COLLOQUIUM MATHEMATICUM

VOL. 74 1997 NO. 2

ON PSEUDOSYMMETRIC PARA-KÄHLER MANIFOLDS

BY

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1. Introduction. Let (M,g,J) be a connected, $n=2m$-dimensional, $m\geq 2$, semi-Riemannian manifold of class C^∞ with a not necessarily definite metric g and an almost complex structure J such that

$$g(JX,JY)=g(X,Y),\qquad X,Y\in \Xi(M), \tag{1}$$

where $\Xi(M)$ is the Lie algebra of C^∞ vector fields on M. A manifold (M,g,J) satisfying (1) is called *almost Hermitian*. The almost Hermitian manifold (M,g,J) is said to be a *para-Kähler manifold* ([12], [13], see also [10], p. 69) if its Riemann–Christoffel curvature tensor R satisfies the Kähler identity

$$R(X,Y,Z,W)=R(X,Y,JZ,JW),\qquad X,Y,Z,W\in \Xi(M). \tag{2}$$

Evidently, every Kähler manifold satisfies (2). The converse statement is not true; see e.g. [13] or [17].

In this paper we consider para-Kähler manifolds which satisfy curvature conditions of pseudosymmetric type. In Section 2 we give precise definitions. Pseudosymmetric manifolds constitute a generalization of spaces of constant (sectional) curvature, along the line of locally symmetric ($\nabla R=0$) and semisymmetric ($R\cdot R=0$, cf. [14]) spaces, consecutively. Profound investigation of several properties of semisymmetric manifolds gave rise to their next generalization: the pseudosymmetric manifolds. Both the study of the intrinsic aspect and the study of the extrinsic aspect led to this concept. We have e.g. the following two theorems. Every manifold M which can be mapped geodesically onto a semisymmetric manifold is pseudosymmetric. Every totally umbilical submanifold, with parallel mean curvature

1991 *Mathematics Subject Classification*: 53B20, 53C25.

The first author is a postdoctoral researcher of N.F.W.O., Belgium.

Research of the second author supported by a research-grant of the Research Council of the Katholieke Universiteit Leuven.

Research of the third author supported by the grant OT/TBA/95/9 of the Research Council of the Katholieke Universiteit Leuven.

vector field, of a semisymmetric manifold is pseudosymmetric. This concept of pseudosymmetry in the proper sense belongs to a larger class of curvature conditions of pseudosymmetric type. For more detailed information on the geometric motivation for the introduction of pseudosymmetry, and for a review of results on different aspects of pseudosymmetric spaces, see e.g. [6] and [16]. We just mention here the following application. Curvature conditions of pseudosymmetric type often appear in the theory of general relativity, which is rather surprising in view of the purely geometrical origin of the concept. For more information on this aspect see e.g. [3].

2. Preliminaries. Let (M,g) be an n-dimensional, $n \geq 3$, semi-Riemannian connected manifold of class C^∞. We denote by ∇, S and κ the Levi-Civita connection, the Ricci tensor and the scalar curvature of (M,g), respectively. We define on M the endomorphisms $\widetilde{R}(X,Y)$, $X \wedge Y$ and $\widetilde{\mathcal{C}}(X,Y)$ by

$$\widetilde{R}(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z, (X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y,$$
$$\widetilde{\mathcal{C}}(X,Y) = \widetilde{\mathcal{R}}(X,Y) + \frac{1}{n-2}\left(\frac{\kappa}{n-1} X \wedge Y - (X \wedge \widetilde{\mathcal{S}}Y + \widetilde{\mathcal{S}}X \wedge Y)\right),$$

respectively, where $X,Y,Z \in \Xi(M)$, $\Xi(M)$ being the Lie algebra of vector fields on M, and the Ricci operator $\widetilde{\mathcal{S}}$ of (M,g) is defined by $S(X,Y) = g(X, \widetilde{\mathcal{S}}Y)$. The $(0,4)$-tensor G is defined by

$$G(X_1, \ldots, X_4) = g((X_1 \wedge X_2)X_3, X_4).$$

The *Riemann curvature tensor* R and the *Weyl curvature tensor* C of (M,g) are defined by

$$R(X_1,X_2,X_3,X_4) = g(\widetilde{R}(X_1,X_2)X_3,X_4),$$
$$C(X_1,X_2,X_3,X_4) = g(\widetilde{\mathcal{C}}(X_1,X_2)X_3,X_4).$$

Further, for a symmetric $(0,2)$-tensor field A on M, we define the endomorphism $X \wedge_A Y$ of $\Xi(M)$ by $(X \wedge_A Y)Z = A(Z,Y)X - A(Z,X)Y$, where $X,Y,Z \in \Xi(M)$. Evidently, we have $X \wedge_g Y = X \wedge Y$. For a $(0,k)$-tensor field T on M, $k \geq 1$, and a symmetric $(0,2)$-tensor field A on M, we define the $(0,k+2)$-tensor fields $R \cdot T$ and $Q(A,T)$ by

$$\begin{aligned}(R \cdot T)(X_1, \ldots, X_k; X, Y) = & -T(\widetilde{\mathcal{R}}(X,Y)X_1, X_2, \ldots, X_k) \\ & - \ldots - T(X_1, \ldots, X_{k-1}, \widetilde{\mathcal{R}}(X,Y)X_k), \\ Q(A,T)(X_1, \ldots, X_k; X, Y) = & -T((X \wedge_A Y)X_1, X_2, \ldots, X_k) \\ & - \ldots - T(X_1, \ldots, X_{k-1}, (X \wedge_A Y)X_k).\end{aligned}$$

Curvature conditions involving tensors of the form $R \cdot T$ and $Q(A,T)$ are called *curvature conditions of pseudosymmetric type*. E.g. manifolds satisfying the condition

$$R \cdot R = L_R Q(g, R) \tag{3}$$

on the set $U_R = \{x \in M \mid R - \frac{\kappa}{n(n-1)} G \neq 0 \text{ at } x\}$ are called *pseudosymmetric*; in particular, for $L_R = 0$ they contain the *semisymmetric spaces* ($R \cdot R = 0$). Manifolds satisfying the condition

$$R \cdot S = L_S Q(g, S) \tag{4}$$

on the set $U_S = \{x \in M \mid S - \frac{\kappa}{n} g \neq 0 \text{ at } x\}$ are called *Ricci-pseudosymmetric*; in particular, for $L_S = 0$ they contain the *Ricci-semisymmetric spaces* ($R \cdot S = 0$). Manifolds satisfying the condition

$$R \cdot C = L_C Q(g, C) \tag{5}$$

on the set $U_C = \{x \in M \mid C \neq 0 \text{ at } x\}$ are called *Weyl-pseudosymmetric*; in particular, for $L_C = 0$ they contain the *Weyl-semisymmetric* spaces ($R \cdot C = 0$).

The inclusions among the above mentioned classes of manifolds can be summarized in the following table; in general, for manifolds with dimension ≥ 4, all inclusions are strict [6].

$$\begin{array}{ccccc} R \cdot S = L_S Q(g,S) & \supset & R \cdot R = L_R Q(g,R) & \subset & R \cdot C = L_C Q(g,C) \\ \cup & & \cup & & \cup \\ R \cdot S = 0 & \supset & R \cdot R = 0 & \subset & R \cdot C = 0 \end{array}$$

In the present paper, we prove that on para-Kähler manifolds the curvature conditions of pseudosymmetric type $R \cdot T = LQ(g, T)$ for $T = R, S$ and C reduce to the corresponding curvature conditions of semisymmetric type, e.g. $R \cdot T = 0$ for $T = R, S$ and C, respectively. This question for Ricci-generalized pseudosymmetric para-Kähler manifolds (i.e. manifolds realizing a curvature condition of the form $R \cdot R = LQ(S, R)$) was treated in [2].

Let $\mathcal{L}$ denote the class of all almost Hermitian manifolds (M, g, J). Further, the class of all para-Kähler manifolds will be denoted by $\mathcal{L}_1$. According to [10], we denote by $\mathcal{L}_2$ and $\mathcal{L}_3$ the classes of all almost Hermitian manifolds realizing the relations

$$R(X,Y,Z,W) = R(JX,JY,Z,W) + R(JX,Y,JZ,W) + R(JX,Y,Z,JW)$$

and

$$R(X,Y,Z,W) = R(JX,JY,JZ,JW),$$

respectively, where $X, Y, Z, W \in \Xi(M)$. As shown in [10] (Lemma 5.1, p. 68), we have the following inclusions:

$$\mathcal{L}_1 \subset \mathcal{L}_2 \subset \mathcal{L}_3 \subset \mathcal{L}.$$

Some results on the above classes of manifolds are presented in Chapter II of [10].

3. Para-Kähler manifolds of pseudosymmetric type. Let (M, g, J), $n = \dim M \geq 4$, be a para-Kähler manifold covered by a system of coordinate neighbourhoods $\{U; u^h\}$. We denote by $J_h^k = J_h{}^k$ the local components of the almost complex structure J. Morever, let C_{hijk}, R_{hijk}, S_{ij} and g_{ij} be the local components of the Weyl conformal curvature tensor C, the Riemann–Christoffel curvature tensor R, the Ricci tensor S and the metric tensor g, respectively. Thus, by (1), we have

$$J_k^s J_s^l = -\delta_k^l, \qquad J_k^r J_l^s g_{rs} = g_{kl}, \qquad J^{kl} = g^{ks} J_s^l. \tag{6}$$

Further, (2) takes the form

$$J_h^r J_i^s R_{rsjk} = R_{hijk}, \tag{7}$$

whence, by (6), we obtain

$$J_h^s R_{sljk} - J_l^s R_{shjk} = 0. \tag{8}$$

Further, using the above relations, we find

$$J^{rs} R_{rsjk} = 2A_{jk}, \qquad J^{rs} R_{rijs} = -A_{ij}, \tag{9}$$

where $A_{jk} = -J_j^s S_{sk}$.

3.1. *Ricci-semisymmetric para-Kähler manifolds*

PROPOSITION 3.1. *Every semi-Riemannian Ricci-pseudosymmetric para-Kähler manifold* (M, g, J), $\dim M \geq 4$, *is Ricci-semisymmetric.*

Proof. Let x be a point of the set U_S and let $\tilde{U} \subset U_S$ be a coordinate neighbourhood of x. Then the equality

$$(R \cdot S)_{hirs} = L_S Q(g, S)_{hirs}$$

holds on $\widetilde{U}$. Transvecting this with $J_j^r J_k^s$ and using the Ricci identity and (7) we obtain

$$(R \cdot S)_{hirs} = L_S Q(g, S)_{hirs} J_s^r J_k^s.$$

Thus by (4) we have

$$L_S Q(g, S)_{hirs} J_j^r J_k^s = L_S Q(g, S)_{hijk}.$$

Suppose that the function L_S is non-zero at x. Then the last equality gives

$$Q(g, S)_{hirs} J_j^r J_k^s = Q(g, S)_{hijk},$$

which, by contraction with g^{hk}, yields $S_{ij} = \frac{\kappa}{n} g_{ij}$, a contradiction. Thus the function L_S vanishes identically on U_S, which completes the proof.

The above proposition generalizes Theorem 1 of [11].

3.2. *Pseudosymmetric para-Kähler manifolds*

PROPOSITION 3.2. *Every semi-Riemannian pseudosymmetric para-Kähler manifold* (M, g, J), $\dim M \geq 4$, *is semisymmetric.*

P r o o f. Let x be a point of the set U_R and let $\widetilde{U} \subset U_R$ be a coordinate neighbourhood of x. Then the equality

$$(R \cdot R)_{hijkrs} = L_R Q(g, R)_{hijkrs}$$

holds on $\widetilde{U}$. Now, in the same way as in the proof of Proposition 3.1 we obtain

$$L_R Q(g, R)_{hijkrs} J_l^r J_m^s = L_R Q(g, R)_{hijklm}.$$

Suppose that the function L_R is non-zero at x. Then the last equality gives

$$Q(g, R)_{hijkrs} J_l^r J_m^s = Q(g, R)_{hijklm}. \tag{10}$$

Since every pseudosymmetric manifold is Ricci-pseudosymmetric, from Proposition 3.1 it follows that the equality

$$S_{ij} = \frac{\kappa}{n} g_{ij} \tag{11}$$

holds at x. Contracting (10) with g^{hm} and using (11), (7), and (9), we get

$$\begin{aligned} J_l^r J_j^s R_{rkis} - J_l^r J_k^s R_{rjis} + \frac{2\kappa}{n} J_{li} J_{jk} - \frac{\kappa}{n} J_{lj} J_{ki} + \frac{\kappa}{n} J_{lk} J_{ji} \\ = (n-3) R_{lijk} - \frac{\kappa}{n} G_{lijk}, \end{aligned} \tag{12}$$

where $J_{li} = J_l^s g_{si}$ and $G_{lijk} = g_{lk} g_{ij} - g_{lj} g_{ik}$ are the local components of the tensor G. Transvecting (12) with J_h^j and and using (6) and (7), we find

$$\begin{aligned} -J_l^r R_{rkih} - J_k^s R_{sihl} - \frac{2\kappa}{n} J_{li} g_{hk} - \frac{\kappa}{n} J_{ki} g_{hl} - \frac{\kappa}{n} J_{lk} g_{hi} \\ = (n-3) J_h^s R_{skli} - \frac{\kappa}{n} g_{kl} J_{hi} + \frac{\kappa}{n} g_{ik} J_{hl}. \end{aligned}$$

Contracting this with g^{hi} and applying (6), (7), (8) and (11), we find $\kappa J_{kl} = 0$, whence $\kappa = 0$. Thus (11) yields $S_{ij} = 0$. Now (9) reduces to

$$J^{rs} R_{rsjk} = 0, \qquad J^{rs} R_{rijs} = 0, \tag{13}$$

respectively. Further, from (7), by (8), we have

$$J_h^r J_i^s (R \cdot R)_{rsjkab} = (R \cdot R)_{hijkab},$$

which, by virtue of (3), (7) and the assumption that L_R is non-zero at x, turns into

$$\begin{aligned} g_{ia} R_{bhjk} + g_{hb} R_{aijk} - g_{ha} R_{bijk} - g_{ib} R_{ahjk} \\ = J_{ha} J_i^s R_{sbjk} + J_{ib} J_h^s R_{sajk} - J_{ia} J_h^s R_{sbjk} - J_{hb} J_i^s R_{sajk}. \end{aligned}$$

This yields

$$\begin{aligned} g_{ia} (R \cdot R)_{bhjklm} + g_{hb} (R \cdot R)_{aijklm} - g_{ha} (R \cdot R)_{bijklm} - g_{ib} (R \cdot R)_{ahjklm} \\ = J_{ha} J_i^s (R \cdot R)_{sbjklm} + J_{ib} J_h^s (R \cdot R)_{sajklm} \\ - J_{ia} J_h^s (R \cdot R)_{sbjklm} - J_{hb} J_i^s (R \cdot R)_{sajklm}. \end{aligned}$$

Finally, applying (3) and contracting the resulting equality with g^{am} and g^{hb} we get $R_{lijk} = 0$, a contradiction. Thus L_R vanishes at x. But this completes the proof.

The above proposition generalizes Theorem 4 of [4].

3.3. *Weyl-pseudosymmetric para-Kähler manifolds*

PROPOSITION 3.3. *Every semi-Riemannian Weyl-pseudosymmetric para-Kähler manifold* (M, g, J), $\dim M \geq 4$, *is Weyl-semisymmetric.*

Proof. Let x be a point of U_C. First assume that $\dim M \geq 5$. Thus, in view of Theorem 1 of [7] and our Proposition 3.2, the tensor $R \cdot R$ vanishes at x. Thus $R \cdot C = 0$ holds at x, completing the proof of this case. Now assume that $\dim M = 4$. Transvecting the equality

$$(R \cdot C)_{hijklm} = L_C Q(g, C)_{hijklm}$$

with $J^l_a J^m_b$ and using the Ricci identity and (7) we obtain

$$(R \cdot C)_{hijkab} = L_C J^l_a J^m_b Q(g, C)_{hijklm},$$

whence

$$L_C Q(g, C)_{hijkab} = L_C J^l_a J^m_b Q(g, C)_{hijklm}.$$

Suppose that the function L_C is non-zero at x. Thus the last equality reduces to

$$Q(g, C)_{hijkab} = J^l_a J^m_b Q(g, C)_{hijklm}.$$

This, by contraction with g^{hb}, gives

$$\begin{aligned} 2C_{aijk} = {} & J^s_a J^r_i C_{srjk} + J^s_a J^r_j C_{skir} - J^s_a J^r_k C_{sjir} \\ & - J_{ai} J^{sr} C_{srjk} - J_{aj} J^{sr} C_{skir} + J_{ak} J^{sr} C_{sjir}, \end{aligned} \tag{14}$$

which, by transvection with J^a_h, turns into

$$\begin{aligned} 2J^s_h C_{sijk} = {} & J^r_i C_{rhjk} - J^r_j C_{hkir} + J^r_k C_{hjir} \\ & + g_{hj} J^{sr} C_{skir} - g_{hk} J^{sr} C_{sjir} + g_{hi} J^{sr} C_{srjk}. \end{aligned} \tag{15}$$

Next, contracting (15) with g^{hi} we obtain

$$J^{sr} C_{srjk} = 0, \tag{16}$$

which reduces (15) to

$$2J^s_h C_{sijk} = J^s_i C_{shjk} - J^s_j C_{sikh} + J^s_k C_{sijh}. \tag{17}$$

Next, summing (17) cyclically in h, j, k we get

$$J^s_h C_{sijk} + J^s_j C_{sikh} + J^s_k C_{sihj} = 0.$$

Now (17), by making use of the above relation, yields

$$J^s_k C_{sijk} = J^s_i C_{shjk}.$$

From this, by (8) and (5), it follows that

$$J_{ha}C_{bijk} + g_{ia}J_h^sC_{sbjk} - J_{hb}C_{aijk} - g_{ib}J_h^sC_{sajk} \\ = J_{ia}C_{bhjk} + g_{hl}J_i^sC_{sbjk} - J_{ib}C_{ahjk} - g_{hb}J_i^sC_{sajk}$$

and

$$\begin{aligned} J_{ha}Q(g,C)_{bijklm} &- J_{hb}Q(g,C)_{aijklm} \\ &+ g_{ia}J_h^sQ(g,C)_{sbjklm} - g_{ib}J_h^sQ(g,C)_{sajklm} \\ &= J_{ia}Q(g,C)_{bhjklm} - J_{ib}Q(g,C)_{ahjklm} \\ &+ g_{hl}J_i^sQ(g,C)_{sbjklm} - g_{hb}J_i^sQ(g,C)_{sajklm}. \end{aligned}$$

Contracting the last equality with g^{am} and g^{hb}, after some calculations, we obtain

$$J_i^sC_{sljk} = 0.$$

Applying this and (15) in (14) we get at x the relation $C = 0$, a contradiction. Our theorem is thus proved.

Remark 3.1. An example of a non-conformally flat and non-semisymmetric Weyl-semisymmetric manifold $(M, g), \dim M = 4$, which is a Kähler manifold was described in [5] (Lemme 1.1).

Remark 3.2. Let B be the Bochner curvature tensor ([1], [18], [15]) of a para-Kähler semi-Riemannian manifold (M, g, J), $n = 2m \geq 4$. In [8] Kähler Riemannian manifolds with semisymmetric Bochner tensor ($R \cdot B = 0$) were considered. From the main results of [8] (Theorem) we can conclude that if the tensor B of a para-Kähler Riemannian manifold (M, g, J), $n = 2m \geq 4$, is semisymmetric then the Riemann–Christoffel curvature tensor R of (M, g, J) is semisymmetric on the subset $\mathcal{U}_B \subset M$ consisting of all points of M at which B is non-zero. Most recently this statement was generalized as follows [9]: if the Bochner tensor B of a para-Kähler semi-Riemannian manifold (M, g, J), $n = 2m \geq 4$, is pseudosymmetric, i.e. $R \cdot B = L_B Q(g, B)$ holds on $\mathcal{U}_B$, then the tensor R of (M, g, J) is semisymmetric on the set $\mathcal{U}_B$.

Acknowledgements. The first author (F.D.) acknowledges support from the project "Gauge theories, applied supersymmetry and quantum gravity", contract SC1-CT92-0789 of the European Economic Community.

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Received 13 November 1996;
revised 27 January 1997