COLLOQUIUM MATHEMATICUM

VOL. 76 1998 NO. 1

CERTAIN FUNCTION SPACES RELATED TO THE METAPLECTIC REPRESENTATION

BY

KRZYSZTOF NOWAK (WROCŁAW)

1. Introduction and statements of the results. Let G be a Lie group and let μ be a unitary representation of G acting on $L^2(\mathbb{R}^d)$. For a pair of functions f, ϕ, where f is defined on G and $\phi \in L^2(\mathbb{R}^d)$, the *generalized convolution-product operator* $S_{f,\phi}$ is defined as follows:

$$S_{f,\phi}h(g) = f(g)\langle h, \mu_g\phi\rangle, \tag{1}$$

where $h \in L^2(G)$, $g \in G$ and $\langle\, ,\, \rangle$ denotes the inner product in $L^2(\mathbb{R}^d)$.

If $G = \mathbb{R}^d$ and $\mu_y\phi(x) = \phi(x-y)$, then $S_{f,\phi}$ is just the composition of the operators of convolution with ϕ^*, where $\phi^*(x) = \overline{\phi(-x)}$, and pointwise multiplication with f. Such an operator is called a *convolution-product operator*.

Let us recall that for any compact operator $T : H_1 \to H_2$, where H_1, H_2 are two Hilbert spaces, the Nth *singular value* $s_N(T)$ is the Nth element of the nonincreasing rearrangement of the sequence of the eigenvalues of $(T^*T)^{1/2}$. The *Schatten class* S^p consists of those compact operators $T : H_1 \to H_2$ for which the sequence $\{s_N(T)\} \in l^p$, $0 < p \le \infty$. The norm in S^p (quasi-norm for $p < 1$) is defined as follows:

$$\|T\|_{S^p} = \|\{s_N\}\|_{l^p}.$$

The operator $S_{f,\phi}$ is called *local* if

(i) f is bounded,
(ii) the support of f is compact,
(iii) $|f(g)| \ge \varepsilon > 0$ on some open set.

It is not hard to observe that if f_1, f_2 satisfy conditions (i)–(iii) then the

1991 *Mathematics Subject Classification*: 45C05, 47B10, 46E30.

Key words and phrases: convolution-product operators, local Toeplitz operators, mixed norm spaces, Schatten classes, singular values.

This work was financially supported by KBN Grant 2P 301 051 07. Some of the preliminary work was done during the author's stay at the University of Vienna as a Lise Meitner research fellow.

norms

$$\|S_{f_1,\phi}\|_{S^p}, \qquad \|S_{f_2,\phi}\|_{S^p} \qquad (0 < p \le \infty)$$

are equivalent, i.e. there are constants $c_1, c_2 > 0$, which do not depend on ϕ (but may depend on f_1, f_2 and p), such that

$$c_1\|S_{f_1,\phi}\|_{S^p} \le \|S_{f_2,\phi}\|_{S^p} \le c_2\|S_{f_1,\phi}\|_{S^p}$$

(compare Propositions 3.1 and 3.2 of [N2]). Our target is to find, for local Toeplitz operators $S_{f,\phi}$, explicit descriptions of the norms $\|S_{f,\phi}\|_{S^p}$ in terms of ϕ. As we have remarked, up to equivalence of norms, such a description does not depend on the choice of f.

Observe that whenever f_1, f_2 are bounded functions with compact support and $|f_1| = |f_2|$, then $S_{f_1,\phi}$ and $S_{f_2,\phi}$ have the same singular values, in particular for all $0 < p \le \infty$ the norms (quasi-norms for $p < 1$) $\|S_{f_1,\phi}\|_{S^p}$ and $\|S_{f_2,\phi}\|_{S^p}$ are equal. This follows immediately from definition (1) since for any $h_1, h_2 \in L^2(\mathbb{R})$,

$$\langle S^*_{f,\phi} S_{f,\phi} h_1, h_2\rangle = \int\limits_G |f(g)|^2 \langle h_1, \mu_g\phi\rangle\langle \mu_g\phi, h_2\rangle\, dg.$$

A description of the norms $\|S_{f,\phi}\|_{S^p}$, up to equivalence, in the case of the Euclidean space equipped with translations is given in [N1]. For $0 < p \le 2$, estimates from above, in this case, were known long ago (see [S]). It occurred, in the local case, that the same $l^p(L^2)$ norms that provide estimates from above for $0 < p \le 2$ are also good for two-sided estimates for all $0 < p \le \infty$.

Analogous descriptions for the Heisenberg group equipped with the Schrödinger representation and the "$ax + b$" group with the natural action by translations and dilations are presented in [N2].

This paper provides descriptions of the norms $\|S_{f,\phi}\|_{S^p}$ when one takes for μ the restrictions of the metaplectic representation to one- and two-dimensional subgroups of the double cover of $SL(2, \mathbb{R})$.

Our results deal exclusively with local operators $S_{f,\phi}$. We do not need to define the metaplectic representation on the whole double cover of $SL(2, \mathbb{R})$. It is enough to define it on some open neighbourhood of the identity.

Let us recall ([F], p. 185) that the map

$$\Phi(\mathcal{A})(\xi, x) = \frac{b}{2}\xi^2 - a\xi x - \frac{c}{2}x^2, \qquad \mathcal{A} = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \tag{2}$$

is a Lie algebra isomorphism from the algebra $sl(2, \mathbb{R})$ onto the space $\mathcal{Q}$ of real homogeneous quadratic polynomials on $\mathbb{R}^2$, equipped with the Poisson bracket.

We will treat the metaplectic representation as the double-valued unitary representation of $SL(2, \mathbb{R})$ which on the matrices of the form $e^{\mathcal{A}}$,

$\mathcal{A} \in sl(2,\mathbb{R})$, is defined by the formula ([F], p. 186)

$$\mu(e^{\mathcal{A}}) = \pm e^{2\pi i \Phi(\mathcal{A})^w}. \tag{3}$$

We denote by $\Phi(\mathcal{A})^w$ the operator with symbol $\Phi(\mathcal{A})$ in Weyl's pseudodifferential calculus. The exponent on the right is understood in the sense of symbolic calculus of normal operators. We recall that for a general symbol σ the Weyl pseudodifferential operator is defined by the formula

$$\sigma^w h(x) = \iint \sigma\left(\xi, \frac{x+y}{2}\right) e^{2\pi i(x-y)\xi} h(y)\, dy\, d\xi.$$

In our computations we will take the + sign in (3).

Let A, N, K denote the following subgroups of $SL(2,\mathbb{R})$:

$$A = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t > 0 \right\}, \qquad N = \left\{ \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} : t \in \mathbb{R} \right\},$$
$$K = \left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} : t \in \mathbb{R} \right\}.$$

The elements

$$\mathcal{A}_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathcal{A}_N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \mathcal{A}_K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4}$$

of $sl(2,\mathbb{R})$ correspond to the one-dimensional subgroups A, N, K. The metaplectic representation restricted to the subgroups A, N is given by the following formulas ([F], p. 187):

$$\mu(e^{x\mathcal{A}_A})h(y) = e^{2\pi i x \Phi(\mathcal{A}_A)^w} h(y) = e^{-x/2} h(e^{-x} y), \tag{5}$$
$$\mu(e^{x\mathcal{A}_N})h(y) = e^{2\pi i x \Phi(\mathcal{A}_N)^w} h(y) = e^{-\pi i x y^2} h(y). \tag{6}$$

The corresponding symbols are $\Phi(\mathcal{A}_A) = -\xi x$ and $\Phi(\mathcal{A}_N) = -x^2/2$.

The operator $\Phi(\mathcal{A}_K)^w$ equals $\frac{1}{2}(D^2 + X^2)$, where $Dh(x) = \frac{1}{2\pi i} h'(x)$ and $Xh(x) = xh(x)$. This is just the Hermite operator and

$$\mu(e^{x\mathcal{A}_K}) = e^{2\pi i x \Phi(\mathcal{A}_K)^w}$$

is a group of unitary operators, frequently called the *Hermite group*.

Let H be a connected one-dimensional subgroup of $SL(2,\mathbb{R})$. Take $\mathcal{A}_H \in sl(2,\mathbb{R})$ such that $H = \{e^{t\mathcal{A}_H} : t \in \mathbb{R}\}$. There are precisely three classes of one-parameter subgroups of $SL(2,\mathbb{R})$ under the equivalence associated with inner automorphisms of $SL(2,\mathbb{R})$ (see [HH]). Hence there is a matrix $\mathcal{B}_H \in SL(2,\mathbb{R})$ such that

$$\mathcal{B}_H \mathcal{A}_H \mathcal{B}_H^{-1} = \begin{cases} \mathcal{A}_A & \text{if } \det(\mathcal{A}_H) < 0, \\ \mathcal{A}_N & \text{if } \det(\mathcal{A}_H) = 0, \\ \mathcal{A}_K & \text{if } \det(\mathcal{A}_H) > 0, \end{cases} \tag{7}$$

and

$$\mathcal{B}_H = \mathcal{D}_H \mathcal{O}_H, \tag{8}$$

where $\mathcal{D}_H$ is a diagonal matrix with positive entries, $\mathcal{O}_H$ is an orthogonal matrix, and both have determinant 1.

For $0 < p \le \infty$ we introduce the norms (quasi-norms for $0 < p < 1$)

$$\begin{aligned}\|\phi\|_{W_A^p} &= \Big(\sum_{n\in\mathbb{Z}} \Big(\int_n^{n+1} \Big|\int_0^\infty \phi(x) x^{-1/2} x^{-2\pi i\xi}\, dx\Big|^2 d\xi\Big)^{p/2}\Big)^{1/p} \\ &\quad + \Big(\sum_{n\in\mathbb{Z}} \Big(\int_n^{n+1} \Big|\int_0^\infty \phi(-x) x^{-1/2} x^{-2\pi i\xi}\, dx\Big|^2 d\xi\Big)^{p/2}\Big)^{1/p}, \\ \|\phi\|_{W_N^p} &= \Big(\sum_{n\ge 0} \Big(\int_{\sqrt{n}}^{\sqrt{n+1}} |\phi(x)|^2\, dx\Big)^{p/2}\Big)^{1/p} \\ &\quad + \Big(\sum_{n\ge 0} \Big(\int_{-\sqrt{n}}^{-\sqrt{n+1}} |\phi(x)|^2\, dx\Big)^{p/2}\Big)^{1/p}, \\ \|\phi\|_{W_K^p} &= \Big(\sum_{n\ge 0} |\langle \phi, h_n\rangle|^p\Big)^{1/p},\end{aligned}$$

where h_n is the nth Hermite function,

$$h_n(x) = \frac{2^{1/4}}{\sqrt{j!}} \Big(\frac{-1}{2\sqrt{\pi}}\Big)^j e^{\pi x^2} \frac{d^j}{dx^j}(e^{-2\pi x^2}). \tag{9}$$

Now we are ready to state our results. Let H be a Lie subgroup of $SL(2,\mathbb{R})$. We consider local operators $S_{f,\phi}$, $f \in C_c(H)$. For a representation we take the restriction of the metaplectic representation to H. Since, for the description of $\|S_{f,\phi}\|_{S^p}$, the dependence on f is not essential, we drop the letter f in our notation and write $S_{H,\phi}$ instead of $S_{f,\phi}$.

Our first result deals with one-dimensional subgroups of $SL(2,\mathbb{R})$.

THEOREM 1. *Let H be a one-dimensional, connected Lie subgroup of $SL(2,\mathbb{R})$. Let $\mathcal{A}_H$ be such that $H = \{e^{t\mathcal{A}_H} : t \in \mathbb{R}\}$ and let $\mathcal{B}_H$, $\mathcal{D}_H$, $\mathcal{O}_H$ be matrices satisfying conditions* (7), (8). *For any $0 < p \le \infty$ the following equivalence of norms* (*quasi-norms*) *holds*:

$$\|S_{H,\phi}\|_{S^p} \cong \|\mu(\mathcal{D}_H)\mu(\mathcal{O}_H)\phi\|_{W_B^p},$$

where

$$B = \begin{cases} A & \text{if } \det(\mathcal{A}_H) < 0, \\ N & \text{if } \det(\mathcal{A}_H) = 0, \\ K & \text{if } \det(\mathcal{A}_H) > 0. \end{cases}$$

Our second result provides a description of the norms for two-dimensional Lie subgroups of $SL(2,\mathbb{R})$. Before stating it we need to introduce some more notation and recall some facts.

All two-dimensional subalgebras of $sl(2,\mathbb{R})$ are conjugate under the action of orthogonal matrices with determinant 1 ([HH], p. 22), i.e. for every two-dimensional, connected, Lie subgroup H there is an orthogonal matrix $\mathcal{O}_H$ which has determinant 1 and satisfies

$$\mathcal{O}_H H \mathcal{O}_H^{-1} = NA. \tag{10}$$

Let $m \in C_c^\infty(0,\infty)$ be a nonnegative function which satisfies the condition

$$\sum_{k\in\mathbb{Z}} m_k(x) = 1, \qquad m_k(x) = m(x/e^k), \qquad x > 0. \tag{11}$$

We denote by $\mathcal{M}$ the *Mellin transform*, i.e.

$$\mathcal{M}f(\xi) = \int_0^\infty f(x) e^{-2\pi i \log x \log \xi}\,\frac{dx}{x}, \qquad \xi > 0.$$

Let

$$\begin{aligned} \phi_{k,l}^+ &= \zeta^{-1/2}\mathcal{M}^{-1}(m_l \mathcal{M}(\xi^{1/2} m_k(\xi)\phi(\xi)))(\zeta),\\ \phi_{k,l}^- &= \zeta^{-1/2}\mathcal{M}^{-1}(m_l \mathcal{M}(\xi^{1/2} m_k(\xi)\phi(-\xi)))(-\zeta). \end{aligned} \tag{12}$$

For $0 < p \le \infty$ we introduce

$$\begin{aligned} \|\phi\|_{W_{NA}^p}^p &= \sum_l \Big(\sum_{k\le 0} \|\phi_{k,l}^+\|_{L^2}^2\Big)^p + \sum_l \Big(\sum_{k\le 0} \|\phi_{k,l}^-\|_{L^2}^2\Big)^p \\ &\quad + \sum_{k>0,l} \bigg(\frac{1}{[e^{2k}]}\sum_{r=0}^{[e^{2k}]} \|\phi_{k,l+r}^+\|_{L^2}^2\bigg)^p + \sum_{k>0,l} \bigg(\frac{1}{[e^{2k}]}\sum_{r=0}^{[e^{2k}]} \|\phi_{k,l+r}^-\|_{L^2}^2\bigg)^p. \end{aligned} \tag{13}$$

THEOREM 2. *Let H be a two-dimensional, connected Lie subgroup of $SL(2,\mathbb{R})$. Let $\mathcal{O}_H$ be a matrix satisfying* (10) *and let $0 < p \le \infty$. The following equivalence of norms* (*quasi-norms*) *holds*:

$$\|S_{H,\phi}\|_{S^p} \cong \|\mu(\mathcal{O}_H)\phi\|_{W_{NA}^p}.$$

COMMENTS. (i) The operators $\mu(\mathcal{D}_H)$, $\mu(\mathcal{O}_H)$ are interpreted as phase space dilation and rotation. Theorems 1 and 2 show that all the norms obtained from one- and two-dimensional subgroups of $SL(2,\mathbb{R})$ by phase space dilations and rotations reduce to the norms W_A^p, W_N^p, W_K^p, and W_{NA}^p.

(ii) The norms that show up in Theorem 1 for $\det \mathcal{A}_H \le 0$ are versions of the mixed norm $l^p(L^2)$. They are obtained from it by changes of variables and an application of the Fourier transform.

(iii) One can apply the results of [N1] to obtain, in Theorem 1, two-sided eigenvalue estimates instead of norm equivalence.

We refer the reader to [R] and [N2] for more background and motivation and to [Fe] for a survey on mixed norm spaces.

Acknowledgements. The author thanks Prof. Karl Hofmann for several interesting comments.

2. Preliminaries. In this section we collect several facts about the Fock space, the metaplectic representation and convolution-product operators. These facts are needed for the proofs of Theorems 1 and 2.

Recall that the *Bargmann transform*

$$Bf(z) = 2^{1/4} \int_{-\infty}^{\infty} f(x) e^{2\pi xz - \pi x^2 - (\pi/2)z^2}\, dx \tag{14}$$

is a unitary map from $L^2(\mathbb{R})$ onto the *Fock space*

$$\mathcal{F} = \Big\{ F : F \text{ entire on the complex plane, } \int_{\mathbb{C}} |F(z)|^2 e^{-\pi|z|^2}\, dz < \infty \Big\},$$

and that the system of functions

$$\{\pi^{n/2} z^n/(n!)^{1/2}\}_{n\ge 0} \tag{15}$$

is an orthonormal basis of $\mathcal{F}$.

Let O_t be a rotation matrix, i.e.

$$O_t = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}. \tag{16}$$

FACT 1 ([F], p. 184). *Up to a phase factor* (*i.e. a complex number of absolute value* 1),

$$B\mu(O_t)B^{-1}F(z) = F(e^{it}z), \qquad F \in \mathcal{F}. \tag{17}$$

FACT 2 ([F], p. 51). *Let h_n be the nth Hermite function defined in* (9). *Then*

$$B(h_n) = \pi^{n/2} z^n/(n!)^{1/2}. \tag{18}$$

FACT 3 ([St], p. 578). *For $\mathcal{B} \in SL(2,\mathbb{R})$*,

$$\mu(\mathcal{B})\sigma^w \mu(\mathcal{B}^{-1}) = (\sigma \circ \mathcal{B}^*)^w. \tag{19}$$

LEMMA 1 ([N2], Proposition 3.8). *If $b(l)$ is an absolutely summable, positive definite sequence and $b(0) > 0$, then for $0 < p \le \infty$*,

$$\|b(n-m)a_m\overline{a}_n\|_{S^{p/2}}^{p/2} \cong \sum_{n\ge 0} |a_n|^p. \tag{20}$$

LEMMA 2 ([N1], p. 306). *If $f \in C_{\rm c}(\mathbb{R})$, $f \ne 0$, then*

$$\|f(x)\overline{\phi(y-x)}\|_{S^p} \cong \|\widehat{\phi}\|_{l^p(L^2)}, \qquad \|f(x)e^{-\pi ixy}\overline{\phi(y)}\|_{S^p} \cong \|\phi\|_{l^p(L^2)}, \tag{21}$$

where $\|\phi\|_{l^p(L^2)} = (\sum_n (\int_n^{n+1} |\phi(x)|^2\, dx)^{p/2})^{1/p}$.

LEMMA 3 ([N2], p. 60). *Let τ be the unitary representation of the "ax+b" group $P = \{(b,a) : b \in \mathbb{R},\ a > 0\}$ acting on $L^2(0,\infty)$, given by the formula*

$$\tau(b,a)\phi(y) = e^{-\pi iby}a^{1/2}\phi(ay). \tag{22}$$

Then

$$\|S_{P,\phi}\|_{S^p}^p = \sum_l \Big(\sum_{k\le 0} \|\phi_{k,l}\|_{L^2}^2\Big)^p + \sum_{k>0,l} \bigg(\frac{1}{[e^k]} \sum_{r=0}^{[e^k]} \|\phi_{k,l+r}\|_{L^2}^2\bigg)^p, \tag{23}$$

where

$$\phi_{k,l} = \zeta^{-1/2}\mathcal{M}^{-1}(m_l\mathcal{M}(\xi^{1/2}m_k(\xi)\phi(\xi)))(\zeta).$$

Proofs of Theorems 1 and 2

Proof of Theorem 1. The steps of the proof are the following:

(i) $\|S_{K,\phi}\|_{S^p} \cong \|\phi\|_{W_K^p}$,
(ii) $\|S_{N,\phi}\|_{S^p} \cong \|\phi\|_{W_N^p}$,
(iii) $\|S_{A,\phi}\|_{S^p} \cong \|\phi\|_{W_A^p}$,
(iv) reduction of the general case to (i)–(iii).

(i) Consider the composition of the inverse Bargmann transform B^{-1} and the operator $S_{K,\phi}$. By Fact 1 we obtain

$$\begin{aligned} S_{K,\phi}B^{-1}F(x) &= f(x)\langle B^{-1}F, \mu(e^{x\mathcal{A}_K})\phi\rangle \\ &= f(x)\langle F, B\mu(e^{x\mathcal{A}_K})B^{-1}B\phi\rangle \\ &= f(x)\int_{\mathbb{C}} F(z)\overline{\Phi(e^{ix}z)}e^{-\pi|z|^2}\,dz, \qquad \Phi = B\phi. \end{aligned}$$

It follows that the integral kernel of $S_{K,\phi}B^{-1}$ is

$$f(x)\overline{\Phi(e^{ix}z)}. \tag{24}$$

Expand $\Phi(z)$ with respect to the orthonormal basis $\pi^{n/2}z^n/(n!)^{1/2}$:

$$\Phi(z) = \sum_{n=0}^{\infty} a_n \frac{\pi^{n/2}z^n}{(n!)^{1/2}}. \tag{25}$$

We may assume that $f \in C_c^\infty$ and that $f \ge 0$. An obvious calculation shows that the integral kernel of $BS_{K,\phi}^*S_{K,\phi}B^{-1}$ equals

$$\sum_{n,m} \int f(t)^2 e^{-i(n-m)t}\,dt\, a_m\overline{a}_n \frac{\pi^{m/2}w^m}{(m!)^{1/2}} \cdot \frac{\pi^{n/2}\overline{z}^n}{(n!)^{1/2}}. \tag{26}$$

It follows that the matrix of the operator $BS_{K,\phi}^*S_{K,\phi}B^{-1}$ with respect to the basis $\{\pi^{n/2}z^n/(n!)^{1/2}\}_{n\ge 0}$ equals

$$\widehat{f^2}(n-m)a_m\overline{a}_n. \tag{27}$$

By Lemma 1 and (27) we obtain

$$\|S_{K,\phi}\|_{S^p}^p = \|S_{K,\phi}^* S_{K,\phi}\|_{S^{p/2}}^{p/2} = \|\{\widehat{f^2}(n-m)a_m\overline{a}_n\}_{n,m\ge 0}\|_{S^{p/2}}^{p/2} \cong \sum_{n=0}^{\infty}|a_n|^p. \tag{28}$$

By Fact 2 we have

$$a_n = \left\langle B\phi, \frac{\pi^{n/2}z^n}{(n!)^{1/2}}\right\rangle = \left\langle \phi, B^{-1}\left(\frac{\pi^{n/2}z^n}{(n!)^{1/2}}\right)\right\rangle = \langle \phi, h_n\rangle. \tag{29}$$

Part (i) follows from (28) and (29).

(ii) Since the formula (6) holds we may assume that $\phi = 0$ on $(-\infty, 0)$.

Let $U_N : L^2((0,\infty)) \to L^2((0,\infty))$ be the unitary map defined by the formula

$$U_N\phi(y) = 2^{1/2}y^{1/2}\phi(y^2). \tag{30}$$

Its inverse equals

$$U_N^{-1}\phi(y) = 2^{-1/2}y^{-1/4}\phi(y^{1/2}). \tag{31}$$

It follows from (6), (30) and (31) that

$$U_N^{-1}\mu(e^{x\mathcal{A}_N})U_N\phi(y) = e^{\pi ixy}\phi(y). \tag{32}$$

By (32) we see that

$$\begin{aligned} S_{N,\phi}U_N h(x) &= f(x)\langle U_N h, \mu(e^{x\mathcal{A}_N})\phi\rangle \\ &= f(x)\langle h, U_N^{-1}\mu(e^{x\mathcal{A}_N})U_N U_N^{-1}\phi\rangle \\ &= f(x)\int_{\mathbb{R}} h(y)\overline{e^{\pi ixy}(U_N^{-1}\phi)(y)}\,dy, \end{aligned}$$

thus the integral kernel of $S_{N,\phi}U_N$ equals

$$f(x)\overline{e^{\pi ixy}(U_N^{-1}\phi)(y)}. \tag{33}$$

From Lemma 2 and (33) we conclude that

$$\|S_{N,\phi}\|_{S^p}^p \cong \|U_N^{-1}\phi\|_{l^p(L^2)}^p = \sum_{n\ge 0}\Big(\int_{\sqrt{n}}^{\sqrt{n+1}}|\phi(x)|^2\,dx\Big)^{p/2}.$$

(iii) As in (ii) we may assume that $\phi = 0$ on $(-\infty, 0)$. This is a consequence of the formula (5).

Define a unitary map $U_A : L^2(\mathbb{R}) \to L^2((0,\infty))$ by the formula

$$U_A h(x) = x^{-1/2}h(\log x). \tag{34}$$

Its inverse equals

$$U_A^{-1}h(u) = e^{u/2}h(e^u). \tag{35}$$

From (5), (34) and (35) we conclude that

$$U_A^{-1}\mu(e^{x\mathcal{A}_A})U_A h(y) = h(y-x). \tag{36}$$

By the same argument as in (ii) we check that the integral kernel of $S_{A,\phi}U_A$ is

$$f(x)(U_A^{-1}\phi)(y-x). \tag{37}$$

From Lemma 2 and (37) we obtain

$$\begin{aligned}\|S_{A,\phi}\|_{S^p}^p &\cong \sum_{n\in\mathbb{Z}}\Big(\int_n^{n+1} |(U_A^{-1}\phi)^\wedge(\lambda)|^2\,d\lambda\Big)^{p/2}\\ &= \sum_{n\in\mathbb{Z}}\Big(\int_n^{n+1}\Big|\int_0^\infty \phi(x)x^{-1/2}e^{2\pi i\lambda\log x}\,dx\Big|^2 d\lambda\Big)^{p/2}.\end{aligned}$$

(iv) Let $\mathcal{A}\in SL(2,\mathbb{R})$. By Fact 3 we obtain

$$\begin{aligned}S_{H,\phi}\mu(\mathcal{A})^{-1}h(x) &= f(x)\langle \mu(\mathcal{A})^{-1}h, \mu(e^{x\mathcal{A}_H})\phi\rangle\\ &= f(x)\langle h, \mu(\mathcal{A})e^{2\pi i x\Phi(\mathcal{A}_H)^w}\mu(\mathcal{A})^{-1}\mu(\mathcal{A})\phi\rangle\\ &= f(x)\langle h, e^{2\pi i x(\Phi(\mathcal{A}_H)\circ\mathcal{A}^*)^w}\mu(\mathcal{A})\phi\rangle.\end{aligned} \tag{38}$$

Since

$$\Phi(\mathcal{A}_H)(w) = -\tfrac{1}{2}\langle \mathcal{A}_H Jw, w\rangle, \quad \text{where} \quad J = \begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix},$$

and $J\mathcal{B}_H^* = \mathcal{B}_H^{-1}J$ we obtain

$$\begin{aligned}\Phi(\mathcal{A}_H)\mathcal{B}_H^* w &= -\tfrac{1}{2}\langle \mathcal{A}_H J\mathcal{B}_H^* w, \mathcal{B}_H^* w\rangle\\ &= -\tfrac{1}{2}\langle \mathcal{B}_H\mathcal{A}_H\mathcal{B}_H^{-1}Jw, w\rangle = \Phi(\mathcal{A}_B)w.\end{aligned} \tag{39}$$

Combining (38) and (39) provides

$$\|S_{H,\phi}\|_{S^p} = \|S_{B,\,\mu(\mathcal{B}_H)\phi}\|_{S^p} \cong \|\mu(\mathcal{B}_H)\phi\|_{W_B^p}.$$

Proof of Theorem 2. Observe that the map

$$\Psi((b,a)) = \begin{pmatrix} a^{-1/2} & 0\\ a^{-1/2}b & a^{1/2}\end{pmatrix} \tag{40}$$

is a group isomorphism from the "$ax+b$" group P onto NA. Since

$$\begin{pmatrix} a^{-1/2} & 0\\ a^{-1/2}b & a^{1/2}\end{pmatrix} = \begin{pmatrix} 1 & 0\\ b & 1\end{pmatrix}\begin{pmatrix} a^{-1/2} & 0\\ 0 & a^{1/2}\end{pmatrix} \tag{41}$$

the metaplectic representation is defined on NA as follows:

$$\begin{aligned}&\mu\left(\begin{pmatrix} a^{-1/2} & 0\\ a^{-1/2}b & a^{1/2}\end{pmatrix}\right)\phi(y)\\ &= \mu\left(\begin{pmatrix} 1 & 0\\ b & 1\end{pmatrix}\right)\mu\left(\begin{pmatrix} a^{-1/2} & 0\\ 0 & a^{1/2}\end{pmatrix}\right)\phi(y) = e^{\pi i b y^2}a^{1/4}\phi(a^{1/2}y).\end{aligned} \tag{42}$$

It follows easily from (42) that

$$U_N \mu_{NA} U_{NA}^{-1} = \tau, \tag{43}$$

where U_N is the map defined in part (ii) of the proof of Theorem 1, μ_{NA} is the restriction of μ to the subgroup NA, and τ is defined in Lemma 3. We combine (43) and Lemma 3 to get

$$\|S_{NA,\phi}\|_{S^p} \cong \|\phi\|_{W^p_{NA}}. \tag{44}$$

The last part follows in the same way as part (iv) of the proof of Theorem 1.

REFERENCES

[Fe] H. G. Feichtinger, *Wiener amalgams over Euclidean spaces and some of their applications*, in: Lecture Notes in Pure and Appl. Math. 136, K. Jarosz (ed.) Dekker, 1992, 123–137.

[F] G. B. Folland, *Harmonic Analysis in Phase Space*, Ann. Math. Stud. 122, Princeton Univ. Press, Princeton, N.J., 1989.

[HH] J. Hilgert and K. Hofmann, *Old and new on* $SL(2,\mathbb{R})$, Manuscripta Math. 54 (1985), 17–52.

[N1] K. Nowak, *Singular value estimates for certain convolution-product operators*, J. Fourier Anal. Appl. 3 (1995), 297–310.

[N2] —, *Local Toeplitz operators based on wavelets: phase space patterns for rough wavelets*, Studia Math. 119 (1996), 37–64.

[R] R. Rochberg, *The use of decomposition theorems in the study of operators*, in: Wavelets: Mathematics and Applications, J. J. Benedetto and M. Frazier (eds.), CRC Press, 1994, 547–570.

[S] B. Simon, *Trace Ideals and Their Applications*, Cambridge Univ. Press, London, 1979.

[St] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Univ. Press, Princeton, 1993.

Institute of Mathematics
Wrocław University
Pl. Grunwaldzki 2/4
50-384 Wrocław, Poland

Received 19 March 1997;
revised 18 September 1997