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ENDPOINT BOUNDS FOR CONVOLUTION OPERATORS WITH SINGULAR MEASURES

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Let $S \subset \mathbb{R}^{n+1}$ be the graph of the function $\varphi : [-1,1]^n \to \mathbb{R}$ defined by $\varphi(x_1,\ldots,x_n) = \sum_{j=1}^n |x_j|^{\beta_j}$, with $1 < \beta_1 \le \ldots \le \beta_n$, and let μ the measure on $\mathbb{R}^{n+1}$ induced by the Euclidean area measure on S. In this paper we characterize the set of pairs (p,q) such that the convolution operator with μ is L^p-L^q bounded.

1. Introduction. In this paper we study convolution operators with singular measures μ given by $\mu(E) = \int_Q \chi_E(x,\varphi(x))\,dx$ where $Q = [-1,1]^n$ and $\varphi(x) = \sum_{j=1}^n |x_j|^{\beta_j}$ for $x = (x_1,\ldots,x_n)$, $\beta_j > 1$, $1 \le j \le n$. We set, for $y \in \mathbb{R}^{n+1}$, $T_\mu f(y) = (\mu * f)(y)$ and $E_\mu = \{(1/p,1/q) : \|T_\mu\|_{p,q} < \infty,$ $1 \le p,q \le \infty\}$, where the L^p spaces are taken with respect to the Lebesgue measure on $\mathbb{R}^{n+1}$. The set E_μ is known in several cases. If $\beta_j = 2$, $1 \le j \le n$, and the graph of φ has nonzero Gaussian curvature at each point, then a theorem of Littman implies that E_μ is the closed triangle with vertices $(0,0)$, $(1,1)$ and $((n+1)/(n+2), 1/(n+2))$ (see [O]). Now, if the curvature vanishes at some point, E_μ can be strictly contained in the above triangle. Related examples in a more general context can be found in [C], [O] and [R-S]. In [F-G-U] we showed that E_μ is a polygonal region. We gave a complete description of it, as a closed polygon, when $\beta_j \le n+2$, $1 \le j \le n$. In the other cases certain endpoint cases were left unsolved.

In this paper we characterize E_μ completely, using a different argument that follows the ideas developed by M. Christ in [C].

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2. Preliminaries. For $1 \le k \le n$, we consider an even function $\Phi_k \in C_c^\infty(\mathbb{R})$ such that $\operatorname{supp}\Phi_k \subset \{t \in \mathbb{R} : 2^{1/\beta_k} \le |t| \le 2^{4/\beta_k}\}$, $0 \le \Phi_k \le 1$ and

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$\sum_{r\in\mathbb{Z}} \Phi_k(2^{r/\beta_k}t) = 1$ if $t \neq 0$. For $r_1,\dots,r_n \in \mathbb{N}$ and a Borel set E, we set

$$\nu_{r_1,\dots,r_n}(E) = \int \chi_E(x_1,\dots,x_n,\varphi(x_1,\dots,x_n)) \prod_{1\le k\le n} \Phi_k(2^{r_k/\beta_k}x_k)\,dx_1\dots dx_n.$$

Then

$$\mu \le \nu = \sum_{r_1,\dots,r_n\in\mathbb{N}} \nu_{r_1,\dots,r_n}. \tag{2.1}$$

Following the approach in [C], for $1\le k\le n$, we introduce a C^∞ partition of unity $\{m_{k,r}\}_{r\in\mathbb{Z}}$ in $\mathbb{R}^2$ minus the coordinate axes, with $m_{k,r}$ homogeneous of degree zero (with respect to the Euclidean dilations on $\mathbb{R}^2$) such that $m_{k,r}(t_1,t_2) = m_{k,0}(2^{-r/\beta_k}t_1, 2^{-r}t_2)$ and $\operatorname{supp} m_{k,r} \subset \{(t_1,t_2) : 2^{-r/\beta_k-1}|t_1| \le 2^{-r}|t_2| \le 2^{-r/\beta_k+2}|t_1|\}$. Also we set $M_{k,r}(\xi_1,\dots,\xi_{n+1}) = m_{k,r}(\xi_k,\xi_{n+1})$. Let $Q_{k,r}$ be the operator with multiplier $M_{k,r}$, and let C_0 be a constant such that $\widetilde{m}_{k,r} = \sum_{|i-r|\le C_0} m_{k,i}$ is identically one on $\operatorname{supp} m_{k,r}$. We define $\widetilde{Q}_{k,r} = \sum_{|i-r|\le C_0} Q_{k,i}$ and we denote by $\widetilde{M}_{k,r}$ its multiplier.

Let $h \in C_c^\infty(\mathbb{R}^2)$ be identically one in a neighborhood of the origin, let $H_{k,r}(\xi_1,\dots,\xi_{n+1}) = h(2^{-r/\beta_k}\xi_k, 2^{-r}\xi_{n+1})$ and let $P_{k,r}$ be the Fourier multiplier operator with symbol $H_{k,r}$.

Throughout this work, c will denote a positive constant not necessarily the same at each occurrence. For $g:\mathbb{R}^n\to\mathbb{C}$ we set $g^\vee(x) = g(-x)$. If $g\in S(\mathbb{R}^n)$ we denote by $\widehat{g}$ its Fourier transform.

The following lemmas provide a suitable version of arguments contained in [C] adapted to our n-dimensional setting. Lemma 2.2 is the crux of Christ's argument.

LEMMA 2.2. *Let $\{\sigma_r\}_{r\in\mathbb{N}}$ be a sequence of positive measures on $\mathbb{R}^{n+1}$, and let $T_r f = \sigma_r * f$ for $f\in S(\mathbb{R}^{n+1})$. Suppose $1\le k\le n$, $1<p\le 2$ and $p\le q<\infty$. If there exists $A>0$ such that*

$$\sup_{r\in\mathbb{N}} \|T_r\|_{p,q} \le A, \qquad \Big\| \sum_{1\le r\le R} T_r P_{k,r}\Big\|_{p,q} \le A \quad \textit{and}$$

$$\Big\| \sum_{1\le r\le R} T_r (I-P_{k,r})(I-\widetilde{Q}_{k,r})\Big\|_{p,q} \le A \quad \textit{for all } R\in\mathbb{N},$$

then there exists $c>0$, independent of A, R and $\{\sigma_r\}_{r\in\mathbb{N}}$, such that

$$\Big\| \sum_{1\le r\le R} T_r\Big\|_{p,q} \le cA.$$

Proof. We note that, if $\varepsilon_r = \pm 1$ then $\sum_{r\in\mathbb{N}} \varepsilon_r \widetilde{Q}_{k,r}$ satisfies the hypothesis of the Marcinkiewicz multiplier theorem (see [S], p. 109). Thus

$\|\sum_{r\in\mathbb{N}}\varepsilon_r\widetilde{Q}_{k,r}\|_{p,p}\le c$, with c independent of $\{\varepsilon_r\}$. As in [S], 5.3, p. 105, we get the Littlewood–Paley inequality

$$\Big\|\Big(\sum_{r\in\mathbb{N}}|\widetilde{Q}_{k,r}f|^2\Big)^{1/2}\Big\|_p\le c\|f\|_p.$$

Let S_R be the operator given by $S_R(\{g_r\}_r)=\{h_r\}_r$ where $h_r=T_rg_r$ for $1\le r\le R$, and $h_r=0$ otherwise. As usual, we denote by $\|S_R\|_{p,q,s}$ the norm of $S_R:L^p(l^s)\to L^q(l^s)$. As in the proof of Theorem 1 of [C], there exists $c>0$, independent of R, $\{\sigma_r\}_{r\in\mathbb{N}}$ and $f\in S(\mathbb{R}^{n+1})$, such that

$$\Big\|\sum_{1\le r\le R}T_r(I-P_{k,r})\widetilde{Q}_{k,r}f\Big\|_q\le c\|S_R\|_{p,q,2}(\|\{f_r\}_r\|_{L^p(l^2)}+\|\{P_{k,r}f_r\}_r\|_{L^p(l^2)})$$

where $f_r=\widetilde{Q}_{k,r}f$. Let $x=(x_1,\ldots,x_{n+1})$. We have, for $f\in S(\mathbb{R}^{n+1})$,

$$|\widehat{H}^\vee_{k,r_k}*f(x)|=|2^{-r_k(1+\beta_k^{-1})}((2^{-r_k}\bullet\widehat{h}^\vee)*f_{\overline{x}})(x_k,x_{n+1})|$$

where $\overline{x}=(x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_n)$, $f_{\overline{x}}(y_1,y_2)=f(x_1,\ldots,x_{k-1},y_1,x_{k+1},\ldots,x_n,y_2)$ and $(2^{-r_k}\bullet\widehat{h}^\vee)(y_1,y_2)=\widehat{h}^\vee(2^{-r_k/\beta_k}y_1,2^{-r_k}y_2)$. Thus, using a result in [St], p. 85, we see that there exists c independent of k,r such that

$$|P_{k,r}f_r|\le cM(f_r)\tag{2.3}$$

where M is the strong maximal function defined as in [St], p. 83. Let $\overline{M}$ be the vector-valued maximal operator associated with M defined by $\overline{M}(\{g_r\}_{r\in\mathbb{N}})=\{Mg_r\}_{r\in\mathbb{N}}$. Then $\overline{M}$ is bounded on $L^p(l^2)$ for $p\le 2$, so for such p,

$$\Big\|\sum_{1\le r\le R}T_r(I-P_{k,r})\widetilde{Q}_{k,r}f\Big\|_q\le c\|S_R\|_{p,q,2}\|\{f_r\}_r\|_{L^p(l^2)}\le c\|S_R\|_{p,q,2}\|f\|_p.$$

The lemma follows as in the proof of Theorem 1 of [C]. ∎

Lemma 2.4. *For $1<p,q<\infty$ and $R\in\mathbb{N}$,*

$$\Big\|\sum_{1\le r_k\le R}T_{\nu_{r_1,\ldots,r_n}}P_{k,r_k}\Big\|_{p,q}\le c\Big\|\sum_{1\le r_k\le R}T_{\nu_{r_1,\ldots,r_n}}\Big\|_{p,q}$$

with c independent of R.

Proof. Since $\nu_{r_1,\ldots,r_n}$ is a positive measure, the lemma follows from (2.3) and the boundedness of the strong maximal function (see [St], p. 84). ∎

Lemma 2.5. *For $1<p,q<\infty$ and $R\in\mathbb{N}$,*

$$\Big\|\sum_{1\le r_k\le R}T_{\nu_{r_1,\ldots,r_n}}(I-P_{k,r_k})(I-\widetilde{Q}_{k,r_k})\Big\|_{p,q}\le c\Big\|\sum_{1\le r_k\le R}T_{\nu_{r_1,\ldots,r_n}}\Big\|_{p,q}$$

with c independent of R.

Proof. We decompose

$$\sum_{1\le r_k\le R} T_{\nu_{r_1,\ldots,r_n}}(I-P_{k,r_k})(I-\widetilde{Q}_{k,r_k})$$
$$= \sum_{1\le r_k\le R} T_{\nu_{r_1,\ldots,r_n}} - \sum_{1\le r_k\le R} T_{\nu_{r_1,\ldots,r_n}} P_{k,r_k} - \sum_{1\le r_k\le R} T_{\nu_{r_1,\ldots,r_n}} \widetilde{Q}_{k,r_k}$$
$$+ \sum_{1\le r_k\le R} T_{\nu_{r_1,\ldots,r_n}} P_{k,r_k}\widetilde{Q}_{k,r_k}.$$

In view Lemma 2.4, it is enough to study the last two terms. By (2.3), for $f\in S(\mathbb{R}^{n+1})$,

$$\Big\|\sum_{1\le r_k\le R} T_{\nu_{r_1,\ldots,r_n}} P_{k,r_k}\widetilde{Q}_{k,r_k} f\Big\|_q$$
$$\le c\Big\|\sum_{1\le r_k\le R} T_{\nu_{r_1,\ldots,r_n}}\Big\|_{p,q}\|M(\sup_{r\in\mathbb{N}}|\widetilde{Q}_{k,r}f|)\|_p$$
$$\le c\Big\|\sum_{1\le r_k\le R} T_{\nu_{r_1,\ldots,r_n}}\Big\|_{p,q}\|\sup_{r}|\widetilde{Q}_{k,r}f|\|_p$$
$$\le c\Big\|\sum_{1\le r_k\le R} T_{\nu_{r_1,\ldots,r_n}}\Big\|_{p,q}\|\{\widetilde{Q}_{k,r}f\}_r\|_{L^p(l^2)}$$
$$\le c\Big\|\sum_{1\le r_k\le R} T_{\nu_{r_1,\ldots,r_n}}\Big\|_{p,q}\|f\|_p.$$

The estimation of the term $\sum_{1\le r_k\le R} T_{\nu_{r_1,\ldots,r_n}}\widetilde{Q}_{k,r_k}f$ is analogous. ■

LEMMA 2.6. *The kernel of the convolution operator*

$$\sum_{1\le r_k\le R} T_{\nu_{r_1,\ldots,r_n}}(I-P_{k,r_k})(I-\widetilde{Q}_{k,r_k})$$

belongs to weak-$L^{1+\beta_k^{-1}}$ *and its norm is less than* $c2^{-\sum_{j\ne k} r_j/\beta_j}$, *with* c *independent of* R *and* r_j, $j\ne k$.

Proof. We set

$$I_k(t_1,t_2)=\int \Phi_k(s)e^{-ist_1-i|s|^{\beta_k}t_2}\,ds \qquad \text{for } (t_1,t_2)\in\mathbb{R}^2.$$

A computation shows that the kernel $K_{r_1,\ldots,r_n}$ of the convolution operator $T_{\nu_{r_1,\ldots,r_n}}(I-P_{k,r_k})(I-\widetilde{Q}_{k,r_k})$ is the function given by

$$K^{\vee}_{r_1,\ldots,r_n}(x_1,\ldots,x_{n+1})$$
$$= 2^{r_k}G_k\Big(2^{r_k/\beta_k}x_k, 2^{r_k}\Big(x_{n+1}+\sum_{j\ne k}|x_j|^{\beta_j}\Big)\Big)\prod_{j\ne k}\Phi_j(2^{r_j/\beta_j}x_j)$$

where

$$G_k = (I_k(1-h)(1-\widetilde{m}_{k,0}))^\wedge.$$

Taking account of Proposition 1 of [St], p. 331, we note that if we choose, in the definition of $\widetilde{m}_{k,0}$, C_0 large enough, we find that $G_k \in S(\mathbb{R}^2)$.

For $1 \le k \le n$ and $r_1, \ldots, r_{k-1}, r_{k+1}, \ldots, r_n \in \mathbb{N}$, we set

$$V^k_{r_1,\ldots,r_{k-1},r_{k+1},\ldots,r_n} = \{(x_1,\ldots,x_n) \in Q : 2^{-(r_j-1)/\beta_j} \le |x_j| \le 2^{-(r_j-4)/\beta_j},\ j \ne k\}.$$

Since $G_k \in S(\mathbb{R}^2)$, we obtain

$$\sum_{1\le r_k\le R} |K^\vee_{r_1,\ldots,r_n}(x_1,\ldots,x_{n+1})| \le c\frac{\chi_{V^k_{r_1,\ldots,r_{k-1},r_{k+1},\ldots,r_n}}(x_1,\ldots,x_n)}{|x_k|^{\beta_k} + |\sum_{j\ne k}|x_j|^{\beta_j} + x_{n+1}|}$$

with c independent of R and r_j, $j \ne k$. Thus

$$\Big|\Big\{x \in \mathbb{R}^{n+1} : \sum_{1\le r_k\le R} |K^\vee_{r_1,\ldots,r_n}(x_1,\ldots,x_{n+1})| > \lambda\Big\}\Big| \le c2^{-\sum_{j\ne k} r_j/\beta_j}\frac{1}{\lambda^{1+1/\beta_k}}$$

and the lemma follows. ∎

LEMMA 2.7. *The kernel of the convolution operator*

$$\sum_{1\le r_k\le R} T_{\nu_{r_1,\ldots,r_n}} P_{k,r_k}$$

belongs to weak-$L^{1+\beta_k^{-1}}$ *with norm less than* $c2^{-\sum_{j\ne k} r_j/\beta_j}$, *with* c *independent of* R *and* r_j, $j \ne k$.

Proof. As in Lemma 2.6 we can see that the kernel of $T_{\nu_{r_1,\ldots,r_n}} P_{k,r_k}$ is given by

$$\Big(\prod_{j\ne k} \varPhi_j(2^{r_j/\beta_j} x_j^j) G_k\Big(2^{r_k/\beta_k} x_k, 2^{r_k}\Big(x_{n+1} + \sum_{j\ne k}|x_j|^{\beta_j}\Big)\Big)\Big)^\vee$$

where now $G_k = (I_k h)^\wedge$. Since $G_k \in S(\mathbb{R}^2)$, as before, the lemma follows. ∎

3. The main result. Let Q, φ, μ and E_μ be defined as in the introduction. Without loss of generality we suppose $1 < \beta_1 \le \ldots \le \beta_n$. It is easy to check that E_μ contains the principal diagonal, and the Riesz–Thorin theorem implies that E_μ is a convex subset of $[0,1]\times[0,1]$. It is well known that if $(1/p, 1/q) \in E_\mu$ then $p \le q$ (see [S-W], p. 33).

For $1 \le k \le n$, we set $S_k = \sum_{j=k}^n \beta_j^{-1}$, also we set $S_{n+1} = 0$. We denote by L_k, $0 \le k \le n$, the lines given by

$$\frac{1}{q} = \frac{k+1+S_{k+1}}{1+S_{k+1}}\cdot\frac{1}{p} - \frac{k+S_{k+1}}{1+S_{k+1}}$$

Also we denote by A_k, $0 \le k \le n$, the intersection of L_k with the nonprincipal diagonal $\{(x, 1-x) : 0 \le x \le 1\}$ and by B_k, $1 \le k \le n$, the intersection of L_{k-1} with L_k. A computation shows that for $0 \le k \le n$,

$$A_k = \left(\frac{1+k+2S_{k+1}}{k+2+2S_{k+1}}, \frac{1}{k+2+2S_{k+1}} \right),$$

and for $1 \le k \le n$,

$$B_k = \left(\frac{1+S_{k+1}+(k-1)\beta_k^{-1}}{1+k\beta_k^{-1}+S_{k+1}}, \frac{1-\beta_k^{-1}}{1+k\beta_k^{-1}+S_{k+1}} \right).$$

Let $\Sigma^{(\beta_1,\ldots,\beta_n)}$ be the closed convex polygonal region contained in $[0,1] \times [0,1]$, given by the intersection of the lower half space determined by the principal diagonal with all the upper half spaces determined by the lines L_k, $0 \le k \le n$, and all the upper half spaces determined by their symmetric images with respect to the nonprincipal diagonal. Lemma 2.1 and Remark 2.4 of [F-G-U] say that $E_\mu \subset \Sigma^{(\beta_1,\ldots,\beta_n)}$. Let k_0 be defined by $k_0 = 0$ if $\beta_1 > 2$ and $k_0 = \max\{k : 1 \le k \le n,\ \beta_k \le 2\}$ if $\beta_1 \le 2$. Remark 2.6 of [F-G-U] says that, for $k_0 < n$, $\Sigma^{(\beta_1,\ldots,\beta_n)} = \Sigma^{(2,\ldots,2,\beta_{k_0+1},\ldots,\beta_n)}$ is the closed convex polygonal region with vertices A_{k_0}, $(0,0)$, $(1,1)$, $B_n, B_{n-1}, \ldots, B_{k_0+1}$ and their symmetric images $B'_n, B'_{n-1}, \ldots B'_{k_0+1}$ with respect to the nonprincipal diagonal, and for $k_0 = n$, $\Sigma^{(\beta_1,\ldots,\beta_n)}$ is the closed triangular region with vertices $(0,0)$, $(1,1)$ and A_n. Our aim is to prove that $E_\mu = \Sigma^{(\beta_1,\ldots,\beta_n)}$ for $k_0 < n$. The remaining case is done in [F-G-U].

For $B = (1/p, 1/q) \in (0,1) \times (0,1)$ and $T : L^p \to L^q$ we write, to simplify the notation, $\|T\|_B$ instead of $\|T\|_{p,q}$.

LEMMA 3.2. *There exists $c > 0$, independent of $r_1, \ldots, r_{k-1}$, such that for $R \in \mathbb{N}$ and $k_0 + 1 \le k \le n$,*

$$\Big\| \sum_{1 \le r_k, \ldots, r_n \le R} T_{\nu_{r_1,\ldots,r_n}} (I - P_{k,r_k})(I - \widetilde{Q}_{k,r_k}) \Big\|_{B_k} \le c \exp_2 \left(- \sum_{j=1}^{k-1} \frac{r_j}{\beta_j} \cdot \frac{\beta_j(\beta_j^{-1} - \beta_k^{-1})}{1 + S_{k+1} + k\beta_k^{-1}} \right)$$

where $\exp_2(x) = 2^x$.

Proof. We fix k and consider the operator

$$\sum_{1 \le r_k \le R} T_{\nu_{r_1,\ldots,r_n}} (I - P_{k,r_k})(I - \widetilde{Q}_{k,r_k}).$$

Lemma 2.6 and the weak Young inequality imply that it is of weak type $(1, 1 + \beta_k^{-1})$ with weak constant less than $c \exp_2(-\sum_{j \ne k} r_j/\beta_j)$, with c independent of R and r_j, $j \ne k$. We set $D = (1, 1/(1 + \beta_k^{-1}))$.

We now study the behavior of this operator on the nonprincipal diagonal. We note that $\nu_{r_1,\ldots,r_n} \le \mu_{r_1,\ldots,r_n}$ where $\mu_{r_1,\ldots,r_n}$ is the measure μ restricted to

$$\prod_{1\le j\le n} \{t \in \mathbb{R} : 2^{-(r_j-1)/\beta_j} \le |t| \le 2^{-(r_j-4)/\beta_j}\}.$$

Let $J_z = \delta \otimes \ldots \otimes \delta \otimes I_z$, where I_z is the analytic extension to $\mathbb{C}$ of the fractional integration kernel

$$\frac{2^{-z/2}}{\Gamma(z/2)}|t|^{z-1}.$$

We consider the analytic family of operators given by

$$T_z f = \sum_{1\le r_k\le R} \mu_{r_1,\ldots,r_n} * J_z * f, \qquad z \in \mathbb{C},\ f \in S(\mathbb{R}^{n+1}).$$

A computation shows that $\|T_z\|_{1,\infty} \le c$ if $\mathrm{Re}(z) = 1$. Reasoning as in the proof of Theorem 3.2 of [F-G-U], using Lemma 2.2 of [R-S] and the van der Corput Lemma (see [St], p. 332), we obtain

$$\Big|\sum_{1\le r_k\le R} \widehat{\mu}_{r_1,\ldots,r_n}(y_1,\ldots,y_{n+1})\Big| \le c\exp_2\left(\sum_{j\ne k} \frac{r_j}{\beta_j}\cdot\frac{\beta_j-2}{2}\right)|y_{n+1}|^{-(n-1)/2-1/\beta_k}.$$

Thus the complex interpolation theorem, applied on the strip $-(n-1)/2-1/\beta_k \le \mathrm{Re}(z) \le 1$, gives us

$$\Big\|\sum_{1\le r_k\le R} T_{\mu_{r_1,\ldots,r_n}}\Big\|_{A^{n-1}} \le c\exp_2\left(\sum_{j\ne k}\frac{r_j}{\beta_j}\cdot\frac{\beta_j-2}{n+1+2\beta_k^{-1}}\right)$$

where

$$A^{n-1} = \left(\frac{n+2\beta_k^{-1}}{1+n+2\beta_k^{-1}}, \frac{1}{1+n+2\beta_k^{-1}}\right).$$

Since $\nu_{r_1,\ldots,r_n} \le \mu_{r_1,\ldots,r_n}$, Lemma 2.5 implies that

$$\Big\|\sum_{1\le r_k\le R} T_{\nu_{r_1,\ldots,r_n}}(I-P_{k,r_k})(I-\widetilde{Q}_{k,r_k})\Big\|_{A^{n-1}} \le c\exp_2\left(\sum_{j\ne k}\frac{r_j}{\beta_j}\cdot\frac{\beta_j-2}{n+1+2\beta_k^{-1}}\right). \tag{3.3}$$

We set, for $t \in (0,1]$, $B_t^n = tA^{n-1} + (1-t)D$. The Marcinkiewicz interpolation theorem (see [B-S], p. 227, Remark 4.15(d)) gives us

$$\Big\| \sum_{1\le r_k\le R} T_{\nu_{r_1,\dots,r_n}}(I-P_{k,r_k})(I-\widetilde{Q}_{k,r_k})\Big\|_{B_t^n} \le c\exp_2\Big(-\sum_{j\ne k}\frac{r_j}{\beta_j}\Big((1-t)-\frac{\beta_j-2}{n+1+2\beta_k^{-1}}t\Big)\Big) \tag{3.4}$$

for some positive constant c independent of t, R and r_j, $j\ne k$.

If $k=n$, we check that there exists $t\in(0,1)$ such that $B_t^n = B_n$. Using this t in the above expression, we get the lemma in this case.

If $k_0+1\le k\le n-1$, we will construct inductively an open polygonal region that contains B_k and such that at each of its points,

$$\Big\| \sum_{1\le r_k,\dots,r_n\le R} T_{\nu_{r_1,\dots,r_n}}(I-P_{k,r_k})(I-\widetilde{Q}_{k,r_k})\Big\| \le c\exp_2\Big(-\sum_{j=1}^{k-1}\frac{r_j}{\beta_j}\cdot\frac{\beta_j(\beta_j^{-1}-\beta_k^{-1})}{1+S_{k+1}+k\beta_k^{-1}}\Big).$$

We define $t_n\in(0,1)$ as the value of t that annihilates the coefficient of r_n/β_n in (3.4). Now we set $B^n(\varepsilon)=B^n_{t_n-\varepsilon}$. So a computation shows that

$$\Big\| \sum_{1\le r_n\le R}\sum_{1\le r_k\le R} T_{\nu_{r_1,\dots,r_n}}(I-P_{k,r_k})(I-\widetilde{Q}_{k,r_k})\Big\|_{B^n(\varepsilon)} \le c_\varepsilon \exp_2\Big(-\sum_{j=1,\, j\ne k}^{n-1}\frac{r_j}{\beta_j}\Big(\frac{\beta_j(\beta_j^{-1}-\beta_n^{-1})}{n\beta_n^{-1}-\beta_n^{-1}+2\beta_k^{-1}\beta_n^{-1}+1} + \varepsilon\Big(1+\frac{\beta_j-2}{n+1+2\beta_k^{-1}}\Big)\Big)\Big). \tag{3.5}$$

We set, for $k-1\le m\le n-1$,

$$A^m=\Big(\frac{1+m+2\beta_k^{-1}+2S_{m+2}}{2+m+2\beta_k^{-1}+2S_{m+2}},\frac{1}{2+m+2\beta_k^{-1}+2S_{m+2}}\Big).$$

We note that $A^{k-1}=A_{k-1}$. Reasoning as in the proof of (3.3), but now using the complex interpolation theorem on the strip $-m/2-1/\beta_k-S_{m+2}\le \mathrm{Re}(z)\le 1$, we obtain

$$\Big\| \sum_{1\le r_{m+2},\dots,r_n\le R}\sum_{1\le r_k\le R} T_{\nu_{r_1,\dots r_n}}(I-P_{k,r_k})(I-\widetilde{Q}_{k,r_k})\Big\|_{A^m} \le c\exp_2\Big(\sum_{j=1,\, j\ne k}^{m+1}\frac{r_j}{\beta_j}\cdot\frac{\beta_j-2}{m+2+2\beta_k^{-1}+2S_{m+2}}\Big). \tag{3.6}$$

For $1\le j\le m-1$, $k\le m\le n$ and $\varepsilon>0$ small enough, we define $\delta(m,j,\varepsilon)$ and $B^m(\varepsilon)$ recursively on m. These definitions will be done in such a way that, for $k+1 \leq m$,

$$(3.7) \quad \Big\| \sum_{r_m,\dots,r_n} \sum_{r_k} T_{\nu_{r_1,\dots,r_n}} (I - P_{k,r_k})(I - \widetilde{Q}_{k,r_k}) \Big\|_{B^m(\varepsilon)}$$

$$\leq c_\varepsilon \exp_2\bigg(- \sum_{j=1,\, j\neq k}^{m-1} \frac{r_j}{\beta_j} \bigg[\frac{\beta_j(\beta_j^{-1} - \beta_m^{-1})}{(m-1)\beta_m^{-1} + 2\beta_k^{-1}\beta_m^{-1} + S_{m+1} + 1} + \delta(m,j,\varepsilon) \bigg] \bigg)$$

for some positive constant c_ε.

(3.5) is (3.7) with $m = n$,

$$c_\varepsilon = c \sum_{r_n \in \mathbb{N}} \exp_2\bigg(-\frac{r_n}{\beta_n} \varepsilon \bigg(1 + \frac{\beta_n - 2}{n + 1 + 2\beta_k^{-1}} \bigg) \bigg)$$

and

$$\delta(n,j,\varepsilon) = \varepsilon \bigg(1 + \frac{\beta_j - 2}{n + 1 + 2\beta_k^{-1}} \bigg).$$

Suppose that we have defined $B^{m+1}(\varepsilon)$ and $\delta(m+1,j,\varepsilon)$ for $1 \leq j \leq m$ so that (3.7) holds for $m+1$ instead of m. We set, for $t \in [0,1]$, $B_t^m(\varepsilon) = tA^{m-1} + (1-t)B^{m+1}(\varepsilon)$. The Marcinkiewicz interpolation theorem and (3.6) applied to $m-1$ instead of m give us

$$(3.8) \quad \Big\| \sum_{1 \leq r_{m+1},\dots,r_n \leq R} \sum_{1 \leq r_k \leq R} T_{\nu_{r_1,\dots,r_n}} (I - P_{k,r_k})(I - \widetilde{Q}_{k,r_k}) \Big\|_{B_t^m(\varepsilon)}$$

$$\leq c_\varepsilon \exp_2\bigg(- \sum_{j=1,\, j\neq k}^{m} \frac{r_j}{\beta_j} \bigg[(1-t)$$

$$\times \bigg(\frac{\beta_j(\beta_j^{-1} - \beta_{m+1}^{-1})}{m\beta_{m+1}^{-1} + 2\beta_k^{-1}\beta_{m+1}^{-1} + S_{m+2} + 1} + \delta(m+1,j,\varepsilon) \bigg)$$

$$- t \frac{\beta_j - 2}{m + 1 + 2\beta_k^{-1} + 2S_{m+1}} \bigg] \bigg).$$

We define t_m by

$$(1-t_m) \frac{\beta_m(\beta_m^{-1} - \beta_{m+1}^{-1})}{m\beta_{m+1}^{-1} + 2\beta_k^{-1}\beta_{m+1}^{-1} + S_{m+2} + 1} - t_m \frac{\beta_m - 2}{m + 1 + 2\beta_k^{-1} + 2S_{m+1}} = 0.$$

Taking account of $1 < \beta_1 \leq \ldots \leq \beta_n$, we easily check that $t_m \in [0,1)$. We set

$$B_m(\varepsilon) = t_m A^{m-1} + (1-t_m) B^{m+1}(\varepsilon).$$

A computation shows that t_m satisfies, for $1 \leq j \leq m$,

$$(1-t_m)\frac{\beta_j(\beta_j^{-1}-\beta_{m+1}^{-1})}{m\beta_{m+1}^{-1}+2\beta_k^{-1}\beta_{m+1}^{-1}+S_{m+2}+1}-t_m\frac{\beta_j-2}{m+1+2\beta_k^{-1}+2S_{m+1}}$$
$$=\frac{\beta_j(\beta_j^{-1}-\beta_m^{-1})}{(m-1)\beta_m^{-1}+2\beta_k^{-1}\beta_m^{-1}+S_{m+1}+1}.$$

Then from (3.8) we obtain (3.7) if $m \geq k+1$, with

$$\delta(m,j,\varepsilon)=(1-t_m)\delta(m+1,j,\varepsilon)$$

and some positive constant c_ε. Thus

$$\Big\|\sum_{1\le r_k,\dots,r_n\le R} T_{\nu_{r_1,\dots,r_n}}(I-P_{k,r_k})(I-\widetilde{Q}_{k,r_k})\Big\|_{B^m(\varepsilon)} \tag{3.9}$$
$$\le c_\varepsilon \sum_{r_{k+1},\dots,r_{m-1}} \exp_2\bigg(-\sum_{j=1,\,j\ne k}^{m-1}\frac{r_j}{\beta_j}\bigg(\frac{\beta_j(\beta_j^{-1}-\beta_m^{-1})}{(m-1)\beta_m^{-1}+2\beta_k^{-1}\beta_m^{-1}+S_{m+1}+1}$$
$$+\delta(m,j,\varepsilon)\bigg)\bigg)$$
$$\le c_\varepsilon \exp_2\bigg(-\sum_{j=1}^{k-1}\frac{r_j}{\beta_j}\bigg(\frac{\beta_j(\beta_j^{-1}-\beta_m^{-1})}{(m-1)\beta_m^{-1}+2\beta_k^{-1}\beta_m^{-1}+S_{m+1}+1}+\delta(m,j,\varepsilon)\bigg)\bigg)$$
$$\le c_\varepsilon \exp_2\bigg(-\sum_{j=1}^{k-1}\frac{r_j}{\beta_j}\bigg(\frac{\beta_j(\beta_j^{-1}-\beta_k^{-1})}{(k-1)\beta_k^{-1}+2\beta_k^{-1}\beta_k^{-1}+S_{k+1}+1}+\delta(k,j,\varepsilon)\bigg)\bigg)$$
$$\le c_\varepsilon \exp_2\bigg(-\sum_{j=1}^{k-1}\frac{r_j}{\beta_j}\cdot\frac{\beta_j(\beta_j^{-1}-\beta_k^{-1})}{1+S_{k+1}+k\beta_k^{-1}}\bigg)$$

where $\delta(k,j,\varepsilon)=(1-t_k)\delta(k+1,j,\varepsilon)$.

Also, (3.8) with $m=k$ and $t=t_k$ gives us

$$\Big\|\sum_{1\le r_k,\dots,r_n\le R} T_{\nu_{r_1,\dots,r_n}}(I-P_{k,r_k})(I-\widetilde{Q}_{k,r_k})\Big\|_{B^k(\varepsilon)} \tag{3.10}$$
$$=c_\varepsilon \exp_2\bigg(-\sum_{j=1}^{k-1}\frac{r_j}{\beta_j}\bigg[\bigg(\frac{\beta_j(\beta_j^{-1}-\beta_k^{-1})}{(k-1)\beta_k^{-1}+2\beta_k^{-1}\beta_k^{-1}+S_{k+1}+1}+\delta(k,j,\varepsilon)\bigg)\bigg]\bigg)$$
$$\le c_\varepsilon \exp_2\bigg(-\sum_{j=1}^{k-1}\frac{r_j}{\beta_j}\cdot\frac{\beta_j(\beta_j^{-1}-\beta_k^{-1})}{1+S_{k+1}+k\beta_k^{-1}}\bigg).$$

Now,

$$\frac{\beta_j(\beta_j^{-1}-\beta_k^{-1})}{1+S_{k+1}+k\beta_k^{-1}}\le 1,$$

so the same bound holds for the norm of

$$\sum_{1\le r_k,\dots,r_n\le R} T_{\nu_{r_1,\dots,r_n}}(I-P_{k,r_k})(I-\widetilde{Q}_{k,r_k})$$

at the points D and $(1/2,1/2)$.

We set $B^m=\lim_{\varepsilon\to 0}B^m(\varepsilon)$. Taking account of the definition of t_m one can check inductively on m that

$$B^m=\Bigg(\frac{1+S_{m+1}+(m-2)\beta_m^{-1}+2\beta_m^{-1}\beta_k^{-1}}{1+S_{m+1}+(m-1)\beta_m^{-1}+2\beta_m^{-1}\beta_k^{-1}},\\ \frac{(1+\beta_k^{-1})^{-1}(1-\beta_m^{-1}+\beta_m^{-1}\beta_k^{-1})}{1+S_{m+1}+(m-1)\beta_m^{-1}+2\beta_m^{-1}\beta_k^{-1}}\Bigg).$$

Now, it is easy to see that B_k belongs to the open segment that joins B^k and D, so for ε small enough, it belongs to the open convex polygonal region with vertices D, $B^n(\varepsilon),\dots,B^k(\varepsilon)$ and $(1/2,1/2)$. Therefore the lemma follows from (3.9), (3.10) and the Marcinkiewicz interpolation theorem. ■

LEMMA 3.11. *There exists $c>0$, independent of $r_1,\dots,r_{k-1}$, such that for each $R\in\mathbb{N}$ and for $k_0+1\le k\le n$,*

$$\Big\|\sum_{1\le r_k\le R}\dots\sum_{1\le r_n\le R}T_{\nu_{r_1,\dots,r_n}}P_{k,r_k}\Big\|_{B_k}\\ \le c\exp_2\Bigg(-\sum_{j=1}^{k-1}\frac{r_j}{\beta_j}\cdot\frac{\beta_j(\beta_j^{-1}-\beta_k^{-1})}{1+S_{k+1}+k\beta_k^{-1}}\Bigg).$$

Proof. In view of Lemmas 2.4 and 2.7, the proof follows as in Lemma 3.2. ■

THEOREM 3.12. *E_μ is the closed convex polygonal region with vertices $(1,1)$, $B_n,\dots,B_{k_0+1}$, A_{k_0}, $B'_{k_0+1},\dots,B'_n$ and $(0,0)$.*

Proof. Since $A_{k_0}\in E_\mu$ (see [F-G-U], Lemma 3.1). Taking account of $E_\nu\subset E_\mu\subset\Sigma^{(2,\dots,2,\beta_{k_0+1},\dots,\beta_n)}$, we first prove that $B_n,\dots,B_{k_0+1}\in E_\nu$. Let $R\in\mathbb{N}$. We prove inductively on k that, if $k_0+1\le k\le n$, then

$$\Big\|\sum_{1\le r_k,\dots,r_n\le R}T_{\nu_{r_1,\dots,r_n}}\Big\|_{B_k}\\ \le c\exp_2\Bigg(-\sum_{j=1}^{k-1}\frac{r_j}{\beta_j}\cdot\frac{\beta_j^{-1}-\beta_k^{-1}}{\beta_j^{-1}(1+S_{k+1}+k\beta_k^{-1})}\Bigg)\tag{3.13}$$

with c independent of $r_1, \dots, r_{k-1}$ and R. Indeed, if $k = n$ we decompose

$$\sum_{r_n} T_{\nu_{r_1,\dots,r_n}} = \sum_{r_n} T_{\nu_{r_1,\dots,r_n}} P_{n,r_n} + \sum_{r_n} T_{\nu_{r_1,\dots,r_n}} (I - P_{n,r_n})(I - \widetilde{Q}_{n,r_n}) + \sum_{r_n} T_{\nu_{r_1,\dots,r_n}} (I - P_{n,r_n})\widetilde{Q}_{n,r_n}.$$

Reasoning as in the proof of (3.3), we obtain

$$\|T_{\nu_{r_1,\dots,r_n}}\|_{A_n} \le c \exp_2 \left(\sum_{j=1}^{n} \frac{r_j}{\beta_j} \cdot \frac{\beta_j - 2}{n+2} \right).$$

Using the Riesz–Thorin interpolation theorem between A_n and $(1,1)$ we get

$$\sup_{r_n} \|T_{\nu_{r_1,\dots,r_n}}\|_{B_n} < c \exp_2 \left(- \sum_{j=1}^{n-1} \frac{\beta_n - \beta_j}{n + \beta_n} \cdot \frac{r_j}{\beta_j} \right).$$

So, Lemmas 2.2, 3.2 and 3.11 imply

$$\Big\| \sum_{1 \le r_n \le R} T_{\nu_{r_1,\dots,r_n}} \Big\|_{B_n} \le c \exp_2 \left(- \sum_{j=1}^{n-1} \frac{r_j}{\beta_j} \cdot \frac{\beta_n - \beta_j}{n + \beta_n} \right)$$

with c independent of $r_1, \dots, r_{n-1}$ and R. Suppose (3.13) holds for k. Let us prove it for $k - 1$. We decompose

$$\begin{aligned} \sum_{1 \le r_{k-1},\dots,r_n \le R} T_{\nu_{r_1,\dots,r_n}} &= \sum_{1 \le r_{k-1},\dots,r_n \le R} T_{\nu_{r_1,\dots,r_n}} (I - P_{k-1,r_{k-1}})(I - \widetilde{Q}_{k-1,r_{k-1}}) \\ &\quad + \sum_{1 \le r_{k-1},\dots,r_n \le R} T_{\nu_{r_1,\dots,r_n}} P_{k-1,r_{k-1}} \\ &\quad + \sum_{1 \le r_{k-1},\dots,r_n \le R} T_{\nu_{r_1,\dots,r_n}} (I - P_{k-1,r_{k-1}})\widetilde{Q}_{k-1,r_{k-1}}. \end{aligned}$$

Again, reasoning as in the proof of (3.3), we obtain

$$\Big\| \sum_{1 \le r_k,\dots,r_n \le R} T_{\nu_{r_1,\dots,r_n}} \Big\|_{A_{k-1}} \le c \exp_2 \left(\sum_{j=1}^{k-1} \frac{r_j}{\beta_j} \cdot \frac{\beta_j - 2}{k + 1 + 2S_k} \right) \tag{3.14}$$

and so (3.13), (3.14) and the Riesz–Thorin theorem imply

$$\sup_{r_{k-1}} \Big\| \sum_{1 \le r_k,\dots,r_n \le R} T_{\nu_{r_1,\dots,r_n}} \Big\|_{B_{k-1}} \le c \exp_2 \left(- \sum_{j=1}^{k-2} \frac{r_j}{\beta_j} \cdot \frac{\beta_j(\beta_j^{-1} - \beta_{k-1}^{-1})}{1 + S_k + (k-1)\beta_{k-1}^{-1}} \right).$$

This inequality and Lemmas 2.2, 3.2 and 3.11 give us (3.13) with k replaced by $k-1$. So (3.13) holds.

Now, it is easy to see that $B_k \in E_\nu$ for $k_0+1 \le k \le n$. Indeed, if $\beta_{k-1} \ne \beta_k$, we can sum over $r_1, \dots, r_{k-1} \in \mathbb{N}$ in (3.13). In the other case, let $s = \min\{j \ge k_0+1 : \beta_j = \beta_k\}$. Then $B_k = B_s$ and we can sum over $r_1, \dots, r_{s-1} \in \mathbb{N}$ in (3.13). Since c is independent of R we conclude that, in both cases, $B_k \in E_\nu$.

A simple computation shows that $(T_\mu)^* = T_{\mu^*}$ where

$$\mu^*(E) = \mu(-E) = \int_Q \chi_E(x_1, \dots, x_n, -\varphi(x_1, \dots, x_n))\, dx_1 \dots dx_n.$$

Reasoning as before, we deduce, by duality that $B'_n, \dots, B'_{k_0+1}$ belong to E_μ. ■

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