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COMULTIPLICATIONS OF THE WEDGE OF TWO MOORE SPACES

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Throughout, we work in the category of connected pointed topological spaces which have the homotopy type of finite-dimensional CW-complexes. All maps and homotopies preserve base points. Here, it is convenient to ignore the distinction between a map and its homotopy class. Thus we ambiguously regard a map $f : X \to Y$ as an element of $[X, Y]$, the homotopy classes of maps from X to Y.

Recall that a *comultiplication* or a *co-H-structure* on a space X is a map $\phi : X \to X \vee X$ such that $j\phi = \Delta$, where $j : X \vee X \to X \times X$ is the inclusion map and $\Delta : X \to X \times X$ is the diagonal map. Equivalently, $\phi : X \to X \vee X$ is a comultiplication if and only if $q_1\phi = \mathrm{id}_X = q_2\phi : X \to X$, where $q_1, q_2 : X \vee X \to X$ are the projections onto the first and second summands of the wedge. A space X together with a comultiplication ϕ is called a *co-H-space*.

Let $\mathcal{C}(X)$ denote the set of homotopy classes of comultiplications of X. A number of authors (e.g. [1, 2, 3, 9]) have computed the set $\mathcal{C}(X)$ for some spaces X and investigated the basic properties of its elements. The primary example of a co-H-space is the suspension of a space with the natural pinching map. Then, as shown in [9], the set $\mathcal{C}(X)$ can be described by means of the Hilton–Milnor Theorem (see e.g. [10, Chapter 11]). It is well known that a rational co-H-space X has the homotopy type of the wedge of rational spheres. The latter space admits a standard comultiplication arising from the pinching map. Basic properties of comultiplications of this space have been investigated in [2, 3]. On the other hand, Moore spaces are a natural generalization of ordinary spheres.

The aim of this paper is to study the set of comultiplications of the wedge $M(\mathbb{A}, n) \vee M(\mathbb{B}, m)$ of two Moore spaces and then describe this set by means of the groups $\mathbb{A}$ and $\mathbb{B}$ from a wide class of abelian groups. An example of the wedge of two Moore spaces is a co-Moore space $M'(\mathbb{A}, n)$

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of type $(\mathbb{A}, n)$ (considered e.g. in [6]), i.e. a simply connected space with a single non-vanishing reduced integral cohomology group $\mathbb{A}$ in dimension n. Section 1 establishes the basic framework. We recall a description of the set $\mathcal{C}(X)$ presented in [9], where X is the suspension of a space. Then in Proposition 1.3 we present a formula for the set $\mathcal{C}(M(\mathbb{A}, n) \vee M(\mathbb{B}, m))$ and deduce that its description for $m = 2n-1$ leads to a computation of the group $[M(\mathbb{B}, 2n-1), \Sigma(M(\mathbb{A}, n-1) \wedge M(\mathbb{A}, n-1))]$ for $n \geq 2$. The Universal Coefficient Theorem for homotopy groups in [7] implies an exact sequence

$$0 \to \operatorname{Ext}(\mathbb{B}, \pi_{2n}(X)) \to [M(\mathbb{B}, 2n-1), X] \to \operatorname{Hom}(\mathbb{B}, \mathbb{A} \otimes \mathbb{A}) \to 0,$$

where $X = \Sigma(M(\mathbb{A}, n-1) \wedge M(\mathbb{A}, n-1))$.

In Section 2 we show first that there is an extension

$$0 \to \mathbb{A} \otimes \mathbb{A} \otimes \mathbb{Z}_2 \to \pi_{2n}(X) \to \operatorname{Tor}(\mathbb{A}, \mathbb{A}) \to 0$$

determined by the dual Steenrod square $\mathrm{Sq}_2 : H_{2n+1}(X, \mathbb{Z}_2) = {}_2\operatorname{Tor}(\mathbb{A}, \mathbb{A}) \to H_{2n-1}(X, \mathbb{Z}_2) = \mathbb{A} \otimes \mathbb{A} \otimes \mathbb{Z}_2$. In Corollary 2.2 we infer that $[M(\mathbb{B}, 2n-1), X] = \operatorname{Ext}(\mathbb{B}, \pi_{2n}(X)) \oplus \operatorname{Hom}(\mathbb{B}, \mathbb{A} \otimes \mathbb{A})$ provided the p-primary component of $\mathbb{A} \otimes \mathbb{A}$ is finitely generated and $\mathbb{B}$ is a cyclic group of order p^m, and $m > 1$ or $p > 2$ and $m = 1$. Next we restrict the computation of the group $[M(\mathbb{B}, 2n-1), X]$ to abelian groups $\mathbb{A} = \mathbb{A}' \oplus T_2(\mathbb{A})$ with $T_2(\mathbb{A}') = 0$ and $T_2(\mathbb{A})$ a finitely generated group, where T_2 is the 2-component functor. In particular, from Corollaries 1.5 and 2.4 a description of $\mathcal{C}(M'(\mathbb{A}, n))$ of a co-Moore space $M'(\mathbb{A}, n)$, for $n \geq 3$ and $\mathbb{A}$ as above, follows. If $\mathbb{A}$ is a finite direct sums $\bigoplus_k \bigoplus_{I_k} \mathbb{Z}_{2^k}$ of cyclic 2-groups and $\mathbb{B}$ any abelian group we observe that

$$\begin{aligned}&[M(\mathbb{B}, 2n-1), \Sigma(M(\mathbb{A}, n-1) \wedge M(\mathbb{A}, n-1))]\\ &\qquad = \bigoplus_{k,l} \bigoplus_{I_k \times I_l} [M(\mathbb{B}, 2n-1), \Sigma(M(\mathbb{Z}_{2^k}, n-1) \wedge M(\mathbb{Z}_{2^l}, n-1))].\end{aligned}$$

Section 3 contains our study of the group $[M(\mathbb{Z}_{2^m}, 2n–1), \Sigma(M(\mathbb{Z}_{2^k}, n-1) \wedge M(\mathbb{Z}_{2^l}, n-1))]$. In Proposition 3.2 the homotopy properties of the space $X = \Sigma(M(\mathbb{Z}_{2^k}, n-1) \wedge M(\mathbb{Z}_{2^l}, n-1))$ are presented to derive our main result of this section, Theorem 3.3:

$$\begin{aligned}&[M(\mathbb{Z}_{2^m}, 2n-1), X]\\ &\quad = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2, & 1 = k = l = m;\\ \mathbb{Z}_4 \oplus \mathbb{Z}_2, & 1 = k < l,\ m = 1 \text{ or } 1 = k = l,\ m > 1;\\ \mathbb{Z}_2 \oplus \mathbb{Z}_{2^{\min(k,m)}} \oplus \mathbb{Z}_{2^{\min(k,m)}}, & \text{otherwise}\end{cases}\end{aligned}$$

for $n \geq 2$. But $[M(\mathbb{Z}_2, 2n-1), M(\mathbb{Z}_2, 2n-1)] = \mathbb{Z}_4$ by means of Barratt's results in [4], [7, Chapter 12], so we may deduce that the group $[M(\mathbb{B}, 2n-1), \Sigma(M(\mathbb{A}, n-1) \wedge M(\mathbb{A}, n-1))]$ can be computed for an abelian group $\mathbb{A}$ as above and $\mathbb{B}$ a direct sum of cyclic groups; in particular, for finitely generated abelian groups $\mathbb{A}$ and $\mathbb{B}$.

1. Comultiplications. We begin by interpreting the set $\mathcal{C}(X)$ of comultiplications of X in terms of homotopy sets. If X is a cogroup with comultiplication ϕ then it induces a group structure (denoted multiplicatively) on the set $[X, Y]$, for any space Y. Now let $Y\flat Y$ be the space of paths in $Y \times Y$ which begin in $Y \vee Y$ and end at the base point of $Y \times Y$ and let $\lambda : Y\flat Y \to Y \vee Y$ be the map that projects a path onto its initial point. In other words, $Y\flat Y$ is the homotopy fibre (called also a *flat product*) of the inclusion map $j : Y \vee Y \to Y \times Y$. Write $j_* : [X, Y \vee Y] \to [X, Y \times Y]$ and $\lambda_* : [X, Y\flat Y] \to [X, Y \times Y]$ for the induced maps. Then there is the following description of the set $\mathcal{C}(X)$ presented in e.g. [1, 3].

PROPOSITION 1.1. *If X is a cogroup then there is a split short exact sequence*

$$1 \to [X, Y\flat Y] \xrightarrow{\lambda_*} [X, Y \vee Y] \xrightarrow{j_*} [X, Y \times Y] \to 1$$

for any space Y and the map $\Phi : [X, X\flat X] \to \mathcal{C}(X)$ defined by $\Phi(\beta) = \phi\cdot(\lambda\beta)$ for $\beta \in [X, X\flat X]$ is a bijection.

There is another interesting link in [9] between the set of co-H-structures on a suspension and some sets of homotopy classes. Namely, let X_1 and X_2 be CW-complexes, Σ the suspension and Ω the loop functors. Then, by the Hilton–Milnor Theorem [10], $\Omega\Sigma(X_1 \vee X_2)$ is homotopy equivalent to the weak product $\prod_{k\ge1}^* \Omega\Sigma P_{\omega_k}(X_1, X_2)$, where ω_k runs through a set of basic products for the set $\{1, 2\}$. The space $P_{\omega_k}(X_1, X_2)$ has the homotopy type of the smash product $X_1^{(\alpha_1)} \wedge X_2^{(\alpha_2)}$, where, for any space X, $X^{(\alpha)}$ is the smash product of α copies of X; the integer α_i is just the number of occurrences of i in the word ω_k for $i = 1, 2$. The homotopy equivalence is given by a map of the form $\prod_k^* \Omega g_k$, where $g_k : \Sigma P_{\omega_k}(X_1, X_2) \to \Sigma(X_1 \vee X_2)$ is an iterated generalized Whitehead product which is associated with the basic product ω_k. In particular, $P_1(X_1, X_2) = X_1$, $P_2(X_1, X_2) = X_2$ and the maps $g_i : \Sigma X_i \to \Sigma(X_1 \vee X_2)$ $(i = 1, 2)$ are inclusions. All g_k with $k \ge 2$ are generalized Whitehead products involving both the first and second factors of $\Sigma(X_1 \vee X_2)$.

PROPOSITION 1.2 [9]. (a) *There is a split short exact sequence*

$$1 \to \bigoplus_{k\ge2} [\Sigma Y, \Sigma P_{\omega_k}(X_1, X_2)] \to [\Sigma Y, \Sigma X_1 \vee \Sigma X_2] \to [\Sigma Y, \Sigma X_1 \times \Sigma X_2] \to 1$$

for any space Y.

(b) *The set $\mathcal{C}(\Sigma X)$ of comultiplications of the suspension ΣX is in one-one correspondence with elements of the group $\bigoplus_{k\ge2}[\Sigma X, \Sigma P_{\omega_k}(X, X)]$.*

If $\mathbb{A}$ is an abelian group and n an integer ≥ 2 then a *Moore space of type* $(\mathbb{A}, n)$ is a simply connected space $M(\mathbb{A}, n)$ with a single non-vanishing reduced homology group $\mathbb{A}$ in dimension n. In particular, $M(\mathbb{A}, n)$ is an

$(n-1)$-connected CW-complex and from its construction it follows that $\dim M(\mathbb{A},n) \leq n+1$. By uniqueness of the Moore space we get $M(\mathbb{A},n) = \Sigma M(\mathbb{A},n-1)$ for $n \geq 2$, where $M(\mathbb{A},1)$ is any connected space with a single non-vanishing reduced homology group $\mathbb{A}$ in dimension 1. In [1] it is shown that for $n > 2$ the set $\mathcal{C}(M(\mathbb{A},n))$ has one element and for $n = 2$ it is in one-one correspondence with $\mathrm{Ext}(\mathbb{A}, \mathbb{A} \otimes \mathbb{A})$. Throughout this section and the next one, for convenience, we denote the Moore space $M(\mathbb{A},n)$ by $\mathbb{A}_n$ in the proofs.

Now let $\mathbb{A}$ and $\mathbb{B}$ be abelian groups, $n,m \geq 2$ and $X = M(\mathbb{A},n) \vee M(\mathbb{B},m)$. If $m = n$ then $X = M(\mathbb{A} \oplus \mathbb{B}, n)$ and the set $\mathcal{C}(X)$ is described in [1]. Therefore we may assume that $2 \leq n < m$. But $X = M(\mathbb{A},n) \vee M(\mathbb{B},m) = \Sigma(M(\mathbb{A},n-1) \vee M(\mathbb{B},m-1))$ so from Propositions 1.1 and 1.2 we derive

Proposition 1.3. *The set* $\mathcal{C}(M(\mathbb{A},n) \vee M(\mathbb{B},m))$ *of comultiplications of* $M(\mathbb{A},n) \vee M(\mathbb{B},m)$ *is in one-one correspondence with the group*:

(a) $\ker([M(\mathbb{B},m), M(\mathbb{A},n) \vee M(\mathbb{A},n)] \to [M(\mathbb{B},m), M(\mathbb{A},n) \times M(\mathbb{A},n)]) \oplus \mathrm{Ext}(\mathbb{A}, \mathbb{A} \otimes \mathbb{A}) \oplus \mathrm{Ext}(\mathbb{B}, \mathbb{A} \otimes \mathbb{B}) \oplus \mathrm{Ext}(\mathbb{B}, \mathbb{A} \otimes \mathbb{B})$ *for* $2 = n < m$,

(b) $\ker([M(\mathbb{B},m), M(\mathbb{A},n) \vee M(\mathbb{A},n)] \to [M(\mathbb{B},m), M(\mathbb{A},n) \times M(\mathbb{A},n)])$ *for* $2 < n < m$.

Proof. By Proposition 1.1, the set $\mathcal{C}(\mathbb{A}_n \vee \mathbb{B}_m)$ is in one-one correspondence with the group $\ker j_*$, where $j_* : [\mathbb{A}_n \vee \mathbb{B}_m, \mathbb{A}_n \vee \mathbb{B}_m \vee \mathbb{A}_n \vee \mathbb{B}_m] \to [\mathbb{A}_n \vee \mathbb{B}_m, (\mathbb{A}_n \vee \mathbb{B}_m) \times (\mathbb{A}_n \vee \mathbb{B}_m)]$ is the map induced by the inclusion. But the inclusion maps $\mathbb{A}_n \vee \mathbb{B}_m \to \mathbb{A}_n \times \mathbb{B}_m$ and $\mathbb{A}_n \vee \mathbb{B}_m \vee \mathbb{A}_n \vee \mathbb{B}_m \to (\mathbb{A}_n \vee \mathbb{A}_n) \times (\mathbb{B}_m \vee \mathbb{B}_m)$ are $(n+m-1)$-homology isomorphisms of simply connected spaces; however, the inclusion maps $\mathbb{A}_n \vee \mathbb{A}_n \to \mathbb{A}_n \times \mathbb{A}_n$ and $\mathbb{B}_m \vee \mathbb{B}_m \to \mathbb{B}_m \times \mathbb{B}_m$ are $(2n-1)$- and $(2m-1)$-homology isomorphisms of simply connected spaces, respectively. Therefore in the light of the Whitehead Theorem the maps above are $(n+m-2)$-, $(2n-2)$- and $(2m-2)$-homotopy isomorphisms, respectively.

(a) If $2 = n < m$ then $3 \leq m < m+1 \leq 2m-2$ and the induced maps $[\mathbb{A}_2, \mathbb{A}_2 \vee \mathbb{B}_m \vee \mathbb{A}_2 \vee \mathbb{B}_m] \to [\mathbb{A}_2, \mathbb{A}_2 \vee \mathbb{A}_2] \oplus [\mathbb{A}_2, \mathbb{B}_m \vee \mathbb{B}_m] \to [\mathbb{A}_2, \mathbb{A}_2 \vee \mathbb{A}_2] \oplus [\mathbb{A}_2, \mathbb{B}_m] \oplus [\mathbb{A}_2, \mathbb{B}_m]$, and $[\mathbb{A}_2, \mathbb{A}_2 \vee \mathbb{B}_m] \to [\mathbb{A}_2, \mathbb{A}_2] \oplus [\mathbb{A}_2, \mathbb{B}_m]$ are isomorphisms. But $\dim \mathbb{A}_2 \leq 3$ so, by Propositions 1.1 and 1.2, $[\mathbb{A}_2, \mathbb{A}_2 \vee \mathbb{A}_2] = [\mathbb{A}_2, \mathbb{A}_2] \oplus [\mathbb{A}_2, \mathbb{A}_2] \oplus [\mathbb{A}_2, \Sigma(\mathbb{A}_1 \wedge \mathbb{A}_1)]$. The space $\Sigma(\mathbb{A}_1 \wedge \mathbb{A}_1)$ is 2-connected and $H_3(\Sigma(\mathbb{A}_1 \wedge \mathbb{A}_1), \mathbb{Z}) = H_2(\mathbb{A}_1 \wedge \mathbb{A}_1, \mathbb{Z}) = \mathbb{A} \otimes \mathbb{A}$. Therefore the Eilenberg–MacLane space $K(\mathbb{A} \otimes \mathbb{A}, 3)$ is the 3-stage of its Postnikov tower and $[\mathbb{A}_2, \Sigma(\mathbb{A}_1 \wedge \mathbb{A}_1)] = H^3(\mathbb{A}_2, \mathbb{A} \otimes \mathbb{A}) = \mathrm{Ext}(\mathbb{A}, \mathbb{A} \otimes \mathbb{A})$ by the Universal Coefficient Theorem.

Moreover, $\dim \mathbb{B}_m \leq m+1$, so again by Propositions 1.1 and 1.2 we get $[\mathbb{B}_m, \mathbb{A}_2 \vee \mathbb{B}_m] = [\mathbb{B}_m, \mathbb{A}_2] \oplus [\mathbb{B}_m, \mathbb{B}_m] \oplus [\mathbb{B}_m, \Sigma(\mathbb{A}_1 \wedge \mathbb{B}_{m-1})]$. The space $\Sigma(\mathbb{A}_1 \wedge \mathbb{B}_{m-1})$ is m-connected and $H_{m+1}(\Sigma(\mathbb{A}_1 \wedge \mathbb{B}_{m-1}), \mathbb{Z}) = H_m(\mathbb{A}_1 \wedge$

$\mathbb{B}_{m-1},\mathbb{Z}) = \mathbb{A}\otimes\mathbb{B}$. Therefore, the Eilenberg–MacLane space $K(\mathbb{A}\otimes\mathbb{B}, m+1)$ is the $(m+1)$-stage of its Postnikov tower and $[\mathbb{B}_m, \Sigma(\mathbb{A}_1 \wedge \mathbb{B}_{m-1})] = H^{m+1}(\mathbb{B}_m, \mathbb{A}\otimes\mathbb{B}) = \operatorname{Ext}(\mathbb{B}, \mathbb{A}\otimes\mathbb{B})$ by the Universal Coefficient Theorem. Thus we get $\ker j_* = \ker([\mathbb{B}_m, \mathbb{A}_n \vee \mathbb{A}_n] \to [\mathbb{B}_m, \mathbb{A}_n \times \mathbb{A}_n]) \oplus \operatorname{Ext}(\mathbb{A}, \mathbb{A}\otimes\mathbb{A}) \oplus \operatorname{Ext}(\mathbb{B}, \mathbb{A}\otimes\mathbb{B}) \oplus \operatorname{Ext}(\mathbb{B}, \mathbb{A}\otimes\mathbb{B})$.

(b) If $2 < n < m$ then $n+1 \le 2n-2 < n+m-2$ and $n+1, m+1 \le n+m-2 < 2m-2$. Then the induced maps $[\mathbb{A}_n, \mathbb{A}_n \vee \mathbb{B}_m \vee \mathbb{A}_n \vee \mathbb{B}_m] \to [\mathbb{A}_n, \mathbb{A}_n \vee \mathbb{A}_n] \oplus [\mathbb{A}_n, \mathbb{B}_m \vee \mathbb{B}_m] \to [\mathbb{A}_n, \mathbb{A}_n] \oplus [\mathbb{A}_n, \mathbb{A}_n] \oplus [\mathbb{A}_n, \mathbb{B}_m] \oplus [\mathbb{A}_n, \mathbb{B}_m]$, $[\mathbb{A}_n, \mathbb{A}_n \vee \mathbb{B}_m] \to [\mathbb{A}_n, \mathbb{A}_n] \oplus [\mathbb{A}_n, \mathbb{B}_m]$ and $[\mathbb{B}_m, \mathbb{A}_n \vee \mathbb{B}_m] \to [\mathbb{B}_m, \mathbb{A}_n] \oplus [\mathbb{B}_m, \mathbb{B}_m]$ are isomorphisms. Finally, we get $\ker j_* = \ker([\mathbb{B}_m, \mathbb{A}_n \vee \mathbb{A}_n] \to [\mathbb{B}_m, \mathbb{A}_n \times \mathbb{A}_n])$. ∎

Corollary 1.4. (a) *If $m < 2n-2$ then on $M(\mathbb{A}, n) \vee M(\mathbb{B}, m)$ there is a unique comultiplication determined by the natural pinching map for $2 < n < m$.*

(b) *If $m = 2n-2$ then $\mathcal{C}(M(\mathbb{A}, n) \vee M(\mathbb{B}, 2n-2)) = \operatorname{Ext}(\mathbb{B}, \mathbb{A}\otimes\mathbb{A})$ for $2 < n$.*

(c) *If $m = 2n-1$ then*

$$\mathcal{C}(M(\mathbb{A}, n) \vee M(\mathbb{B}, 2n-1)) = \begin{cases} [M(\mathbb{B}, 2n-1), \Sigma(M(\mathbb{A}, n-1) \wedge M(\mathbb{A}, n-1))] \\ \quad \oplus \operatorname{Ext}(\mathbb{B}, \mathbb{A}\otimes\mathbb{A}\otimes\mathbb{A}), & n = 3; \\ [M(\mathbb{B}, 2n-1), \Sigma(M(\mathbb{A}, n-1) \wedge M(\mathbb{A}, n-1))], & n > 3. \end{cases}$$

(d) $\mathcal{C}(M(\mathbb{A}, 2) \vee M(\mathbb{B}, 3))$

$$\begin{aligned} &= [M(\mathbb{B}, 3), \Sigma(M(\mathbb{A}, 1) \wedge M(\mathbb{A}, 1))] \\ &\quad \oplus \operatorname{Ext}(\mathbb{B}, \mathbb{A}\otimes\mathbb{A}\otimes\mathbb{A}) \oplus \operatorname{Ext}(\mathbb{A}, \mathbb{A}\otimes\mathbb{A}) \\ &\quad \oplus \operatorname{Ext}(\mathbb{B}, \mathbb{A}\otimes\mathbb{B}) \oplus \operatorname{Ext}(\mathbb{B}, \mathbb{A}\otimes\mathbb{B}). \end{aligned}$$

Proof. (a) The inclusion map $\mathbb{A}_n \vee \mathbb{A}_n \to \mathbb{A}_n \times \mathbb{A}_n$ is a $(2n-1)$-homology isomorphism so it is a $(2n-2)$-homotopy isomorphism by the Whitehead Theorem. But $\dim \mathbb{B}_m \le m+1 < 2n-1$ thus the induced map $[\mathbb{B}_m, \mathbb{A}_n \vee \mathbb{A}_n] \to [\mathbb{B}_m, \mathbb{A}_n \times \mathbb{A}_n]$ is an isomorphism and the result follows from Proposition 1.3.

(b) If $m = 2n-2$ then, by Propositions 1.2 and 1.3, $\mathcal{C}(\mathbb{A}_n \vee \mathbb{B}_{2n-2}) = [\mathbb{B}_{2n-2}, \Sigma(\mathbb{A}_{n-1} \wedge \mathbb{A}_{n-1})]$. But the space $\Sigma(\mathbb{A}_{n-1} \wedge \mathbb{A}_{n-1})$ is $(2n-2)$-connected and $H_{2n-1}(\Sigma(\mathbb{A}_{n-1} \wedge \mathbb{A}_{n-1}), \mathbb{Z}) = \mathbb{A}\otimes\mathbb{A}$. Hence the Eilenberg–MacLane space $K(\mathbb{A}\otimes\mathbb{A}, 2n-1)$ is the $(2n-1)$th stage of its Postnikov tower. Finally, $\mathcal{C}(\mathbb{A}_n \vee \mathbb{B}_{2n-2}) = [\mathbb{B}_{2n-2}, K(\mathbb{A}\otimes\mathbb{A}, 2n-1)] = H^{2n-1}(\mathbb{B}_{2n-2}, \mathbb{A}\otimes\mathbb{A}) = \operatorname{Ext}(\mathbb{B}, \mathbb{A}\otimes\mathbb{A})$ by the Universal Coefficient Theorem.

(c) If $2 < n < m = 2n-1$ then, by Propositions 1.2 and 1.3, $\mathcal{C}(\mathbb{A}_n \vee \mathbb{B}_{2n-1}) = [\mathbb{B}_{2n-1}, \Sigma(\mathbb{A}_{n-1} \wedge \mathbb{A}_{n-1})] \oplus [\mathbb{B}_{2n-1}, \Sigma(\mathbb{A}_{n-1} \wedge \mathbb{A}_{n-1} \wedge \mathbb{A}_{n-1})]$. But the space $\Sigma(\mathbb{A}_{n-1} \wedge \mathbb{A}_{n-1} \wedge \mathbb{A}_{n-1})$ is $(3n-3)$-connected and $H_{3n-2}(\Sigma(\mathbb{A}_{n-1} \wedge$

$\mathbb{A}_{n-1}\wedge\mathbb{A}_{n-1}),\mathbb{Z})=H_{3n-3}(\Sigma(\mathbb{A}_{n-1}\wedge\mathbb{A}_{n-1}\wedge\mathbb{A}_{n-1}),\mathbb{Z})=\mathbb{A}\otimes\mathbb{A}\otimes\mathbb{A}$. Hence the Eilenberg–MacLane space $K(\mathbb{A}\otimes\mathbb{A}\otimes\mathbb{A},3n-3)$ is the $(3n-3)$th stage of its Postnikov tower and $[M(\mathbb{B},2n-1),\Sigma(\mathbb{A}_{n-1}\wedge\mathbb{A}_{n-1}\wedge\mathbb{A}_{n-1})]=[\mathbb{B}_{2n-1},K(\mathbb{A}\otimes\mathbb{A}\otimes\mathbb{A},3n-3)]=\mathrm{Ext}(H_{3n-4}(\mathbb{B}_{2n-1},\mathbb{Z}),\mathbb{A}\otimes\mathbb{A}\otimes\mathbb{A})$. Thus the result follows.

(d) Again by Propositions 1.2 and 1.3 we get

$$\begin{aligned}\mathcal{C}(\mathbb{A}_2\vee\mathbb{B}_3)=[\mathbb{B}_3,\Sigma(\mathbb{A}_1\wedge\mathbb{A}_1)]\oplus[\mathbb{B}_3,\Sigma(\mathbb{A}_1\wedge\mathbb{A}_1\wedge\mathbb{A}_1)]\\ \oplus\,\mathrm{Ext}(\mathbb{A},\mathbb{A}\otimes\mathbb{A})\oplus\mathrm{Ext}(\mathbb{B},\mathbb{A}\otimes\mathbb{B})\oplus\mathrm{Ext}(\mathbb{B},\mathbb{A}\otimes\mathbb{B}).\end{aligned}$$

But the space $\Sigma(\mathbb{A}_1\wedge\mathbb{A}_1\wedge\mathbb{A}_1)$ is 3-connected and $H_4(\Sigma(\mathbb{A}_1\wedge\mathbb{A}_1\wedge\mathbb{A}_1),\mathbb{Z})=\mathbb{A}\otimes\mathbb{A}\otimes\mathbb{A}$. Hence the Eilenberg–MacLane space $K(\mathbb{A}\otimes\mathbb{A}\otimes\mathbb{A},4)$ is the 4th stage of its Postnikov tower and $[\mathbb{B}_3,\Sigma(\mathbb{A}_1\wedge\mathbb{A}_1\wedge\mathbb{A}_1)]=[\mathbb{B}_3,K(\mathbb{A}\otimes\mathbb{A}\otimes\mathbb{A},4)]=\mathrm{Ext}(\mathbb{B},\mathbb{A}\otimes\mathbb{A}\otimes\mathbb{A})$ by the Universal Coefficient Theorem. This completes the proof. ∎

On the other hand, for an abelian group $\mathbb{A}$ and an integer $n\ge 3$, a simply connected space $M'(\mathbb{A},n)$ with a single non-vanishing reduced integral cohomology group $\mathbb{A}$ in dimension n is called a *co-Moore* space of type $(\mathbb{A},n)$. In [6] it is shown that a co-Moore space of type $(\mathbb{A},n)$ has the homotopy type of the wedge $M(\mathbb{A}',n-1)\vee M(\mathbb{A}'',n)$ of Moore spaces, for some abelian groups $\mathbb{A}'$, $\mathbb{A}''$ with $\mathbb{A}=\mathrm{Ext}(\mathbb{A}',\mathbb{Z})\oplus\mathrm{Hom}(\mathbb{A}'',\mathbb{Z})$ and $\mathrm{Hom}(\mathbb{A}',\mathbb{Z})=\mathrm{Ext}(\mathbb{A}'',\mathbb{Z})=0$. Therefore Corollary 1.4 yields

COROLLARY 1.5. *Let $M'(\mathbb{A},n)=M(\mathbb{A}',n-1)\vee M(\mathbb{A}'',n)$ be a co-Moore space of type $(\mathbb{A},n)$ for $n\ge 3$.*

(a) *If $n>4$ then on $M'(\mathbb{A},n)$ there is a unique comultiplication determined by the natural pinching map.*

(b) $\mathcal{C}(M'(\mathbb{A},4))=\mathrm{Ext}(\mathbb{A}'',\mathbb{A}'\otimes\mathbb{A}')$.

(c) $$\begin{aligned}\mathcal{C}(M'(\mathbb{A},3))=[M(\mathbb{A}'',3),\Sigma(M(\mathbb{A}',1)\wedge M(\mathbb{A}',1))]\\ \oplus\,\mathrm{Ext}(\mathbb{A}'',\mathbb{A}'\otimes\mathbb{A}'\otimes\mathbb{A}')\oplus\mathrm{Ext}(\mathbb{A}',\mathbb{A}'\otimes\mathbb{A}')\\ \oplus\,\mathrm{Ext}(\mathbb{A}'',\mathbb{A}'\otimes\mathbb{A}'')\oplus\mathrm{Ext}(\mathbb{A}'',\mathbb{A}'\otimes\mathbb{A}'').\end{aligned}$$

In the remaining part of the paper we describe the group $[M(\mathbb{B},2n-1),\Sigma(M(\mathbb{A},n-1)\wedge M(\mathbb{A},n-1))]$ for some abelian groups $\mathbb{A}$, $\mathbb{B}$ and $n\ge 2$.

2. Restriction to abelian 2-groups. Let $M(\mathbb{A},n)$ be the Moore space of type $(\mathbb{A},n)$ for $n\ge 2$ and X a simply connected pointed space. Then, by the Universal Coefficient Theorem for homotopy groups in [7], we have the following exact sequence:

$$0\to\mathrm{Ext}(\mathbb{A},\pi_{n+1}(X))\to[M(\mathbb{A},n),X]\xrightarrow{\eta}\mathrm{Hom}(\mathbb{A},\pi_n(X))\to 0,$$

where η associates with a homotopy class the induced homomorphism on the nth homotopy groups. Note that we can easily derive the sequence above from the cofibre sequence for some map of wedges of spheres. In particular, if X is the Moore space $M(\mathbb{B}, n)$ of type $(\mathbb{B}, n)$ then by [5], $\pi_{n+1}(M(\mathbb{B}, n)) = \Gamma(\mathbb{B})$ for $n = 2$ and $\pi_{n+1}(M(\mathbb{B}, n)) = \mathbb{B} \otimes \mathbb{Z}_2$ for $n \geq 3$, where Γ is the Whitehead quadratic functor. Thus we get the following short exact sequence:

$$0 \to \operatorname{Ext}(\mathbb{A}, \mathbb{B} \otimes \mathbb{Z}_2) \to [M(\mathbb{A}, n), M(\mathbb{B}, n)] \xrightarrow{\eta} \operatorname{Hom}(\mathbb{A}, \mathbb{B}) \to 0$$

for $n \geq 3$.

For the reduced integral homology groups of the space $\Sigma(M(\mathbb{A}, n-1) \wedge M(\mathbb{A}, n-1))$, from the Künneth formula, we derive

$$H_m(\Sigma(M(\mathbb{A}, n-1) \wedge M(\mathbb{A}, n-1)), \mathbb{Z}) = \begin{cases} \mathbb{A} \otimes \mathbb{A}, & m = 2n-1; \\ \operatorname{Tor}(\mathbb{A}, \mathbb{A}), & m = 2n; \\ 0, & \text{otherwise.} \end{cases}$$

From the homology decomposition in [7, Chapter 8] it follows that the space $\Sigma(M(\mathbb{A}, n-1) \wedge M(\mathbb{A}, n-1))$ has the homotopy type of the mapping cone $M(\mathbb{A} \otimes \mathbb{A}, 2n-1) \cup_\tau c(M(\operatorname{Tor}(\mathbb{A}, \mathbb{A}), 2n-1))$ of a homologically trivial map $\tau : M(\operatorname{Tor}(\mathbb{A}, \mathbb{A}), 2n-1) \to M(\mathbb{A} \otimes \mathbb{A}, 2n-1)$. Thus by the Universal Coefficient Theorem for homotopy groups it follows that the map τ is determined by an element of the group $\operatorname{Ext}(\operatorname{Tor}(\mathbb{A}, \mathbb{A}), \mathbb{A} \otimes \mathbb{A} \otimes \mathbb{Z}_2)$.

Subsequent results require some lemmas and comments. Let X be a CW-complex and $X^{(n)}$ its nth skeleton for $n \geq 0$. We define $\Gamma_n(X) = \operatorname{im}(\pi_n(X^{(n-1)}) \to \pi_n(X^{(n)}))$. There is then ([7, Chapter 8]) the exact Whitehead sequence

$$\ldots \to H_{n+1}(X, \mathbb{Z}) \xrightarrow{\nu} \Gamma_n(X) \xrightarrow{\lambda} \pi_n(X) \xrightarrow{\mu} H_n(X, \mathbb{Z}) \to \ldots,$$

where μ is the Hurewicz homomorphism, λ is induced by inclusion and ν by the homotopy boundary $\pi_{n+1}(X^{(n+1)}, X^{(n)}) \to \pi_n(X^{(n)})$. Let $\operatorname{Sq}_2 : H_{n+2}(X, \mathbb{Z}_2) \to H_n(X, \mathbb{Z}_2)$ be the dual Steenrod square. We point out that for the existence of this map, which is dual to a map of linearly compact vector spaces, we do not need to require that X is of finite type.

Recall from [7, Chapter 8] that an *A_n^2-polyhedron* is an $(n-1)$-connected, $(n+2)$-dimensional polyhedron for $n > 2$. In particular, the space $X = \Sigma(M(\mathbb{A}, n-1) \wedge M(\mathbb{A}, n-1))$ is an A_{2n-1}^2-polyhedron being the mapping cone of a homologically trivial map $\tau : M(\operatorname{Tor}(\mathbb{A}, \mathbb{A}), 2n-1) \to M(\mathbb{A} \otimes \mathbb{A}, 2n-1)$.

LEMMA 2.1 [7, Chapter 8]. *Let X be an A_n^2-polyhedron with $H_{n+2}(X, \mathbb{Z}) = 0$. Then there is a short exact sequence*

$$0 \to \Gamma_{n+1}(X) \xrightarrow{\lambda} \pi_{n+1}(X) \xrightarrow{\mu} H_{n+1}(X, \mathbb{Z}) \to 0$$

with an isomorphism $\Gamma_{n+1}(X) \approx H_n(X) \otimes \mathbb{Z}_2$. Furthermore, $\pi_{n+1}(X)$ is determined by an element $\chi \in \operatorname{Ext}(H_{n+1}(X, \mathbb{Z}), \Gamma_{n+1}(X)) = \operatorname{Hom}({}_2H_{n+1}(X, \mathbb{Z}), \Gamma_{n+1}(X))$ such that $\chi\partial = \operatorname{Sq}_2$, where $\partial : H_{n+2}(X, \mathbb{Z}_2) \to$

${}_2H_{n+1}(X,\mathbb{Z})$ *is the Bockstein map to the subgroup of* $H_{n+1}(X,\mathbb{Z})$ *consisting of elements of order* 2.

In particular, if $X=\Sigma(M(\mathbb{A},n-1)\wedge M(\mathbb{A},n-1))$ then there is an exact sequence

$$0\to \mathbb{A}\otimes\mathbb{A}\otimes\mathbb{Z}_2\to \pi_{2n}(X)\to \mathrm{Tor}(\mathbb{A},\mathbb{A})\to 0$$

and the group $\pi_{2n}(X)$ is determined by a map $\chi: {}_2\mathrm{Tor}(\mathbb{A},\mathbb{A})\to \mathbb{A}\otimes\mathbb{A}\otimes\mathbb{Z}_2$ such that $\chi\partial=\mathrm{Sq}_2$. But $H_{2n+1}(X,\mathbb{Z}_2)=\mathrm{Tor}(H_{2n}(X,\mathbb{Z}),\mathbb{Z}_2)=\mathrm{Tor}(\mathrm{Tor}(\mathbb{A},\mathbb{A}),\mathbb{Z}_2)={}_2\mathrm{Tor}(\mathbb{A},\mathbb{A})$ so the Bockstein map ∂ is the identity on the group ${}_2\mathrm{Tor}(\mathbb{A},\mathbb{A})$. Thus $\pi_{2n}(X)$ is determined by the map $\mathrm{Sq}_2: {}_2\mathrm{Tor}(\mathbb{A},\mathbb{A})\to\mathbb{A}\otimes\mathbb{A}\otimes\mathbb{Z}_2$. Now if $\mathbb{B}$ is another abelian group then from the Universal Coefficient Theorem for homotopy groups we have the following short exact sequence:

$$0\to \mathrm{Ext}(\mathbb{B},\pi_{2n}(X))\to [M(\mathbb{B},2n-1),X]\to \mathrm{Hom}(\mathbb{B},\mathbb{A}\otimes\mathbb{A})\to 0.$$

In particular, from [7, Chapter 12] we infer

Corollary 2.2. *If* $\mathbb{B}$ *is a cyclic group of order* p^m, *with* p *a prime and* $m\ge 1$ *then the sequence above splits provided the* p*-primary component of* $\mathbb{A}\otimes\mathbb{A}$ *is finitely generated and either* $m>1$ *or* $p>2$ *and* $m=1$.

Let $T(\mathbb{A})$ be the torsion subgroup of an abelian group $\mathbb{A}$ and $T_2(\mathbb{A})$ its 2-component. Then $\mathrm{Tor}(\mathbb{A},\mathbb{A})=\mathrm{Tor}(T(\mathbb{A}),T(\mathbb{A}))$ and the map $2\times - : T(\mathbb{A})\to T(\mathbb{A})$ given by multiplication by 2 is an isomorphism provided $T_2(\mathbb{A})=0$. Thus we deduce that $\mathrm{Ext}(\mathrm{Tor}(\mathbb{A},\mathbb{A}),\mathbb{A}\otimes\mathbb{A}\otimes\mathbb{Z}_2)=0$ and the space $\Sigma(M(\mathbb{A},n-1)\wedge M(\mathbb{A},n-1))$ has the homotopy type of the wedge $M(\mathbb{A}\otimes\mathbb{A},2n-1)\vee M(\mathrm{Tor}(\mathbb{A},\mathbb{A}),2n)$. The inclusion map

$$M(\mathbb{A}\otimes\mathbb{A},2n-1)\vee M(\mathrm{Tor}(\mathbb{A},\mathbb{A}),2n)\to M(\mathbb{A}\otimes\mathbb{A},2n-1)\times M(\mathrm{Tor}(\mathbb{A},\mathbb{A}),2n)$$

is a $(4n-2)$-homotopy isomorphism. Hence

$$\begin{aligned}&[M(\mathbb{B},2n-1),\Sigma(M(\mathbb{A},n-1)\wedge M(\mathbb{A},n-1))]\\&\quad=[M(\mathbb{B},2n-1),M(\mathbb{A}\otimes\mathbb{A},2n-1)]\oplus[M(\mathbb{B},2n-1),M(\mathrm{Tor}(\mathbb{A},\mathbb{A}),2n)]\\&\quad=[M(\mathbb{B},2n-1),M(\mathbb{A}\otimes\mathbb{A},2n-1)]\oplus\mathrm{Ext}(\mathbb{B},\mathrm{Tor}(\mathbb{A},\mathbb{A})).\end{aligned}$$

If $\mathbb{B}$ has no elements of order 2 then by [7, Chapter 8], $\mathrm{Ext}(\mathbb{B},\mathbb{A}\otimes\mathbb{A}\otimes\mathbb{Z}_2)=0$ and from the Universal Coefficient Theorem for homotopy groups $[M(\mathbb{B},2n-1),M(\mathbb{A}\otimes\mathbb{A},2n-1)]=\mathrm{Hom}(\mathbb{B},\mathbb{A}\otimes\mathbb{A})$.

In particular, let $\mathbb{B}$ be an infinite cyclic group or of order p^m, where p is a prime and $m\ge 1$. Then from the Universal Coefficient Theorem for homotopy groups and [7, Chapter 8] we derive

$$[M(\mathbb{B},2n-1),M(\mathbb{A}\otimes\mathbb{A},2n-1)]=\begin{cases}\mathbb{A}\otimes\mathbb{A}, & \mathbb{B}=\mathbb{Z};\\ {}_{p^m}(\mathbb{A}\otimes\mathbb{A}), & \mathbb{B}=\mathbb{Z}_{p^m},\ p>2;\\ \mathbb{A}\otimes\mathbb{A}\otimes\mathbb{Z}_2, & \mathbb{B}=\mathbb{Z}_{2^m},\ p=2,\end{cases}$$

where ${}_{p^m}(\mathbb{A}\otimes\mathbb{A})$ is the subgroup of $\mathbb{A}\otimes\mathbb{A}$ consisting of elements annihilated by p^m.

More generally, we have

LEMMA 2.3. *If* $\mathbb{A}=\mathbb{A}'\oplus T_2(\mathbb{A})$ *with* $T_2(\mathbb{A}')=0$ *then*

$$\begin{aligned}[M(\mathbb{B},2n-1),\Sigma(M(\mathbb{A},n-1)\wedge M(\mathbb{A},n-1))]\\ =[M(\mathbb{B},2n-1),M(\mathbb{A}'\otimes\mathbb{A}',2n-1)]\oplus \operatorname{Ext}(\mathbb{B},\operatorname{Tor}(\mathbb{A}',\mathbb{A}'))\\ \oplus \operatorname{Ext}(\mathbb{B},\mathbb{A}'\otimes T_2(\mathbb{A})\oplus T_2(\mathbb{A})\otimes\mathbb{A}')\\ \oplus [M(\mathbb{B},2n-1),\Sigma(M(T_2(\mathbb{A}),n-1)\wedge M(T_2(\mathbb{A}),n-1))]\end{aligned}$$

for any abelian group $\mathbb{B}$.

Proof. As in the previous section, we write $\mathbb{A}_n$ for the Moore space $M(\mathbb{A},n)$. Then observe that

$$\begin{aligned}\mathbb{A}_{n-1}\wedge\mathbb{A}_{n-1}=\mathbb{A}'_{n-1}\wedge\mathbb{A}'_{n-1}\vee\mathbb{A}'_{n-1}\wedge(T_2(\mathbb{A}))_{n-1}\vee(T_2(\mathbb{A}))_{n-1}\wedge\mathbb{A}'_{n-1}\\ \vee(T_2(\mathbb{A}))_{n-1}\wedge(T_2(\mathbb{A}))_{n-1}\end{aligned}$$

and $\mathbb{A}'_{n-1}\wedge\mathbb{A}'_{n-1}=(\mathbb{A}'\otimes\mathbb{A}')_{2n-1}\vee(\operatorname{Tor}(\mathbb{A}',\mathbb{A}'))_{2n}$. But $H_{2n-1}(\Sigma(\mathbb{A}'_{n-1}\wedge(T_2(\mathbb{A}))_{n-1}),\mathbb{Z})=\mathbb{A}'\otimes T_2(\mathbb{A})$ and $H_{2n}(\Sigma(\mathbb{A}'_{n-1}\wedge(T_2(\mathbb{A}))_{n-1}),\mathbb{Z})=\operatorname{Tor}(\mathbb{A}',T_2(\mathbb{A}))=\operatorname{Tor}(T(\mathbb{A}'),T_2(\mathbb{A}))=0$, since $\operatorname{Tor}(T(\mathbb{A}'),T_2(\mathbb{A}))=\varinjlim_{i\in I}\operatorname{Tor}(T(\mathbb{A}'),T_2(\mathbb{A})^i)=0$, where $T_2(\mathbb{A})^i$ runs over all finite subgroups of $T_2(\mathbb{A})$. So $\Sigma(\mathbb{A}'_{n-1}\wedge(T_2(\mathbb{A}))_{n-1})=(\mathbb{A}'\otimes T_2(\mathbb{A}))_{2n-1}$ and

$$\begin{aligned}\Sigma(\mathbb{A}_{n-1}\wedge\mathbb{A}_{n-1})=\Sigma(\mathbb{A}'_{n-1}\wedge\mathbb{A}'_{n-1})\vee(\mathbb{A}'\otimes T_2(\mathbb{A})\oplus\mathbb{A}'\otimes T_2(\mathbb{A}))_{2n}\\ \vee\Sigma((T_2(\mathbb{A}))_{n-1}\wedge(T_2(\mathbb{A}))_{n-1}).\end{aligned}$$

Then we deduce that

$$\begin{aligned}[\mathbb{B}_{2n-1},\Sigma(\mathbb{A}_{n-1}\wedge\mathbb{A}_{n-1})]&=[\mathbb{B}_{2n-1},(\mathbb{A}'\otimes\mathbb{A}')_{2n-1}]\\ &\quad\oplus[\mathbb{B}_{2n-1},(\operatorname{Tor}(\mathbb{A}',\mathbb{A}'))_{2n}]\\ &\quad\oplus[\mathbb{B}_{2n-1},(\mathbb{A}'\otimes T_2(\mathbb{A})\oplus\mathbb{A}'\otimes T_2(\mathbb{A}))_{2n}]\\ &\quad\oplus[\mathbb{B}_{2n-1},\Sigma((T_2(\mathbb{A}))_{n-1}\wedge(T_2(\mathbb{A}))_{n-1})]\\ &=[\mathbb{B}_{2n-1},(\mathbb{A}'\otimes\mathbb{A}')_{2n-1}]\oplus\operatorname{Ext}(\mathbb{B},\operatorname{Tor}(\mathbb{A}',\mathbb{A}'))\\ &\quad\oplus\operatorname{Ext}(\mathbb{B},\mathbb{A}'\otimes T_2(\mathbb{A})\oplus\mathbb{A}'\otimes T_2(\mathbb{A}))\\ &\quad\oplus[\mathbb{B}_{2n-1},\Sigma((T_2(\mathbb{A}))_{n-1}\wedge(T_2(\mathbb{A}))_{n-1})].\ \blacksquare\end{aligned}$$

Furthermore, observe that $\pi_{2n}(\Sigma(M(T_2(\mathbb{A}),n-1)\wedge M(T_2(\mathbb{A}),n-1)))$ is an abelian 2-group and $\operatorname{Ext}(\mathbb{B},\pi_{2n}(\Sigma(M(T_2(\mathbb{A}),n-1)\wedge M(T_2(\mathbb{A}),n-1))))=0$ provided $T_2(\mathbb{A})$ is a finite abelian group and $\operatorname{Ext}(\mathbb{B},\mathbb{Z})=0$, i.e. $\mathbb{B}$ is a Whitehead group. Then

$$[M(\mathbb{B},2n-1),\Sigma(M(T_2(\mathbb{A}),n-1)\wedge M(T_2(\mathbb{A}),n-1))]=\operatorname{Hom}(\mathbb{B},\mathbb{A}'\otimes\mathbb{A}')$$

and we get the following complement of Corollary 1.5.

Corollary 2.4. *Let $M'(\mathbb{A},3) = M(\mathbb{A}_1,2) \vee M(\mathbb{A}_2,3)$ be a co-Moore space of type $(\mathbb{A},3)$, where $\mathbb{A} = \mathrm{Ext}(\mathbb{A}_1,\mathbb{Z}) \oplus \mathrm{Hom}(\mathbb{A}_2,\mathbb{Z})$ with $\mathrm{Hom}(\mathbb{A}_2,\mathbb{Z}) = \mathrm{Ext}(\mathbb{A}_2,\mathbb{Z}) = 0$. If $\mathbb{A}_1 = \mathbb{A}_1' \oplus T_2(\mathbb{A}_1)$ with $T_2(\mathbb{A}_1') = 0$ and $T_2(\mathbb{A}_1)$ is a finitely generated abelian group then*

$$\begin{aligned}
&[M(\mathbb{A}_2,3),\Sigma(M(\mathbb{A}_1,1)\wedge M(\mathbb{A}_1,1))]\\
&\quad = \mathrm{Hom}(\mathbb{A}_2,\mathbb{A}_1'\otimes\mathbb{A}_1') \oplus \mathrm{Ext}(\mathbb{A}_2,\mathrm{Tor}(\mathbb{A}_1',\mathbb{A}_1'))\\
&\qquad \oplus \mathrm{Ext}(\mathbb{A}_2,\mathbb{A}_1'\otimes T_2(\mathbb{A}_1)\oplus T_2(\mathbb{A}_1)\otimes\mathbb{A}_1') \oplus \mathrm{Hom}(\mathbb{A}_2,T_2(\mathbb{A}_1')\otimes T_2(\mathbb{A}_1')).
\end{aligned}$$

Moreover, $\mathrm{Ext}(\mathbb{Z}_{p^m},\pi_{2n}(\Sigma(M(T_2(\mathbb{A}),n-1)\wedge M(T_2(\mathbb{A}),n-1)))) = 0$ since $\pi_{2n}(\Sigma(M(T_2(\mathbb{A}),n-1)\wedge M(T_2(\mathbb{A}),n-1)))$ is a 2-group and $\mathrm{Hom}(\mathbb{Z}_{p^m},T_2(\mathbb{A})\otimes T_2(\mathbb{A})) = 0$ for $p>2$. Therefore

$$[M(\mathbb{B},2n-1),\Sigma(M(T_2(\mathbb{A}),n-1)\wedge M(T_2(\mathbb{A}),n-1))] = \begin{cases} T_2(\mathbb{A})\otimes T_2(\mathbb{A}), & \mathbb{B}=\mathbb{Z};\\ 0, & \mathbb{B}=\mathbb{Z}_{p^m},\ p>2.\end{cases}$$

In the sequel we compute this group if $T_2(\mathbb{A})$ is a finitely generated abelian group (i.e. a finite direct sum of cyclic 2-groups) and $\mathbb{B} = \mathbb{Z}_{2^m}$. Then we obtain a description of the group $[M(\mathbb{B},2n-1),\Sigma(M(\mathbb{A},n-1)\wedge M(\mathbb{A},n-1))]$ for $\mathbb{A} = \mathbb{A}'\oplus T_2(\mathbb{A})$ with $T_2(\mathbb{A}')=0$ and $T_2(\mathbb{A})$ a finitely generated abelian group, and $\mathbb{B}$ a direct sum of cyclic groups; in particular, for finitely generated abelian groups $\mathbb{A}$ and $\mathbb{B}$.

On the other hand, if $\mathbb{A} = \bigoplus_k \bigoplus_{I_k} \mathbb{Z}_{2^k}$ is a finite direct sum of cyclic 2-groups and $\mathbb{B}$ an abelian group then $M(\mathbb{A},n-1) = \bigvee_k \bigvee_{I_k} M(\mathbb{Z}_{2^k},n-1)$ and $\Sigma(M(\mathbb{A},n-1)\wedge M(\mathbb{A},n-1)) = \bigvee_{k,l}\bigvee_{I_k\times I_l}\Sigma(M(\mathbb{Z}_{2^k},n-1)\wedge M(\mathbb{Z}_{2^l},n-1))$. Thus

$$\begin{aligned}
&[M(\mathbb{B},2n-1),\Sigma(M(\mathbb{A},n-1)\wedge M(\mathbb{A},n-1))]\\
&\qquad = \bigoplus_{k,l}\bigoplus_{I_k\times I_l}[M(\mathbb{B},2n-1),\Sigma(M(\mathbb{Z}_{2^k},n-1)\wedge M(\mathbb{Z}_{2^l},n-1))].
\end{aligned}$$

3. Cyclic 2-groups. Write $X = \Sigma(M(\mathbb{Z}_{2^k},n-1)\wedge M(\mathbb{Z}_{2^l},n-1))$. The aim of this section is to compute the group $[M(\mathbb{Z}_{2^m},2n-1),X]$ for $1\le k\le l$ and $m\ge 1$. In the sequel some cohomology groups of the spaces involved will be needed. Observe that by the Universal Coefficient Theorem,

$$H^m(X,\mathbb{Z}) = \begin{cases}\mathbb{Z}, & m=0;\\ \mathbb{Z}_{2^k}, & m=2n,2n+1;\\ 0, & \text{otherwise},\end{cases}$$

$$H^m(X,\mathbb{Z}_2) = \begin{cases}\mathbb{Z}_2, & m=0;\\ \mathbb{Z}_2=(a_{2n-1}), & m=2n-1;\\ \mathbb{Z}_2\oplus\mathbb{Z}_2=(a_{2n}')\oplus(a_{2n}''), & m=2n;\\ \mathbb{Z}_2=(a_{2n+1}), & m=2n+1;\\ 0, & \text{otherwise}\end{cases}$$

and

$$H^m(M(\mathbb{Z}_{2^k}, n-1), \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2, & m = 0; \\ \mathbb{Z}_{2^k}, & m = n-1, n; \\ 0, & \text{otherwise.} \end{cases}$$

Let $\iota^k_{n-1} \in H^{n-1}(M(\mathbb{Z}_{2^k}, n-1), \mathbb{Z}_2)$ be the generator and β_r the rth power Bockstein operation [8, Chapter 7]. Then $\beta_k(\iota^k_{n-1})$ is the generator of $H^n(M(\mathbb{Z}_{2^k}, n-1), \mathbb{Z}_2)$. Furthermore, $a_{2n-1} = \sigma(\iota^k_{n-1} \otimes \iota^l_{n-1})$, $a'_{2n} = \sigma(\beta_k(\iota^k_{n-1}) \otimes \iota^l_{n-1})$, $a''_{2n} = \sigma(\iota^k_{n-1} \otimes \beta_l(\iota^l_{n-1}))$ and $a_{2n+1} = \sigma(\beta_k(\iota^k_{n-1}) \otimes \beta_l(\iota^l_{n-1}))$, where $\sigma : H^*(M(\mathbb{Z}_{2^k}, n-1) \wedge M(\mathbb{Z}_{2^l}, n-1), \mathbb{Z}_2) \to H^*(X, \mathbb{Z}_2)$ is the suspension isomorphism.

LEMMA 3.1. *Let $X = \Sigma(M(\mathbb{Z}_{2^k}, n-1) \wedge M(\mathbb{Z}_{2^l}, n-1))$ and $1 \le k \le l$.*

(a) *If $k = l = 1$ then the action of the Steenrod algebra $\mathcal{A}_2$ on $H^*(X, \mathbb{Z}_2)$ is given by the formulae*: $\mathrm{Sq}^1(a_{2n-1}) = a'_{2n} + a''_{2n}$, $\mathrm{Sq}^1(a'_{2n}) = \mathrm{Sq}^1(a''_{2n}) = a_{2n+1}$ *and* $\mathrm{Sq}^2(a_{2n-1}) = a_{2n+1}$.

(b) *Otherwise the action of the Steenrod algebra $\mathcal{A}_2$ and higher power Bockstein operations on $H^*(X, \mathbb{Z}_2)$ are given by the formulae*: $\beta_r(a_{2n-1}) = 0$ *for* $r < k$, $\beta_k(a_{2n-1}) = a'_{2n}$, $\beta_r(a''_{2n}) = 0$ *for* $r < k$, $\beta_k(a''_{2n}) = a_{2n+1}$ *and* $\mathrm{Sq}^2(a_{2n-1}) = 0$.

Proof. (a) The action of the Steenrod algebra $\mathcal{A}_2$ on $H^*(M(\mathbb{Z}_2, n-1) \wedge M(\mathbb{Z}_2, n-1), \mathbb{Z}_2)$ is stable, so by the Cartan formula the result follows.

(b) From the long exact cohomology sequence

$$\ldots \to H^m(X, \mathbb{Z}) \to H^m(X, \mathbb{Z}) \to H^m(X, \mathbb{Z}_2) \xrightarrow{\delta} H^{m+1}(X, \mathbb{Z}) \to \ldots$$

determined by the short one $0 \to \mathbb{Z} \xrightarrow{2\times} \mathbb{Z} \to \mathbb{Z}_2 \to 0$ we get

$$0 \to \mathbb{Z}_2 \xrightarrow{\delta} \mathbb{Z}_{2^k} \xrightarrow{2\times} \mathbb{Z}_{2^k} \to \mathbb{Z}_2 \oplus \mathbb{Z}_2 \xrightarrow{\delta} \mathbb{Z}_{2^k} \xrightarrow{2\times} \mathbb{Z}_{2^k} \to \mathbb{Z}_2 \to 0.$$

But $\mathrm{im}(\delta) = \ker(\times 2) = 2^{k-1}\mathbb{Z}_{2^k}$ so $\delta(e)$ is divisible by 2^{k-1}, where $e \in \mathbb{Z}_2$ is the non-zero element. Thus by [8, Chapter 7] we get $\beta_r(a_{2n-1}) = 0$ for $r < k$ and $\beta_k(a_{2n-1}) = a'_{2n}$. The pair (a'_{2n}, a''_{2n}) is a basis for $H^{2n}(X, \mathbb{Z}_2)$ so $\delta(a''_{2n}) = 2^{k-1}e$ and $\beta_r(a''_{2n}) = 0$ for $r < k$, and $\beta_k(a''_{2n}) = a_{2n+1}$. Moreover, by the Cartan formula,

$$\begin{aligned} &\mathrm{Sq}^2(\iota^k_{n-1} \otimes \iota^l_{n-1}) \\ &\quad = \mathrm{Sq}^2(\iota^k_{n-1}) \otimes \iota^l_{n-1} + \mathrm{Sq}^1(\iota^k_{n-1}) \otimes \mathrm{Sq}^1(\iota^l_{n-1}) + \iota^k_{n-1} \otimes \mathrm{Sq}^2(\iota^l_{n-1}). \end{aligned}$$

But $\mathrm{Sq}^2(\iota^k_{n-1}) = \mathrm{Sq}^2(\iota^l_{n-1}) = 0$ by dimension reasons and $\mathrm{Sq}^1(\iota^l_{n-1}) = 0$ for $l > 1$. This completes the proof. ∎

PROPOSITION 3.2. *Let $X = \Sigma(M(\mathbb{Z}_{2^k}, n-1) \wedge M(\mathbb{Z}_{2^l}, n-1))$ and $1 \le k \le l$. Then*

(a) $$\pi_{2n}(X) = \begin{cases} \mathbb{Z}_4, & k = l = 1; \\ \mathbb{Z}_2 \oplus \mathbb{Z}_{2^k}, & \textit{otherwise.} \end{cases}$$

(b) $X = \begin{cases} M(\mathbb{Z}_2, 2n-1) \cup_{2\,\mathrm{id}_{M(\mathbb{Z}_2,2n-1)}} c(M(\mathbb{Z}_2, 2n-1)), & k = l = 1; \\ M(\mathbb{Z}_{2^k}, 2n-1) \vee M(\mathbb{Z}_{2^k}, 2n), & \textit{otherwise} \end{cases}$

for $n \geq 2$.

Proof. (a) The space X is an A^2_{2n-1}-polyhedron, $\Gamma_{2n}(X) = H_{2n-1}(X, \mathbb{Z}) \otimes \mathbb{Z}_2 = \mathbb{Z}_2$ and $H_{2n}(X, \mathbb{Z}) = \mathbb{Z}_{2^k}$. By Lemma 2.1 there is a short exact sequence

$$0 \to \mathbb{Z}_2 \to \pi_{2n}(X) \to \mathbb{Z}_{2^k} \to 0$$

and the group $\pi_{2n}(X)$ is determined by the map

$$H_{2n+1}(X, \mathbb{Z}_2) \xrightarrow{\mathrm{Sq}_2} H_{2n-1}(X, \mathbb{Z}_2) = \mathbb{Z}_2.$$

If $k = l = 1$ then $\mathrm{Sq}^2 \neq 0$ by Lemma 3.1. Thus Sq_2 is the identity map and $\pi_{2n}(X) = \mathbb{Z}_4$.

If $1 \leq k \leq l$ and $1 < l$ then $\mathrm{Sq}^2 = 0$, by Lemma 3.1, and $\pi_{2n}(X) = \mathbb{Z}_2 \oplus \mathbb{Z}_{2^k}$ again by Lemma 2.1.

(b) By [7, Chapter 8] the space X has the homotopy type of the mapping cone $M(\mathbb{Z}_{2^k}, 2n-1) \cup_\tau c(M(\mathbb{Z}_{2^k}, 2n-1))$ of a homologically trivial map $\tau : M(\mathbb{Z}_{2^k}, 2n-1) \to M(\mathbb{Z}_{2^k}, 2n-1)$. But a non-zero Steenrod square occurs whenever a cone over a Moore space is attached essentially. Therefore, by Lemma 3.1, the map $\tau : M(\mathbb{Z}_{2^k}, 2n-1) \to M(\mathbb{Z}_{2^k}, 2n-1)$ is essential for $k = l = 1$ and trivial otherwise. Thus the space X has the homotopy type of the wedge $M(\mathbb{Z}_{2^k}, 2n-1) \vee M(\mathbb{Z}_{2^k}, 2n)$ for $1 \leq k \leq l$ and $1 < l$. However, for $1 = k = l$ by [4, 7, Chapter 12] we have $[M(\mathbb{Z}_2, 2n-1), M(\mathbb{Z}_2, 2n-1)] = \mathbb{Z}_4$, where the identity map $\mathrm{id}_{M(\mathbb{Z}_2,2n-1)}$ is a generator of this group. On the other hand, by the Universal Coefficient Theorem for homotopy groups, the homologically trivial map τ is determined by an element of the subgroup $\mathrm{Ext}(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2 \subseteq [M(\mathbb{Z}_2, 2n-1), M(\mathbb{Z}_2, 2n-1)] = \mathbb{Z}_4$. Thus $\tau = 2\,\mathrm{id}_{M(\mathbb{Z}_2,2n-1)}$ and the proof is finished. ■

We can now compute the group $[M(\mathbb{Z}_{2^m}, 2n-1), \Sigma(M(\mathbb{Z}_{2^k}, n-1) \wedge M(\mathbb{Z}_{2^l}, n-1))]$ for $1 \leq k \leq l$ and $m \geq 1$. Namely the following result holds.

THEOREM 3.3. *Let* $1 \leq k \leq l$ *and* $X = \Sigma(M(\mathbb{Z}_{2^k}, n-1) \wedge M(\mathbb{Z}_{2^l}, n-1))$ *for* $n \geq 2$. *Then*

$$[M(\mathbb{Z}_{2^m}, 2n-1), X] = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2, & 1 = k = l = m; \\ \mathbb{Z}_4 \oplus \mathbb{Z}_2, & 1 = k < l,\ m = 1 \ \textit{or}\ 1 = k = l,\ m > 1; \\ \mathbb{Z}_2 \oplus \mathbb{Z}_{2^{\min(k,m)}} \oplus \mathbb{Z}_{2^{\min(k,m)}}, & \textit{otherwise}. \end{cases}$$

Proof. First observe that for $m > 1$, by Corollary 2.2 and Proposition 3.2, we get

$$\begin{aligned}[M(\mathbb{Z}_{2^m}, 2n-1), X] &= \mathrm{Ext}(\mathbb{Z}_{2^m}, \pi_{2n}(X)) \oplus \mathrm{Hom}(\mathbb{Z}_{2^m}, \mathbb{Z}_{2^k}) \\ &= \begin{cases} \mathbb{Z}_4 \oplus \mathbb{Z}_2, & 1 = k = l; \\ \mathbb{Z}_2 \oplus \mathbb{Z}_{2^{\min(k,m)}} \oplus \mathbb{Z}_{2^{\min(k,m)}}, & 1 \le k \le l,\ l > 1. \end{cases}\end{aligned}$$

Now let $1 = k = l = m$ and $i : M(\mathbb{Z}_2, 2n-1) \to X$ be the inclusion map. Then we obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}(\mathbb{Z}_2, \mathbb{Z}_2) \arrow[r] \arrow[d, "i'_*"] & {[M(\mathbb{Z}_2, 2n-1), M(\mathbb{Z}_2, 2n-1)]} \arrow[r, "\eta"] \arrow[d, "i_*"] & \mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}_2) \arrow[r] \arrow[d, "i''_*"] & 0 \\
0 \arrow[r] & \mathrm{Ext}(\mathbb{Z}_2, \mathbb{Z}_4) \arrow[r] & {[M(\mathbb{Z}_2, 2n-1), X]} \arrow[r, "\eta'"] & \mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}_2) \arrow[r] & 0
\end{tikzcd}$$

Observe that the map $i''_* : \mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2 \to \mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$ is an isomorphism. However, $i'_* : \mathrm{Ext}(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2 \to \mathrm{Ext}(\mathbb{Z}_2, \mathbb{Z}_4) = \mathbb{Z}_2$ is trivial. In the light of [4], [7, Chapter 12] we have $[M(\mathbb{Z}_2, 2n-1), M(\mathbb{Z}_2, 2n-1)] = \mathbb{Z}_4$ so it is easy to deduce that $[M(\mathbb{Z}_2, 2n-1), X] = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

If now $1 \le k \le l$ and $1 < l$ then by Theorem 3.2 we have $X = M(\mathbb{Z}_{2^k}, 2n-1) \vee M(\mathbb{Z}_{2^k}, 2n)$. But the inclusion map $M(\mathbb{Z}_{2^k}, 2n-1) \vee M(\mathbb{Z}_{2^k}, 2n) \to M(\mathbb{Z}_{2^k}, 2n-1) \times M(\mathbb{Z}_{2^k}, 2n)$ is a $(4n-2)$-homotopy isomorphism so

$$\begin{aligned}[M(\mathbb{Z}_2, 2n-1), X] &= [M(\mathbb{Z}_2, 2n-1), M(\mathbb{Z}_{2^k}, 2n-1)] \\ &\quad \oplus [M(\mathbb{Z}_2, 2n-1), M(\mathbb{Z}_{2^k}, 2n)].\end{aligned}$$

By the Universal Coefficient Theorem for homotopy groups $[M(\mathbb{Z}_2, 2n-1), M(\mathbb{Z}_{2^k}, 2n)] = \mathrm{Ext}(\mathbb{Z}_2, \mathbb{Z}_{2^k}) = \mathbb{Z}_2$ and by [4], [7, Chapter 12] we have

$$[M(\mathbb{Z}_2, 2n-1), M(\mathbb{Z}_{2^k}, 2n-1)] = \begin{cases} \mathbb{Z}_4, & k = 1; \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2, & \text{otherwise} \end{cases}$$

and this completes the proof. ∎

We close the paper with the following problem.

Problem 3.4. For $n \ge 2$ and any abelian groups $\mathbb{A}$ and $\mathbb{B}$, describe the group $[M(\mathbb{B}, 2n-1), \Sigma(M(\mathbb{A}, n-1) \wedge M(\mathbb{A}, n-1))]$.

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