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# CONGRUENCE LATTICES OF FREE LATTICES IN NON-DISTRIBUTIVE VARIETIES

BY

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We prove that for any free lattice $F$ with at least $\aleph_2$ generators in any non-distributive variety of lattices, there exists no sectionally complemented lattice $L$ with congruence lattice isomorphic to the one of $F$. This solves a question formulated by Grätzer and Schmidt in 1962. This in turn yields further examples of simply constructed distributive semilattices that are not isomorphic to the semilattice of finitely generated two-sided ideals in any von Neumann regular ring.

**Introduction.** One of the oldest and most famous open problems in lattice theory, the *Congruence Lattice Problem*, is to decide whether for every distributive (join-) semilattice $S$ with zero, there exists a lattice $L$ such that the semilattice $\mathbf{C}(L)$ of compact congruences of $L$ (the *congruence semilattice* of $L$) is isomorphic to $S$. Although the answer is not known yet, many partial results have been obtained (see [7] for a survey). Among these are positive solutions of the Congruence Lattice Problem in the case where $S$ has size at most $\aleph_1$, or is a distributive lattice. In addition, it turns out that in several cases, the solution lattice $L$ to the problem is *sectionally complemented* (e.g., for the finite case, see [3]; the case where $S$ is *countable* results from unpublished work [1] of G. M. Bergman and results in [9]). As there seems to be a growing evidence that in all *known* cases, there exists a sectionally complemented solution lattice $L$, one may be tempted to formulate the even stronger conjecture that every distributive semilattice

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with zero is isomorphic to the congruence semilattice of a *sectionally complemented* lattice. This conjecture had in fact already been formulated in [2, Problem II.8].

In [9], using a construction presented in [8], the third author proves that it cannot be so, by giving a distributive semilattice of size $\aleph_2$ that is not isomorphic to the congruence semilattice of any lattice that is, in the terminology of [9], *congruence splitting*. In particular, every lattice which is either sectionally complemented, or relatively complemented, or a direct limit of atomistic lattices, is congruence splitting. Therefore, if one could prove that for every lattice $L$ there exists a sectionally complemented (or, more generally, congruence splitting) lattice $K$ such that $\mathbf{C}(L) \cong \mathbf{C}(K)$, then one would obtain a negative solution to the Congruence Lattice Problem.

This turns out to be an open problem as well, more specifically the second part of [3, Problem 1, p. 181]. In this paper we give a strong negative solution to this problem, by proving (Corollary 4.2) that in any non-distributive variety of lattices, if $F$ is any (bounded or not) free lattice with at least $\aleph_2$ generators, then there exists no congruence splitting lattice $L$ such that $\mathbf{C}(F) \cong \mathbf{C}(L)$; in particular, $F$ has no congruence-preserving embedding into any sectionally complemented lattice. By earlier results in [9], this implies that $\mathbf{C}(F)$ is never isomorphic to the semilattice of finitely generated two-sided ideals in a von Neumann regular ring. The restrictions on the lattice variety are optimal, because of the classical result saying that every distributive lattice embeds congruence-preservingly into a generalized Boolean algebra.

The strategy of the proof is the following: by the results of [9], the congruence semilattice of any congruence splitting lattice satisfies a certain infinite axiom, the *Uniform Refinement Property* (URP). In this paper, we introduce a slight weakening of URP, the *Weak Uniform Refinement Property* (WURP), that is not satisfied by the congruence semilattice of any free lattice with at least $\aleph_2$ generators in any non-distributive lattice variety $\mathcal{V}$. The two cases into which the proof splits, namely whether the diamond $M_3$ or the pentagon $N_5$ belongs to $\mathcal{V}$, are treated in a similar fashion: they are decorated with three 2-element chains that somewhat concentrate into a finite pattern, the combinatorial core of the original infinite WURP. As in [8], the reduction of the infinite case to the finite case is done *via* Kuratowski's free set property ([5]; see also [8, Proposition 2.5] for a short proof).

**Notation and terminology.** We consider semilattices of compact congruences of lattices. The semilattices are join semilattices with 0. The mapping assigning to every lattice $L$ its congruence semilattice $\mathbf{C}(L)$ can be extended to a functor from the category of lattices and lattice homomorphisms to the category of semilattices with homomorphisms of semilattices;

in addition, this functor preserves direct limits. The least and largest congruence on $L$ will be respectively denoted by $\mathbf{0}$ and $\mathbf{1}$.

For all elements $a$ and $b$ of a lattice $L$, we will denote by $\Theta(a,b)$ the least congruence containing the pair $(a,b)$ and we will then put $\Theta^+(a,b) = \Theta(a \wedge b, a)$; thus $\Theta^+(a,b)$ is the least congruence $\theta$ on $L$ such that $\theta(a) \leq \theta(b)$.

We say that a homomorphism of semilattices $\mu : S \to T$ is *weak-distributive* [9, Section 1] when for all $e \in S$ and all $b_0, b_1 \in T$ such that $\mu(e) = b_0 \vee b_1$, there are elements $a_0$ and $a_1$ of $S$ such that $\mu(a_0) \leq b_0$, $\mu(a_1) \leq b_1$ and $e = a_0 \vee a_1$.

For every non-negative integer $n$, we will identify $n$ with the finite set (initial ordinal) $\{0, 1, \ldots, n-1\}$.

**1. Congruence splitting lattices; uniform refinement properties.** We shall recall in this section some of the definitions and results of [8, 9] as well as a few new ones. Recall first [9, Section 3] that a lattice $L$ is *congruence splitting* when for all $a \leq b$ in $L$ and all congruences $\boldsymbol{a}_0$ and $\boldsymbol{a}_1$ in $L$, if $\Theta(a,b) = \boldsymbol{a}_0 \vee \boldsymbol{a}_1$, then there exist elements $x_0$ and $x_1$ of $[a,b]$ such that $x_0 \vee x_1 = b$, and $\Theta(a, x_i) \subseteq \boldsymbol{a}_i$ for all $i < 2$.

In [9, Proposition 3.2], a list of sufficient conditions is given for a lattice to be congruence splitting; this can be recorded here in the following fashion:

PROPOSITION 1.1. (a) *Every lattice that is either relatively complemented or sectionally complemented is congruence splitting.*

(b) *Every atomistic lattice is congruence splitting.*

(c) *The class of congruence splitting lattices is closed under direct limits.* ■

There are easy examples of non-congruence splitting lattices, as for instance any chain with at least three elements. However, it is to be noted that two lattices may have isomorphic congruence lattices while one is congruence splitting and the other is not. Our next definition is related to the effect of the congruence splitting property on the congruence lattice alone.

DEFINITION 1.2 (see [9, Definition 2.1]). Let $S$ be a semilattice, and $e$ be an element of $S$. We say that the *uniform refinement property* (URP) holds at $e$ when for all families $(a_i)_{i \in I}$ and $(b_i)_{i \in I}$ of elements of $S$ such that $a_i \vee b_i = e$ (all $i \in I$), there are families $(a_i^*)_{i \in I}$, $(b_i^*)_{i \in I}$ and $(c_{ij})_{(i,j) \in I \times I}$ of elements of $S$ such that for all $i, j, k \in I$, we have

(i) $a_i^* \leq a_i$ and $b_i^* \leq b_i$ and $a_i^* \vee b_i^* = e$.
(ii) $c_{ij} \leq a_i^*, b_j^*$ and $a_i^* \leq a_j^* \vee c_{ij}$.
(iii) $c_{ik} \leq c_{ij} \vee c_{jk}$.

We say that $S$ has the URP when the URP holds at every element of $S$.

Then we similarly define the *weak uniform refinement property* (WURP) *at* $e$ when, under the same hypotheses on the $a_i$, $b_i$ $(i \in I)$ and $e$, there are $c_{ij}$ such that for all $i, j, k \in I$, we have

(i′) $c_{ij} \leq a_i, b_j$.
(ii′) $c_{ij} \vee a_j \vee b_i = e$.
(iii′) $c_{ik} \leq c_{ij} \vee c_{jk}$.

We say that $S$ has the WURP when the WURP holds at every element of $S$.

LEMMA 1.3. *In the context above*, *the URP implies the WURP.* ∎

PROPOSITION 1.4. *Every distributive lattice has the URP.*

Proof. Let $(a_i)_{i\in I}$ and $(b_i)_{i\in I}$ be two families of elements of a distributive lattice $D$ such that $a_i + b_i =$ constant. It is easy to verify that the elements $a_i^* = a_i$, $b_i^* = b_i$ and $c_{ij} = a_i \wedge b_j$ are as required. ∎

We will later need the following straightforward lemma (see also [9, Proposition 2.3] for the URP):

LEMMA 1.5. *Let* $\mu : S \to T$ *be a weak-distributive homomorphism of semilattices and let* $e$ *be an element of* $S$. *If URP* (*resp. WURP*) *holds at* $e$ *in* $S$, *then it also holds at* $\mu(e)$ *in* $T$. ∎

COROLLARY 1.6. *Let* $S$ *be a distributive semilattice. If* $S$ *is the image of a distributive lattice under a weak-distributive homomorphism*, *then* $S$ *has the URP* (*thus the WURP*). ∎

In particular, any distributive semilattice that is the image of a generalized Boolean algebra under a weak-distributive homomorphism (this is E. T. Schmidt's sufficient condition for being isomorphic to the congruence semilattice of a lattice; see [6, 7]) satisfies the WURP.

We end this section by recording one of the main results of [9]:

THEOREM 1.7 [9, Theorem 3.3]. *Let* $L$ *be a congruence splitting lattice. Then* $\mathbf{C}(L)$ *has the URP* (*thus the WURP*). ∎

Hence, by Lemma 1.3, if $L$ is a congruence splitting lattice, then $\mathbf{C}(L)$ also has the WURP. In particular, if we manage to find a lattice such that its congruence semilattice does not have the WURP, then this lattice cannot be embedded congruence-preservingly into a congruence splitting lattice.

**2. The decorations of $M_3$ and $N_5$.** From now on until Theorem 3.3, we shall fix a non-distributive lattice variety $\mathcal{V}$. Let $\mathfrak{C}_2$ denote the two-element chain. For every set $X$, denote by $\mathbf{E}(X)$ the free product (= coproduct) of $X$ copies of $\mathfrak{C}_2$ in $\mathcal{V}$. Denote by $\mathbf{B}(X)$ the bounded lattice obtained from $\mathbf{E}(X)$ by adding a new largest element 1 and a new least element 0; write $\mathbf{E}_{\mathcal{V}}(X)$ (resp. $\mathbf{B}_{\mathcal{V}}(X)$) if $\mathcal{V}$ needs to be specified. Thus $\mathbf{B}(X)$ is generated as a bounded lattice by chains $s_i < t_i$ (all $i \in X$). Note that if $Y$ is a subset of $X$, then there is a canonical retraction from $\mathbf{B}(X)$ onto $\mathbf{B}(Y)$, sending each $s_i$ (resp. $t_i$) to 0 for every $i \in X \setminus Y$. Thus, we shall often identify $\mathbf{B}(Y)$ with the bounded sublattice of $\mathbf{B}(X)$ generated by all $s_i$ and $t_i$ ($i \in Y$). Moreover, the above-mentioned retraction from $\mathbf{B}(X)$ onto $\mathbf{B}(Y)$ induces a retraction from $\mathbf{C}(\mathbf{B}(X))$ onto $\mathbf{C}(\mathbf{B}(Y))$. Hence, we shall also identify $\mathbf{C}(\mathbf{B}(Y))$ with the corresponding subsemilattice of $\mathbf{C}(\mathbf{B}(X))$.

From now on until Theorem 3.3, we shall fix a set $X$ such that $|X| \geq \aleph_2$. We denote, for all $i \in X$, by $\boldsymbol{a}_i$ and $\boldsymbol{b}_i$ the compact congruences of $\mathbf{B}(X)$ defined by

$$\boldsymbol{a}_i = \Theta(0, s_i) \vee \Theta(t_i, 1), \qquad \boldsymbol{b}_i = \Theta(s_i, t_i). \tag{2.1}$$

In particular, note that $\boldsymbol{a}_i \vee \boldsymbol{b}_i = \mathbf{1}$. Now, towards a contradiction, suppose that there are compact congruences $\boldsymbol{c}_{ij}$ ($i, j \in X$) such that for all $i, j, k \in X$, the following holds:

$$\boldsymbol{c}_{ij} \subseteq \boldsymbol{a}_i, \boldsymbol{b}_j, \tag{2.2}$$

$$\boldsymbol{c}_{ij} \vee \boldsymbol{a}_j \vee \boldsymbol{b}_i = \mathbf{1}, \tag{2.3}$$

$$\boldsymbol{c}_{ik} \subseteq \boldsymbol{c}_{ij} \vee \boldsymbol{c}_{jk}. \tag{2.4}$$

Since the $\mathbf{C}$ functor preserves direct limits, there exists, for all $i, j \in X$, a finite subset $U = F(\{i, j\})$ of $X$ such that both $\boldsymbol{c}_{ij}$ and $\boldsymbol{c}_{ji}$ belong to $\mathbf{C}(\mathbf{B}(U))$. By Kuratowski's Theorem, there are mutually distinct elements, which we may denote by 0, 1, 2, of $X$ such that $0 \notin F(\{1, 2\})$, $1 \notin F(\{0, 2\})$, and $2 \notin F(\{0, 1\})$. Let $\pi : \mathbf{B}(X) \twoheadrightarrow \mathbf{B}(3)$ be the canonical retraction. For every $i < 3$, denote by $i'$ and $i''$ the other two elements of 3, arranged in such a way that $i' < i''$. For all $i < 3$, put $\boldsymbol{d}_i = \mathbf{C}(\pi)(\boldsymbol{c}_{i'i''})$.

Therefore, applying the semilattice homomorphism $\mathbf{C}(\pi)$ to the inequalities (2.2)–(2.4) yields

$$\boldsymbol{d}_0 \subseteq \boldsymbol{a}_1, \boldsymbol{b}_2, \qquad \boldsymbol{d}_1 \subseteq \boldsymbol{a}_0, \boldsymbol{b}_2, \qquad \boldsymbol{d}_2 \subseteq \boldsymbol{a}_0, \boldsymbol{b}_1, \tag{2.5}$$

$$\boldsymbol{d}_0 \vee \boldsymbol{a}_2 \vee \boldsymbol{b}_1 = \boldsymbol{d}_1 \vee \boldsymbol{a}_2 \vee \boldsymbol{b}_0 = \boldsymbol{d}_2 \vee \boldsymbol{a}_1 \vee \boldsymbol{b}_0 = \mathbf{1}, \tag{2.6}$$

$$\boldsymbol{d}_1 \subseteq \boldsymbol{d}_0 \vee \boldsymbol{d}_2. \tag{2.7}$$

LEMMA 2.1. *For all $i < 3$, $\boldsymbol{d}_i$ belongs to $\mathbf{C}(\mathbf{B}(3 \setminus \{i\}))$.*

Proof. We only treat the case of $i = 0$. Since $0 \notin F(\{1, 2\})$, $\boldsymbol{c}_{12}$ belongs to $\mathbf{B}(X \setminus \{0\})$, hence $\boldsymbol{d}_0 \in \mathbf{B}(\{1, 2\})$. ■

Now, since $\mathcal{V}$ is a non-distributive variety of lattices, by a classical result of lattice theory, either the diamond $M_3$ or the pentagon $N_5$ belongs to the variety $\mathcal{V}$. Denote by $M$ the one of these lattices that belongs to $\mathcal{V}$ and decorate it with three 2-element chains $x_i < y_i$ ($i < 3$) in the following way:

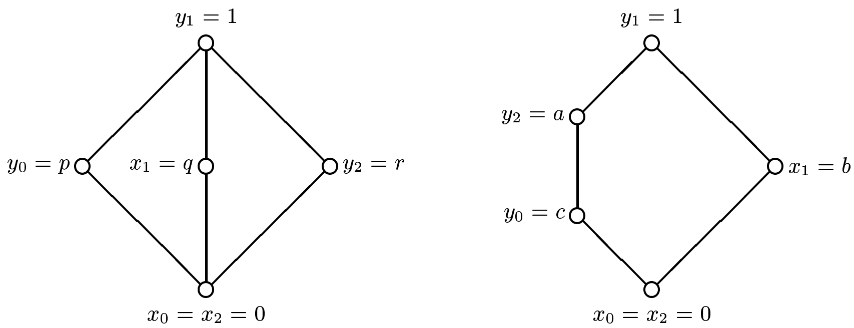
Case 1: $M = M_3$. Let $p, q, r$ be the three atoms of $M_3$. Put

$$x_0 = 0, \quad y_0 = p,$$
$$x_1 = q, \quad y_1 = 1,$$
$$x_2 = 0, \quad y_2 = r.$$

Case 2: $M = N_5$. Let $a > c$ and $b$ be the three join-irreducible elements of $N_5$. Put

$$x_0 = 0, \quad y_0 = c,$$
$$x_1 = b, \quad y_1 = 1,$$
$$x_2 = 0, \quad y_2 = a.$$

Both cases can be described by the following picture:




Case 1. $M = M_3$ Case 2. $M = N_5$

The relevant properties of these decorations are summarized in the following straightforward lemmas:

Lemma 2.2. *The decorations defined above satisfy*

$$x_0 \wedge y_1 \leq x_1, \quad y_1 \leq x_1 \vee y_0, \quad x_1 \wedge y_2 \leq x_2, \quad y_2 \leq x_2 \vee y_1,$$

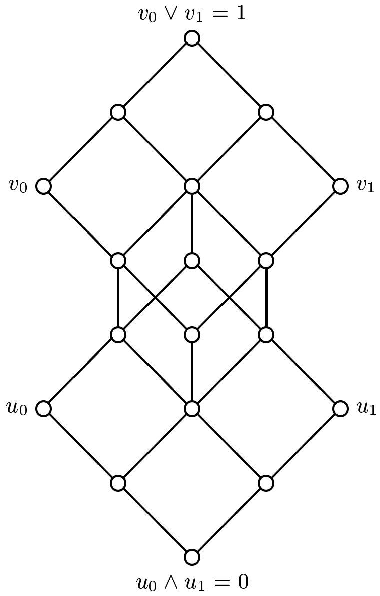
*but* $y_2 \not\leq x_2 \vee y_0$. ∎

Lemma 2.3. *For all* $i < 3$, *the sublattice of* $M$ *generated by the elements* $x_j$ *and* $y_j$ $(j \neq i)$ *is distributive.* ∎

This, along with (2.5)–(2.7), will be sufficient to obtain a contradiction. Note that since the free product of three 2-element chains in the variety generated by either $M_3$ or $N_5$ is finite, the problem is already reduced to a "computable" level. However, the size of the corresponding computations is such that it is useful to reduce (greatly) their complexity to merely computations in $M_3$ and $N_5$. This is what we shall do in Section 3.

**3. Reduction to the distributive world.** From now on, we shall denote by $D$ the free product of two 2-element chains in the variety of all *distributive* lattices. Thus, $D$ is generated by two chains $u_0 < v_0$ and $u_1 < v_1$.

Hence $D$ is a finite distributive lattice that can be represented by the following diagram:




The lattice $D$

For all $i < 3$, let $\pi_i : \mathbf{B}(3\backslash\{i\}) \to D$ be the unique lattice homomorphism sending $s_{i'}$ to $u_0$, $t_{i'}$ to $v_0$, $s_{i''}$ to $u_1$, and $t_{i''}$ to $v_1$. Furthermore, let $\varrho : \mathbf{B}(3) \to M$ be the unique lattice homomorphism sending $s_i$ to $x_i$ and $t_i$ to $y_i$ (for all $i < 3$); let $\varrho_i$ be the restriction of $\varrho$ to $\mathbf{B}(3 \setminus \{i\})$.

LEMMA 3.1. *Let $L$ be any distributive lattice, and $a$, $b$, $a'$, $b'$ be elements of $L$. Then*

$$\Theta^+(a,b) \cap \Theta^+(a',b') = \Theta^+(a \wedge a', b \vee b').$$

Proof. Let $B$ be the generalized Boolean algebra R-generated by $L$ (in the sense of [2, Part II, Section 4]); identify every congruence $\theta$ on $L$ with the unique congruence on $B$ extending $\theta$. For all elements $x$ and $y$ of $B$, denote by $x \smallsetminus y$ the unique relative complement of $x \wedge y$ in the interval $[0, x]$, and then put $x \bigtriangleup y = (x \smallsetminus y) \vee (y \smallsetminus x)$. Then a pair $(x, y)$ belongs to $\Theta^+(a, b)$ (resp. $\Theta^+(a', b')$) if and only if $x \bigtriangleup y \le a \smallsetminus b$ (resp. $x \bigtriangleup y \le a' \smallsetminus b'$). Therefore, $(x, y)$ belongs to $\Theta^+(a, b) \cap \Theta^+(a', b')$ if and only if $x \bigtriangleup y \le (a \smallsetminus b) \wedge (a' \smallsetminus b') = (a \wedge a') \smallsetminus (b \vee b')$. ■

REMARK. In particular, one recovers the classical result that if $a \le b \le c \le d$ are elements of any distributive lattice, then $\Theta(a, b) \cap \Theta(c, d) = \mathbf{0}$.

Now, for all $i < 3$, put $\boldsymbol{e}_i = \mathbf{C}(\pi_i)(\boldsymbol{d}_i)$.

LEMMA 3.2. *For all $i < 3$, we have $\boldsymbol{e}_i = \Theta^+(u_0 \wedge v_1, u_1) \vee \Theta^+(v_1, u_1 \vee v_0)$.*

Proof. Applying $\mathbf{C}(\pi_i)$ to the inequalities (2.5) and (2.6) yields the following inequalities:

$$\boldsymbol{e}_i \subseteq \Theta(0, u_0) \vee \Theta(v_0, 1) \quad \text{and} \quad \boldsymbol{e}_i \subseteq \Theta(u_1, v_1), \tag{3.1}$$

$$\boldsymbol{e}_i \vee \Theta(0, u_1) \vee \Theta(v_1, 1) \vee \Theta(u_0, v_0) = \mathbf{1}. \tag{3.2}$$

However, $D$ is a finite distributive lattice, thus $\mathbf{C}(D)$ is a finite Boolean algebra, and, by the Remark above, for all $j < 2$, the elements $\Theta(0, u_j) \vee \Theta(v_j, 1)$ and $\Theta(u_j, v_j)$ are complemented elements of $\mathbf{C}(D)$; in fact, $\mathbf{1}$ is the disjoint union of $\Theta(0, u_j)$, $\Theta(u_j, v_j)$, and $\Theta(v_j, 1)$. Then, from inequalities (3.1) and (3.2) and a new application of Lemma 3.1, one deduces easily that

$$\begin{aligned} \boldsymbol{e}_i &= (\Theta(0, u_0) \vee \Theta(v_0, 1)) \cap \Theta(u_1, v_1) \\ &= (\Theta(0, u_0) \cap \Theta(u_1, v_1)) \vee (\Theta(v_0, 1) \cap \Theta(u_1, v_1)) \\ &= \Theta^+(u_0 \wedge v_1, u_1) \vee \Theta^+(v_1, u_1 \vee v_0). \ \blacksquare \end{aligned}$$

Now, for all $i < 3$, it results from Lemma 2.3 that there exists a unique lattice homomorphism $\varphi_i : D \to M$ such that $\varphi_i \circ \pi_i = \varrho_i$. The corresponding commutative diagram is the following:

$$\begin{tikzcd} & (D, u_0, v_0, u_1, v_1) \arrow[d, "\varphi_i"] \\ (\mathbf{B}(3 \setminus \{i\}), s_{i'}, t_{i'}, s_{i''}, t_{i''}) \arrow[ur, "\pi_i"] \arrow[r, "\varrho_i"'] & (M, x_{i'}, y_{i'}, x_{i''}, y_{i''}) \end{tikzcd}$$

Since $\mathbf{C}$ is a functor, from this and from Lemma 3.2 for all $i < 3$, we get

$$\begin{aligned} \mathbf{C}(\varrho)(\boldsymbol{d}_i) = \mathbf{C}(\varphi_i)(\boldsymbol{e}_i) &= \mathbf{C}(\varphi_i)(\Theta^+(u_0 \wedge v_1, u_1) \vee \Theta^+(v_1, u_1 \vee v_0)) \\ &= \Theta^+(x_{i'} \wedge y_{i''}, x_{i''}) \vee \Theta^+(y_{i''}, x_{i''} \vee y_{i'}). \end{aligned} \tag{3.3}$$

In particular, we have, using Lemma 2.2,

$$\begin{aligned} \mathbf{C}(\varrho)(\boldsymbol{d}_0) &= \Theta^+(x_1 \wedge y_2, x_2) \vee \Theta^+(y_2, x_2 \vee y_1) = \mathbf{0}, \\ \mathbf{C}(\varrho)(\boldsymbol{d}_2) &= \Theta^+(x_0 \wedge y_1, x_1) \vee \Theta^+(y_1, x_1 \vee y_0) = \mathbf{0}, \\ \text{but} \quad \mathbf{C}(\varrho)(\boldsymbol{d}_1) &= \Theta^+(x_0 \wedge y_2, x_2) \vee \Theta^+(y_2, x_2 \vee y_0) \neq \mathbf{0}. \end{aligned}$$

On the other hand, by applying $\mathbf{C}(\varrho)$ to (2.7), we obtain

$$\mathbf{C}(\varrho)(\boldsymbol{d}_1) \leq \mathbf{C}(\varrho)(\boldsymbol{d}_0) \vee \mathbf{C}(\varrho)(\boldsymbol{d}_2),$$

a contradiction. Therefore, we have proved the following theorem:

THEOREM 3.3. *Let $\mathcal{V}$ be any non-distributive variety of lattices, and $X$ be any set such that $|X| \geq \aleph_2$. Let $\mathbf{B}_{\mathcal{V}}(X)$ be the free product in $\mathcal{V}$ of $X$ copies of a 2-element chain with a least and a largest element added. Then $\mathbf{C}(\mathbf{B}_{\mathcal{V}}(X))$ does not have the WURP at $\mathbf{1}$.* $\blacksquare$

**4. Extensions to further lattices and to regular rings.** This section will be devoted to harvest consequences of Theorem 3.3.

COROLLARY 4.1. *Let $L$ be any lattice that admits a lattice homomorphism onto a free bounded lattice in the variety generated by either $M_3$ or $N_5$ with $\aleph_2$ generators. Then $\mathbf{C}(L)$ does not have the WURP. In particular, there exists no congruence splitting lattice $K$ such that $\mathbf{C}(K) \cong \mathbf{C}(L)$; furthermore, $\mathbf{C}(L)$ does not satisfy Schmidt's condition.*

P r o o f. Let $\mathcal{V}$ be the lattice variety generated by either $M_3$ or $N_5$ and, for any set $X$, let $\mathbf{F}_{\mathcal{V}}(X)$ be the free bounded lattice on $X$ in $\mathcal{V}$. First, if the cardinality of $X$ is infinite, then there exists a surjective lattice homomorphism from $\mathbf{F}_{\mathcal{V}}(X)$ onto $\mathbf{B}_{\mathcal{V}}(X)$ (split $X$ into two disjoint sets $X_0$ and $X_1$ such that $|X_0| = |X_1| = |X|$; send the elements of $X_1$ (resp. $X_2$) onto all $s_i$ (resp. $t_i$)). Therefore, if $|X| = \aleph_2$, then there exists by assumption a surjective lattice homomorphism $f : L \twoheadrightarrow \mathbf{B}_{\mathcal{V}}(X)$. By [9, Proposition 1.2], the corresponding congruence mapping $\mathbf{C}(f) : \mathbf{C}(L) \twoheadrightarrow \mathbf{C}(\mathbf{B}_{\mathcal{V}}(X))$ is weak-distributive. Thus, if $\mathbf{C}(L)$ had the WURP, then, by Lemma 1.5, $\mathbf{C}(\mathbf{B}_{\mathcal{V}}(X))$ would also have the same refinement property, therefore contradicting Theorem 3.3. The last two assertions result from Theorem 1.7 and Corollary 1.6. ∎

This shows, in particular, that there are distributive semilattices that are representable as congruence semilattices of lattices (the $\mathbf{C}(L)$'s, with $L$ a free lattice on at least $\aleph_2$ generators in any non-distributive variety) that, nevertheless, do not satisfy any of the known sufficient conditions implying representability (as Schmidt's condition).

COROLLARY 4.2. *Let $\mathcal{V}$ be any non-distributive variety of lattices and let $F$ be any free (resp. free bounded) lattice with at least $\aleph_2$ generators in $\mathcal{V}$. Then there exists no congruence splitting lattice $K$ such that $\mathbf{C}(K) \cong \mathbf{C}(F)$.* ∎

COROLLARY 4.3. *Let $\mathcal{V}$ and $F$ be as above. Then there exists no von Neumann regular ring $R$ whose semilattice of finitely generated two-sided ideals is isomorphic to $\mathbf{C}(F)$.*

P r o o f. Proposition 1.1 (and the fact that the lattice of principal right ideals of $R$ is sectionally complemented), Corollary 4.2 and [9, Corollary 4.4] imply the result. ∎

We do not know whether every lattice of cardinality at most $\aleph_1$ admits a congruence-preserving extension to a sectionally complemented lattice. On the other hand, G. Grätzer and E. T. Schmidt prove in [4] that every *finite* lattice has a congruence-preserving extension into a sectionally complemented finite lattice. Note that if $\mathcal{V}$ is a non-distributive lattice variety generated by a single finite lattice, then $\mathbf{F}_{\mathcal{V}}(X)$ is a direct limit of a limit system of finite lattices with embeddings having the congruence extension property; nevertheless, its congruence semilattice is complicated in the sense that it does not have the WURP.

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