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FREE OPERATORS WITH OPERATOR COEFFICIENTS

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Let $g_1, \ldots, g_n$ be the generators of the free group $\mathbf{F}_n$. C. Akemann and P. Ostrand proved in [A-O] a formula for the norm of free operators, i.e. operators of the form $\sum \alpha_i \lambda(g_i)$. This formula improved estimates of M. Leinert [L] and M. Bożejko [Bo]. It was previously known in the case of equal coefficients [K] and simpler proofs were found in [W] and [P-P]. In this note we show how a part of the proof of [P-P] can be generalized to obtain estimates for the operator-valued case, which improve the bounds in [H-P, Proposition 1.1] (see also [Bu] for related recent results).

THEOREM 1. *Let $n \ge 2$ and $g_1, \ldots, g_n$ be the generators of the free group $\mathbf{F}_n$, and let further $a_1, \ldots, a_n$ be some operators on a Hilbert space H which can be approximated by invertible operators. Then*

$$(1)\qquad \Big\|\sum_{i=1}^n \lambda(g_i) \otimes a_i\Big\|_{\min} \le \inf_{s>0} \Big\|\sum_{i=1}^n (s^2 I + a_i a_i^*)^{1/2} - (n-2)sI\Big\|^{1/2} \times \Big\|\sum_{i=1}^n (s^2 I + a_i^* a_i)^{1/2} - (n-2)sI\Big\|^{1/2}.$$

Remark. If the Hilbert space H is finite-dimensional, any operator can be approximated by invertible ones. In the infinite-dimensional case this is not generally true; see [H, Problem 140]. However, we have

COROLLARY 2. *For an arbitrary family of operators $a_1, \ldots, a_n$ on a Hilbert space H we have*

$$(2)\qquad \Big\|\sum_{i=1}^n \lambda(g_i) \otimes a_i\Big\|_{\min} \le 2\sqrt{1 - \frac{1}{n}} \left(\frac{\|\sum_{i=1}^n a_i a_i^*\| + \|\sum_{i=1}^n a_i^* a_i\|}{2} \right)^{1/2} \le 2\sqrt{1 - \frac{1}{n}} \max\Big\{ \Big\|\sum_{i=1}^n a_i a_i^*\Big\|^{1/2}, \Big\|\sum_{i=1}^n a_i^* a_i\Big\|^{1/2} \Big\}.$$

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The proof of (1) is an adaptation to the non-commutative situation of the first part of [P-P]. Denote by $\mathrm{HS}(H)$ the space of Hilbert–Schmidt operators on the Hilbert space H and by tr the usual (unbounded) trace. We need the following version of the Cauchy–Schwarz inequality.

LEMMA 3. *For any $x_1,\dots,x_n \in B(H)$ and $y_1,\dots,y_n \in \mathrm{HS}(H)$ we have the inequality*

$$\Big\|\sum_{i=1}^n x_i y_i\Big\|_{\mathrm{HS}} \le \Big\|\sum_{i=1}^n x_i x_i^*\Big\|^{1/2} \operatorname{tr}\Big(\sum_{i=1}^n y_i^* y_i\Big)^{1/2}. \tag{3}$$

P r o o f. After writing the sum as

$$\begin{bmatrix} \sum_{i=1}^n x_i y_i & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} y_1 & 0 & \cdots & 0 \\ y_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ y_n & 0 & \cdots & 0 \end{bmatrix}$$

the claim follows from the operator ideal property of $\mathrm{HS}(H^n)$. ∎

P r o o f o f T h e o r e m 1. We first consider the case where all the operators a_i are invertible. The general case then follows by a topological argument.

Let $T = \sum_{i=1}^n \lambda(g_i) \otimes a_i$ act on the Hilbert space $\ell_2(\mathbf{F}_n; \mathrm{HS}(H))$. For every word $y \in \mathbf{F}_n$, $i \in \{1,\dots,n\}$ and any positive real number s we define the following operator:

$$p_i(y,s) = a_i^{-1}((s^2 + a_i a_i^*)^{1/2} \mp s) = ((s^2 + a_i^* a_i)^{1/2} \mp s) a_i^{-1}$$

with

"−" if there is no cancellation in the word $g_i^{-1}y$, i.e. the first letter of y is different from g_i,

"+" if the first letter of y *is* g_i and there is cancellation.

Here and in the following, scalars appearing in operator expressions mean the corresponding multiple of the identity operator, and the square root of a positive operator is always the unique positive square root. Then one can easily check that $p_i(y,s)$ is invertible and that its inverse is

$$p_i(y,s)^{-1} = ((s^2 + a_i a_i^*)^{1/2} \pm s) a_i^{*-1} = a_i^{*-1}((s^2 + a_i^* a_i)^{1/2} \pm s)$$

(note the change of sign).

Now pick $h \in \ell_2(\mathbf{F}_n; \mathrm{HS}(H))$ with finite support. In order to get the upper bound for $\|Th\|_2^2 = \sum_y \|Th(y)\|_{\mathrm{HS}}^2$ we first give an estimate of

$$\|Th(y)\|_{\mathrm{HS}}^2 = \Big\|\sum_{i=1}^n a_i p_i(y,s) p_i(y,s)^{-1} h(g_i^{-1}y)\Big\|_{\mathrm{HS}}^2$$

for fixed y. Now the operator $a_i p_i(y,s)$ is positive and we can apply the above lemma by letting $x_i = (a_i p_i(y,s))^{1/2}$ be its (positive) square root and $y_i = (a_i p_i(y,s))^{1/2} p_i(y,s)^{-1} h(g_i^{-1}y)$:

$$\|Th(y)\|_{\mathrm{HS}}^2 \le \Big\|\sum_{i=1}^n a_i p_i(y,s)\Big\| \operatorname{tr}\Big(\sum_{i=1}^n h(g_i^{-1}y)^* p_i(y,s)^{*-1} a_i h(g_i^{-1}y)\Big).$$

Note that among the words $\{g_1^{-1}y, \dots, g_n^{-1}y\}$ there is at most one cancellation, so that there is always the "$-$" sign in $p_i(y,s)$ except maybe for one case and since all the summands $a_i p_i(y,s)$ are positive operators, we have the operator inequality

$$\sum_{i=1}^n a_i p_i(y,s) \le 2s + \sum_{i=1}^n ((s^2 + a_i a_i^*)^{1/2} - s) =: c_0(s).$$

This upper bound does not depend on the word y and we can estimate the norm of Th as follows:

$$\begin{aligned}\|Th\|_2^2 &\le \|c_0(s)\| \sum_y \operatorname{tr}\Big(\sum_{i=1}^n p_i(g_i y,s)^{*-1} a_i h(y)h(y)^*\Big)\\ &\le \|c_0(s)\| \sum_y \Big\|\sum_{i=1}^n p_i(g_i y,s)^{*-1} a_i\Big\| \|h(y)\|_{\mathrm{HS}}^2.\end{aligned}$$

Now $p_i(g_i y,s)$ has always the "+" sign with at most one exception possible in the case when there is already cancellation in $g_i y$. Thus $p_i(g_i y,s)^{-1}$ has the "$-$" signs and with the formula for the inverse we get

$$\sum_{i=1}^n p_i(g_i y,s)^{*-1} a_i \le 2s + \sum_{i=1}^n ((s^2 + a_i^* a_i)^{1/2} - s) =: c_1(s).$$

The bound $c_1(s)$ is again independent of y and we finally get the inequality we wanted: For all positive real numbers s,

$$\|Th\|_2^2 \le \|c_0(s)\| \cdot \|c_1(s)\|.$$

Let us now consider the case where the a_i's are not invertible but approximable by invertible operators. Consider the topological space $B(H)^n$ equipped with the product topology and define the functions

$$\begin{aligned} &B(H)^n \to \mathbb{R},\\ g : x = (a_1,\dots,a_n) &\mapsto \inf_{s\ge 0} l(\|c_0(s)\| \cdot \|c_1(s)\| r)^{1/2},\\ f : x = (a_1,\dots,a_n) &\mapsto \Big\|\sum_{i=1}^n \lambda(g_i)\otimes a_i\Big\|.\end{aligned}$$

Observe that g is upper semicontinuous, i.e. the set

$$\{x \in B(H)^n \mid g(x) < t\}$$

is open for any $t \in \mathbb{R}$. Since f is continuous, the set

$$\{x \in B(H)^n \mid g(x) - f(x) \geq 0\}$$

is closed and hence contains the closure of all the n-tuples of invertible operators. ■

Note that the infimum over all positive scalars s could be replaced by an infimum over all positive operators S which commute with the a_i's. However, in view of the examples below this does not seem to improve the inequality very much.

Proof of Corollary 2. We will first prove the case where the a_i's are approximable by invertibles. In fact we will show that the bound (1) is sharper than (2) just as in the commutative case (cf. [A-O]). We recall the following facts from non-commutative analysis. A function f is called *operator-monotone* if for any positive selfadjoint operators a, b,

$$a \geq b \Rightarrow f(a) \geq f(b).$$

It is *operator-concave* if the operator inequality

$$f(\lambda a + (1-\lambda)b) \geq \lambda f(a) + (1-\lambda) f(b)$$

holds for all positive operators a, b and any $0 < \lambda < 1$. By Löwner's theorem [M-O, p. 464], the function $t \mapsto t^\alpha$ is operator-monotone for $0 \leq \alpha \leq 1$ (see also [Ped]). Ando showed in [A] (see also [M]) that any operator-monotone function is necessarily operator-concave. We can now apply this to the function $t \mapsto \sqrt{t}$ and any sequence of positive operators $x_i = a_i^* a_i$:

$$\frac{1}{n}\sum_{i=1}^n x_i^{1/2} \leq \left(\frac{1}{n}\sum_{i=1}^n x_i\right)^{1/2},$$

and our bound becomes

$$\begin{aligned}
(\|c_0(s)\| \cdot \|c_1(s)\|)^{1/2} \leq{} & \left((2-n)s + n\sqrt{s^2 + \frac{1}{n}\Big\|\sum_{i=1}^n a_i a_i^*\Big\|}\right)^{1/2} \\
& \times \left((2-n)s + n\sqrt{s^2 + \frac{1}{n}\Big\|\sum_{i=1}^n a_i^* a_i\Big\|}\right)^{1/2} \\
\leq{} & (2-n)s + \frac{n}{2}\left(\sqrt{s^2 + \frac{1}{n}\Big\|\sum_{i=1}^n a_i a_i^*\Big\|}\right. \\
& \left. + \sqrt{s^2 + \frac{1}{n}\Big\|\sum_{i=1}^n a_i^* a_i\Big\|}\right)
\end{aligned}$$

$$\leq (2-n)s + n\sqrt{s^2 + \frac{1}{2n}\Big(\Big\|\sum_{i=1}^n a_i a_i^*\Big\| + \Big\|\sum_{i=1}^n a_i^* a_i\Big\|\Big)}.$$

The infimum of the last expression over all $s \geq 0$ is the expression in the claim.

Let us now consider general operators $a_1, \ldots, a_n$ which are not necessarily approximable by invertible ones. Denote by $P_{\mathrm{f}}(H)$ the set of all finite-dimensional projections on H. It is easy to see that the embedding

$$\Phi : B(H) \to \bigoplus_{p \in P_{\mathrm{f}}(H)} pB(H)p, \qquad a \mapsto (pap)_{p \in P_{\mathrm{f}}(H)},$$

is a complete isometry, i.e. for any operators $b_1, \ldots, b_n$ acting on some Hilbert space K we have

$$\Big\|\sum_{i=1}^n a_i \otimes b_i\Big\|_{\min} = \Big\|\sum_{i=1}^n \Phi(a_i) \otimes b_i\Big\|_{\min}.$$

Now $\Phi(a_i)$ lying in a direct sum of matrix algebras can be approximated by invertibles, so that inequality (2) holds when we replace a_i by $\Phi(a_i)$. Next we use the fact that the right hand side of (2) comes from an operator space structure. Indeed, denoting by e_{ij} the canonical basis of $\mathbf{M}_n$, it is easy to see that

$$\Big\|\sum_{i=1}^n a_i \otimes e_{1i}\Big\| = \Big\|\sum_{i=1}^n a_i a_i^*\Big\|^{1/2} \quad \text{and} \quad \Big\|\sum_{i=1}^n a_i \otimes e_{i1}\Big\| = \Big\|\sum_{i=1}^n a_i^* a_i\Big\|^{1/2};$$

hence we have

$$\begin{aligned}\Big\|\sum_{i=1}^n \lambda(g_i) \otimes a_i\Big\|_{\min} &= \Big\|\sum_{i=1}^n \lambda(g_i) \otimes \Phi(a_i)\Big\|_{\min} \\ &\leq 2\sqrt{1-\frac{1}{n}}\Big(\frac{\|\sum \Phi(a_i) \otimes e_{1i}\| + \|\sum \Phi(a_i) \otimes e_{i1}\|}{2}\Big)^{1/2} \\ &= 2\sqrt{1-\frac{1}{n}}\Big(\frac{\|\sum a_i \otimes e_{1i}\| + \|\sum a_i \otimes e_{i1}\|}{2}\Big)^{1/2}\end{aligned}$$

and this proves the claim. We do not see how to generalize (1) to non-approximable operators with a similar trick, since the assignment $(a_i) \mapsto \|\sum |a_i|\|$ fails to be a norm and hence is not a complete invariant. ■

Examples. There is actually equality not only for scalar coefficients and it would be interesting to characterize such families of operators.

EXAMPLE 1 (Commuting normal operators). As a simple consequence of Gel'fand's theorem equality holds if the operators $a_1, \ldots, a_n$ generate a commutative C^*-algebra.

Example 2. For unitaries $u_1, \dots, u_n$ there is equality:

$$\begin{aligned}\Bigl\|\sum_{i=1}^n \lambda(g_i) \otimes \alpha_i u_i\Bigr\| &= \Bigl\|\sum_{i=1}^n \alpha_i \lambda(g_i)\Bigr\| \\ &= \min_{s\geq 0} 2s + \sum_{i=1}^n \bigl(\sqrt{s^2 + |\alpha_i|^2} - s\bigr) \\ &= \min_{s\geq 0} (\|c_0(s)\| \cdot \|c_1(s)\|)^{1/2}.\end{aligned}$$

The first identity is Fell's lemma [F] applied to the left regular representation and the unitary representation which is uniquely determined by $\pi(g_i) = u_i$.

The next example uses the following simple identity. For any projection p and positive real numbers σ, α,

$$\bigl(\sigma I + \bigl(\sqrt{\sigma^2 + \alpha^2} - \sigma\bigr)p\bigr)^2 = \sigma^2 I + \alpha^2 p. \tag{4}$$

Example 3. For the basis of the row-space R_n and equal coefficients there is equality:

$$\Bigl\|\sum_{i=1}^n \lambda(g_i) \otimes e_{1i}\Bigr\| = \sqrt{n} = \min_{s\geq 0} (\|c_0(s)\| \cdot \|c_1(s)\|)^{1/2}. \tag{5}$$

However, if the coefficients are different there may be strict inequality, e.g.

$$\begin{aligned}&\|\lambda(g_1) \otimes e_{11} + \lambda(g_2) \otimes e_{12} + 2\lambda(g_3) \otimes e_{13}\| \\ &\qquad = \sqrt{6} < \sqrt{8} = \min_{s\geq 0} (\|c_0(s)\| \cdot \|c_1(s)\|)^{1/2}.\end{aligned} \tag{6}$$

The bound for the general operator $\sum_{i=1}^n \lambda(g_i) \otimes \alpha_i e_{1i}$ is determined by the norms

$$\begin{aligned}\|c_0(s)\| &= \Bigl\|2s + \sum_{i=1}^n ((s^2 + |\alpha_i|^2 e_{11})^{1/2} - s)\Bigr\| \\ &= 2s + \sum_{i=1}^n \bigl(\sqrt{s^2 + |\alpha_i|^2} - s\bigr), \\ \|c_1(s)\| &= \Bigl\|2s + \sum_{i=1}^n ((s^2 + |\alpha_i|^2 e_{ii})^{1/2} - s)\Bigr\| \\ &= \Bigl\|2s + \sum_{i=1}^n ((s^2 + |\alpha_i|^2)^{1/2} - s) e_{ii}\Bigr\| \\ &= s + \sqrt{s^2 + \max_i |\alpha_i|^2}.\end{aligned}$$

We must find the minimum of the function

$$g : s \mapsto \|c_0(s)\| \cdot \|c_1(s)\|.$$

If $\alpha_1 = \ldots = \alpha_n = 1$ this is

$$g(s) = \big(s + \sqrt{s^2+1}\big)\big(2s + n\big(\sqrt{s^2+1} - s\big)\big),$$

with strictly positive derivative

$$g'(s) = \frac{2\big(s + \sqrt{s^2+1}\big)^2}{\sqrt{s^2+1}},$$

and thus the minimum is attained at $s = 0$, which yields (5). For (6) where $\alpha_1 = \alpha_2 = 1$ and $\alpha_3 = 2$ we consider

$$g(s) = \big(2\sqrt{s^2+1} + \sqrt{s^2+4} - s\big)\big(s + \sqrt{s^2+4}\big),$$

which has the derivative

$$g'(s) = \frac{\big(2s + 2\sqrt{s^2+4}\big)\big(s\sqrt{s^2+4} + s^2 + 1\big)}{\sqrt{s^2+1}\sqrt{s^2+4}}.$$

This is again strictly positive and the infimum of g is $g(0) = 8$.

EXAMPLE 4 (The Cuntz algebra). In [H, Problem 140] it is shown that the unilateral shift S on ℓ_2 cannot be approximated by invertible operators. Consider the Cuntz algebra, which is generated by n "free" copies of the shift. A priori we cannot apply Theorem 1 to the sum

$$\sum_{i=1}^{n} \lambda(g_i) \otimes \alpha_i S_i.$$

However, since this norm is trivially equal to $\big(\sum |\alpha_i|^2\big)^{1/2}$, the inequality holds even in this case.

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Added in proof. We are working on an exact formula for norms of free operators with matrix coefficients, which will be the subject of a forthcoming paper.

REFERENCES

[A-O] C. A. Akemann and P. A. Ostrand, *Computing norms in group C^*-algebras*, Amer. J. Math. 98 (1976), 1015–1047.

[A] T. Ando, *Topics on Operator Inequalities*, Hokkaido Univ., Sapporo, 1978; MR 58#2451.

[Bo] M. Bożejko, *On $\Lambda(p)$ sets with minimal constant in discrete noncommutative groups*, Proc. Amer. Math. Soc. 51 (1975), 407–412.

[Bu] A. Buchholz, *Norm of convolution by operator-valued functions on free groups*, preprint.

[F] M. Fell, *Weak containment and induced representations of groups*, Canad. J. Math. 14 (1962), 237–268.

[H-P] U. Haagerup and G. Pisier, *Bounded linear operators between C^*-algebras*, Duke Math. J. 71 (1993), 889–925.

[H] P. R. Halmos, *A Hilbert Space Problem Book*, 2nd ed., Springer, 1982.

[K] H. Kesten, *Symmetric random walks on groups*, Trans. Amer. Math. Soc. 92 (1959), 336–354.

[L] M. Leinert, *Faltungsoperatoren auf gewissen diskreten Gruppen*, Studia Math. 52 (1974), 149–158.

[M-O] A. W. Marshall and I. Olkin, *Inequalities: Theory of Majorization and its Applications*, Academic Press, New York, 1979.

[M] R. Mathias, *Concavity of monotone matrix functions of finite order*, Linear and Multilinear Algebra 27 (1990), 129–138.

[Ped] G. K. Pedersen, *Some operator monotone functions*, Proc. Amer. Math. Soc. 36 (1972), 277–301.

[P-P] M. A. Picardello and T. Pytlik, *Norms of free operators*, ibid. 104 (1988), 257–261.

[W] W. Woess, *A short computation of the norms of free convolution operators*, ibid. 96 (1986), 167–170.

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