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A PROPERTY OF THE UNITARY CONVOLUTION

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The *unitary convolution* $a * b$ of two arithmetic functions (i.e. functions from $\mathbb{N}$ to $\mathbb{C}$) a and b is defined by the formula

$$(a*b)(n) = \sum_{\substack{d\mid n\\ (d,n/d)=1}} a(d)b(n/d).$$

The inverse of a function f under the unitary convolution, if it exists, is a (unique) function g such that

$$(f*g)(n) = \begin{cases} 1 & \text{if } n=1,\\ 0 & \text{if } n>1. \end{cases} \tag{1}$$

W. Narkiewicz proved in [1] the following

Theorem. *Every arithmetic function f such that $f(1)\neq 0$ and*

$$\sum_{n=1}^{\infty} |f(n)| < \infty$$

has an inverse g such that

$$\sum_{n=1}^{\infty} |g(n)| < \infty.$$

The proof was based on the theory of normed rings. Narkiewicz [2] (Problem 12) asked for an elementary, direct proof of this result. It is the aim of this paper to give such a proof.

Lemma 1. *If $f(1)=1$ the inverse function of f exists and is given by the formulae*

$$g(1) = 1, \tag{2}$$

$$g(n) = \sum_{k=1}^{\omega(n)} (-1)^k \sum_{\substack{d_1\ldots d_k = n\\ (d_i,d_j)=1,\, d_i>1}} \prod_{i=1}^{k} f(d_i) \quad \textit{for } n>1, \tag{3}$$

where $\omega(n)$ is the number of distinct prime factors of n.

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P r o o f. We shall check that the function given by the formulae (2) and (3) satisfies (1). For $n = 1$ this is true, thus let $n > 1$. By (2) and (3) we have

$$\begin{aligned}\sum_{\substack{d\mid n\\(d,n/d)=1}} f(d)g\left(\frac{n}{d}\right) &= g(n) + \sum_{\substack{d\mid n\\(d,n/d)=1,\,1<d<n}} f(d)g\left(\frac{n}{d}\right) + f(n)\\ &= -f(n) + \sum_{k=2}^{\omega(n)}(-1)^k \sum_{\substack{d_1\ldots d_k=n\\(d_i,d_j)=1,\,d_i>1}} \prod_{i=1}^{k} f(d_i)\\ &+ \sum_{\substack{d\mid n\\(d,n/d)=1,\,1<d<n}} f(d)\\ &\times \sum_{k=1}^{\omega(n/d)} (-1)^k \sum_{\substack{d_1\ldots d_k=n/d\\(d_i,d_j)=1,\,d_i>1}} \prod_{i=1}^{k} f(d_i) + f(n) = 0,\end{aligned}$$

since the first and the second double sums contain the same terms with opposite signs.

LEMMA 2. *If a sequence $\boldsymbol{a}$ of real numbers $a_i \ge 0$ satisfies*

$$\sum_{i=1}^{\infty} a_i < \infty,$$

then also

$$\sum_{k=1}^{\infty} k! \sum_{i_1<\ldots<i_k} a_{i_1}\ldots a_{i_k} < \infty.$$

P r o o f. Let us denote $\sum_{i_1<\ldots<i_k} a_{i_1}\ldots a_{i_k}$ by $\tau_k(\boldsymbol{a})$ with $\tau_0(\boldsymbol{a}) = 1$ and let l be the least positive integer such that

$$L := \sum_{i=l}^{\infty} a_i < 1.$$

We have the inequalities

$$k!\tau_k(\boldsymbol{a}) \le \tau_1(\boldsymbol{a})^k$$

(from the Newton polynomial formula) and the identity

$$\tau_k(\boldsymbol{a}) = \sum_{j=0}^{\min(k,l-1)} \tau_j(a_1,\ldots,a_{l-1})\tau_{k-j}(a_l,a_{l+1},\ldots).$$

Hence

$$\sum_{k=1}^{\infty} k!\tau_k(\boldsymbol{a}) = \sum_{k=1}^{\infty} k! \sum_{j=0}^{\min(k,l-1)} \tau_j(a_1,\ldots,a_{l-1})\tau_{k-j}(a_l,a_{l+1},\ldots)$$
$$= \sum_{j=0}^{l-1} \tau_j(a_1,\ldots,a_{l-1}) \sum_{k=j}^{\infty} k!\tau_{k-j}(a_l,a_{l+1},\ldots)$$
$$\le \sum_{j=0}^{l-1} \tau_j(a_1,\ldots,a_{l-1}) \sum_{k=j}^{\infty} \frac{k!}{(k-j)!} L^{k-j}.$$

However, for $|x| < 1$,

$$\sum_{k=j}^{\infty} \frac{k!}{(k-j)!} x^{k-j} = \frac{d^j}{dx^j} \sum_{k=0}^{\infty} x^k = \frac{d^j}{dx^j}(1-x)^{-1} = j!(1-x)^{-1-j}.$$

Hence

$$\sum_{j=0}^{l-1} \tau_j(a_1,\ldots,a_{l-1}) \sum_{k=j}^{\infty} \frac{k!}{(k-j)!} L^{k-j}$$
$$= \sum_{j=0}^{l-1} \tau_j(a_1,\ldots,a_{l-1}) j!(1-L)^{-1-j} < \infty.$$

Proof of the Theorem. By the assumption,

$$\sum_{n=1}^{\infty} \left| \frac{f(n)}{f(1)} \right| < \infty.$$

By Lemma 1 the inverse function of $f_0(n) = f(n)/f(1)$, denoted by g_0, exists and satisfies

$$\sum_{n=1}^{\infty} |g_0(n)| \le 1 + \sum_{k=1}^{\infty} \sum_{\substack{d_1\ldots d_k = n \\ (d_i,d_j)=1,\, d_i>1}} |f_0(d_1)\ldots f_0(d_k)|$$
$$\le \sum_{k=0}^{\infty} k!\tau_k(|f_2(2)|, |f_0(3)|, \ldots).$$

On applying Lemma 2 to the sequence

$$a_i = |f_0(i+1)| \quad (i = 1, 2, \ldots)$$

we obtain

$$\sum_{n=1}^{\infty} |g_0(n)| < \infty, \quad \text{hence also} \quad \sum_{n=1}^{\infty} \left| \frac{g_0(n)}{f(1)} \right| < \infty.$$

However, $g(n) = g_0(n)/f(1)$ is the inverse of $f(n)$, which completes the proof.

REFERENCES

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