{1}{4} < 1, -1 < \gamma_2 = \frac{\phi_2}{\pi} - K_2 = -\frac{5}{4} - (-1) = -\frac{1}{4} < 0,$$

thus

$$K_1 = -1, \quad K_2 = 0, \quad K = (K_1 + K_2)/2 = -1/2, \quad \text{or}$$
$$K_1 = 0, \quad K_2 = -1, \quad K = (K_1 + K_2)/2 = -1/2.$$

The unique solution $w(z)$ is continuous in $\overline{D}_1 \setminus \{0, 2\}$. For the second case, $w(z)$ is bounded in the neighbourhood of $t_1 = 2$, and the integral of $w(z)$ is bounded in the neighbourhood of $t_2 = 0$; for the first case, the integral of $w(z)$ is bounded in the neighbourhood of $t_1 = 2$, and $w(z)$ is bounded in the neighbourhood of $t_2 = 0$. If we require that the solution $w(z)$ is bounded in $\overline{D}_1 \setminus \{0, 2\}$, then it suffices to choose the index $K = -1$; in this case there is one solvability condition.

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