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Density of periodic orbit measures for transformations on the interval with two monotonic pieces

by

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This paper is dedicated to the memory of Wiesław Szlenk

Abstract. Transformations $T:[0,1]\to[0,1]$ with two monotonic pieces are considered. Under the assumption that T is topologically transitive and $h_{\mathrm{top}}(T)>0$, it is proved that the invariant measures concentrated on periodic orbits are dense in the set of all invariant probability measures.

Introduction. In order to investigate generic properties of invariant measures for a topological dynamical system R. Bowen [2] introduced the specification property. This is a topological property which implies that the measures concentrated on periodic orbits are dense in the set of all invariant measures. The specification property implies generic properties for different types of invariant measures, e.g. ergodic measures, nonatomic measures, measures with zero entropy and strongly mixing measures (see [3]). It is known that the specification property holds for basic sets of axiom A-diffeomorphisms ([2], [3]), for monotonic mod one transformations ([5]) and for continuous maps on the interval ([1]).

We investigate in this paper dynamical systems generated by piecewise monotonic maps. If these maps have discontinuities, it becomes complicated to prove the density of periodic orbit measures.

Besides generic properties of invariant measures there are two more reasons to consider this problem for piecewise monotonic maps $T:[0,1]\to[0,1]$. We describe these reasons below.

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The first reason occurs in the calculation of the Hausdorff dimension of certain invariant subsets A. Assume that T is piecewise differentiable, the derivative satisfies certain regularity conditions, and there exist no attracting periodic points. Let A be a completely invariant closed subset of $[0,1]$, where "completely invariant" means $x \in A$ is equivalent to $Tx \in A$. Define $\pi(t) := p(T|_A, -t\log|T'|)$, where $p(\cdot,\cdot)$ denotes the pressure. It is shown in [6] that $\mathrm{HD}(A)$ equals the smallest $t_0 \ge 0$ with $\pi(t_0) = 0$, provided that there exists a $t \ge 0$ with $\pi(t) = 0$. Therefore one is interested in showing the existence of a zero of π. The proof of Theorem 1 in [6] shows that there exists a $t \ge 0$ with $\pi(t) = 0$ if the periodic orbit measures are dense in the set of all T-invariant probability measures on $[0,1]$.

Investigating piecewise monotonic maps one sometimes has to exclude the dynamics of the critical orbits. This leads to a modified definition of the pressure (see [7] and [8]). One defines $q(T,f) := \sup p(T|_B, f|_B)$, where the supremum is taken over all T-invariant closed $B \subseteq [0,1]$ for which a Markov partition exists. Naturally the question arises whether $q(T,f) = p(T,f)$. For continuous functions f the proof of Proposition 1 in [7] shows that $q(T,f) = p(T,f)$ if the periodic orbit measures are dense in the set of all T-invariant probability measures on $[0,1]$.

These reasons indicate that the density of periodic orbit measures plays a fundamental role in the investigation of piecewise monotonic maps. For piecewise monotonic maps in general it seems to be rather difficult to find a proof or a counterexample. Therefore we consider only transformations $T : [0,1] \to [0,1]$ with two monotonic pieces. If T is topologically transitive and $h_{\mathrm{top}}(T) > 0$, then we prove in Theorem 2 that the periodic orbit measures are dense in the set of all T-invariant probability measures on $[0,1]$. This result has been proved in [5] if T is strictly increasing on both intervals of monotonicity. The case of three or more monotonic pieces remains open.

1. Piecewise monotonic maps and their Markov diagram. A map $T : [0,1] \to [0,1]$ is called *piecewise monotone* if there exists a set $\mathcal{Z}$ of finitely many pairwise disjoint open intervals with $\bigcup_{Z\in\mathcal{Z}} \overline{Z} = [0,1]$ such that $T|_Z$ is strictly monotone and continuous for all $Z \in \mathcal{Z}$. We call a piecewise monotonic map $T : [0,1] \to [0,1]$ a *transformation with two monotonic pieces* if there exists a $\mathcal{Z}$ with $\operatorname{card}\mathcal{Z} = 2$ such that T is piecewise monotone with respect to $\mathcal{Z}$. Excluding the trivial case we always assume that for a transformation T with two monotonic pieces there exists no partition $\mathcal{Y}$ with $\operatorname{card}\mathcal{Y} = 1$ such that T is piecewise monotone with respect to $\mathcal{Y}$.

Set $E := \{\inf Z, \sup Z : Z \in \mathcal{Z}\} \setminus \{0,1\}$. Then T need not be continuous at x if $x \in E$. We can use a standard doubling points construction as described e.g. in [9] to obtain a dynamical system. For our purpose it is enough to replace each $x \in E$ by x^- and x^+, and define $T^n x^- := \lim_{y\to x^-} T^n y$ and

$T^n x^+ := \lim_{y\to x^+} T^n y$ for $n \in \mathbb{N}_0$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For simplicity of notation we write $[0,1]$, although from now on we mean $([0,1]\setminus E) \cup \{x^-, x^+ : x \in E\}$. An $a \in [0,1]$ is called a *critical point* if $a = x^-$ or $a = x^+$ for an $x \in E$. We call a critical point a an *essential critical point* if $T^{k+n}a \neq T^k a$ for every $k \in \mathbb{N}_0$ and every $n \in \mathbb{N}$. If a is a critical point but not an essential critical point, then let $k(a) \in \mathbb{N}_0$ and $n(a) \in \mathbb{N}$ be minimal with $T^{k(a)+n(a)}a = T^{k(a)}a$.

For the definitions of the topological entropy $h_{\mathrm{top}}(T)$ and of T-invariant measures see e.g. [10]. The set of all T-invariant Borel probability measures is denoted by $M([0,1],T)$. We call $R \subseteq [0,1]$ *topologically transitive* if there exists an $x \in R$ whose ω-limit set equals R. If $[0,1]$ is topologically transitive, then the map T is called *topologically transitive*. A point $p \in [0,1]$ is called a *periodic point* if there exists an $n \in \mathbb{N}$ with $T^n p = p$. Let p be a periodic point with $T^n p = p$, and define $\mu_p(A) := \frac{1}{n}\sum_{j=0}^{n-1} 1_A(T^j p)$ for every Borel set $A \subseteq [0,1]$. Then $\mu_p \in M([0,1],T)$. A measure μ is called a *periodic orbit measure* if there exists a periodic point $p \in [0,1]$ with $\mu = \mu_p$. We say *the periodic orbit measures are dense in* $M([0,1],T)$ if for every nonempty $U \subseteq M([0,1],T)$ which is open in the weak star topology there exists a periodic point $p \in [0,1]$ with $\mu_p \in U$.

Let $C \subseteq [0,1]$ be nonempty. Then D is called a *successor* of C if there exists a $Z \in \mathcal{Z}$ with $D = TC \cap Z$, and we write $C \to D$. Now let $\mathcal{D}$ be the smallest set with $\mathcal{Z} \subseteq \mathcal{D}$ and such that $C \in \mathcal{D}$ and $C \to D$ imply $D \in \mathcal{D}$. We call $(\mathcal{D}, \to)$ the *Markov diagram* of T (with respect to $\mathcal{Z}$).

Define $\mathcal{D}_0 := \mathcal{Z}$, and for $n \in \mathbb{N}$ define $\mathcal{D}_n := \mathcal{D}_{n-1} \cup \{D \in \mathcal{D} : \exists\, C \in \mathcal{D}_{n-1}$ with $C \to D\}$. Then $\mathcal{D}_0 \subseteq \mathcal{D}_1 \subseteq \mathcal{D}_2 \subseteq \ldots$ and $\mathcal{D}_\infty := \mathcal{D} = \bigcup_{n=0}^{\infty} \mathcal{D}_n$. Furthermore, for $n \in \mathbb{N}$ let $\mathcal{Z}_n$ be the set of all Z with $Z = \bigcap_{j=0}^{n-1} T^{-j} Z_j$ and $Z \neq \emptyset$, where $Z_0, Z_1, \ldots, Z_{n-1} \in \mathcal{Z}$.

We call $D_0 \to D_1 \to \ldots \to D_{n-1}$ a *path of length n in $\mathcal{D}$* if $D_{j-1} \to D_j$ for $j = 1, \ldots, n-1$ (a path of length 1 is an element of $\mathcal{D}$). Moreover, $D_0 \to D_1 \to D_2 \to \ldots$ is called an *infinite path in $\mathcal{D}$* if $D_{j-1} \to D_j$ for all $j \in \mathbb{N}$. We say an infinite path $D_0 \to D_1 \to D_2 \to \ldots$ *represents* $x \in [0,1]$ if $T^j x \in D_j$ for all $j \in \mathbb{N}_0$. A subset $\mathcal{C} \subseteq \mathcal{D}$ is called *irreducible* if for every $C, D \in \mathcal{C}$ there exists an $n \in \mathbb{N}$ and a path $D_0 \to D_1 \to \ldots \to D_n$ of length $n+1$ in $\mathcal{C}$ with $D_0 = C$ and $D_n = D$. If $\mathcal{C} \subseteq \mathcal{D}$ is irreducible and every $\mathcal{C}'$ with $\mathcal{C} \subsetneqq \mathcal{C}' \subseteq \mathcal{D}$ is not irreducible, then $\mathcal{C}$ is called *maximal irreducible.*

If $\alpha = D_0 \to D_1 \to \ldots \to D_{n-1}$ is a path of length n in $\mathcal{D}$, $\beta = C_0 \to C_1 \to \ldots \to C_{m-1}$ is a path of length m in $\mathcal{D}$, and $D_{n-1} \to C_0$, then denote by $\alpha \to \beta$ the path $D_0 \to D_1 \to \ldots \to D_{n-1} \to C_0 \to C_1 \to \ldots \to C_{m-1}$ of length $n+m$ in $\mathcal{D}$. A path $\alpha = D_0 \to D_1 \to \ldots \to D_{n-1}$ of length n in $\mathcal{D}$ is called a *periodic path* if $D_{n-1} \to D_0$. Assume that $\alpha = D_0 \to D_1 \to \ldots \to D_{n-1}$ is a periodic path. Then set $\alpha^1 := \alpha$, and for $k \in \mathbb{N}$, $k > 1$, define

$\alpha^k := \alpha^{k-1} \to \alpha$. We say x is *represented* by α if $C_0 \to C_1 \to C_2 \to \ldots$ with $C_{qn+r} := D_r$ for $q \in \mathbb{N}_0$ and $r \in \{0, 1, \ldots, n-1\}$ represents x.

For $x \in [0,1]$ there exists a unique infinite path $C_0^x \to C_1^x \to C_2^x \to \ldots$ in $\mathcal{D}$ with $C_0^x \in \mathcal{D}_0$ which represents x. Define $R_0^x := 0$. If $j \in \mathbb{N}$ and $R_{j-1}^x \neq \infty$, then set

$$R_j^x := \min\{n > R_{j-1}^x : C_{n-1}^x \text{ has at least 2 different successors}\}, \tag{1.1}$$

where we set $R_j^x := \infty$ if C_n^x has only one successor for every $n \geq R_{j-1}^x$. Finally, define $r_j^x := R_j^x - R_{j-1}^x$ if $R_j^x \neq \infty$.

The Markov diagram can be described in the following way (see [4]). We have

$$\mathcal{D} = \{C_n^a : n \in \mathbb{N}_0,\ a \text{ is a critical point or } a \in \{0,1\}\}. \tag{1.2}$$

Suppose that $x \in [0,1]$ and $j \in \mathbb{N}$ with $R_j^x \neq \infty$. Then there exists a critical point a such that $C_{R_{j-1}^x + k}^x \subseteq C_k^a$ for $k \in \{0, 1, \ldots, r_j^x - 1\}$ (choose $a \neq T^{R_{j-1}^x} x$ if this is possible). Hence $C_{R_j^x - 1}^x$ has the two different successors $C_{r_j^x}^a$ and $C(x,j)$, where $C(x,j) \cap \{\inf TC_{R_j^x - 1}^x, \sup TC_{R_j^x - 1}^x\} \neq \emptyset$. If $C_{R_j^x - 1}^x$ has more than two successors, then all other successors (besides $C_{r_j^x}^a$ and $C(x,j)$) are contained in $\mathcal{D}_0$. Furthermore, there exists a $q \in \mathbb{N}$ with $r_j^x = R_q^a$. Obviously, $r_j^x < R_j^x$ if $j > 1$. We have $C(x,j) = C_{R_j^x}^x$ if $j > 1$ and x is a critical point or $x \in \{0,1\}$.

2. Initial segments of critical orbits. In this section we prove that to show the density of periodic orbit measures in $M([0,1],T)$ it suffices to prove that certain initial segments of critical orbits can be approximated by periodic points.

Let $T : [0,1] \to [0,1]$ be a piecewise monotonic map. If $p \in [0,1]$ is a periodic point, then let μ_p be the invariant measure concentrated on the orbit of p. For $x \in [0,1]$, $U \subseteq [0,1]$ and $r, s \in \mathbb{N}_0$ with $0 \leq r < s$ define

$$F_{x,r,s}(U) := \frac{1}{s-r} \sum_{j=r}^{s-1} 1_U(T^j x). \tag{2.1}$$

Recall that we denote the Markov diagram of T by $(\mathcal{D}, \to)$. If T is topologically transitive and $h_{\mathrm{top}}(T) > 0$, then Theorem 11 of [4] implies that there exists a maximal irreducible $\mathcal{D}' \subseteq \mathcal{D}$ such that every $x \in [0,1]$ is represented by an infinite path in $\mathcal{D}'$. Furthermore, there exists no arrow $C \to D$ with $C \in \mathcal{D}'$ and $D \in \mathcal{D} \setminus \mathcal{D}'$, and there exists an $N_1 \in \mathbb{N}$ such that $C_{N_1}^a \in \mathcal{D}'$ for every essential critical point a.

Consider $x, y \in [0,1]$ and $n \in \mathbb{N}$. If C_k^x and C_k^y are contained in the same element of $\mathcal{Z}$ for all $k \in \{0, 1, \ldots, n-1\}$, then $|nF_{x,0,n}(Z) - nF_{y,0,n}(Z)| \leq m$ for all $m \in \mathbb{N}$ and all $Z \in \mathcal{Z}_m$.

In order to prove the density of periodic orbit measures in $M([0,1],T)$ we need the following result.

LEMMA 1. *Let $T:[0,1]\to[0,1]$ be a topologically transitive piecewise monotonic map with $h_{\mathrm{top}}(T)>0$. Fix $k\in\mathbb{N}$, and for $j\in\{1,\ldots,k\}$ let $x_j\in[0,1]$ and $l_j\in\mathbb{N}$. Furthermore, let $q_1,\ldots,q_k\in\mathbb{Q}$ with $q_j\ge 0$ for $j\in\{1,\ldots,k\}$ and $\sum_{j=1}^k q_j=1$. Assume that for every $V\in\bigcup_{m=1}^\infty\mathcal{Z}_m$ there exist $a_V>0$ and $b_V>0$ with the following property: for every $j\in\{1,\ldots,k\}$ there exists a periodic point $p_j\in[0,1]$ such that*

$$\begin{aligned}&|F_{x_j,0,l_j}(V)-\mu_{p_j}(V)|<a_V \quad \textit{for } 2\le j\le k, \quad \textit{and}\\ &|F_{x_1,0,l_1}(V)-\mu_{p_1}(V)|<b_V\end{aligned}\tag{2.2}$$

for every $V\in\bigcup_{m=1}^\infty\mathcal{Z}_m$. Then for every $\eta>0$ there exists a periodic point $p\in[0,1]$ such that

$$\Big|\sum_{j=1}^k q_jF_{x_j,0,l_j}(V)-\mu_p(V)\Big|<(1-q_1)a_V+q_1b_V+\eta m\tag{2.3}$$

for every $m\in\mathbb{N}$ and every $V\in\mathcal{Z}_m$.

Proof. For $j\in\{1,\ldots,k\}$ let α_j be a periodic path in $\mathcal{D}'$ representing p_j. Set $\alpha_{k+1}:=\alpha_1$. Then for every $j\in\{1,\ldots,k\}$ there exists a path υ_j of length u_j in $\mathcal{D}'$ with $\alpha_j\to\upsilon_j\to\alpha_{j+1}$. Define $u:=\max\{u_1,\ldots,u_k\}$. Choose an $n\in\mathbb{N}$ such that

$$\frac{k}{n}\le\frac{2ku}{n}<\frac{\eta}{2},\tag{2.4}$$

and $nq_j/l_j\in\mathbb{N}_0$ and $nq_j/a_j\in\mathbb{N}_0$ for every $j\in\{1,\ldots,k\}$, where a_j is the length of α_j.

We define the periodic path α in $(\mathcal{D},\to)$ by

$$\alpha:=\alpha_1^{nq_1/a_1}\to\upsilon_1\to\alpha_2^{nq_2/a_2}\to\upsilon_2\to\ldots\to\alpha_k^{nq_k/a_k}\to\upsilon_k.\tag{2.5}$$

Then α represents a periodic point $p\in[0,1]$. Set $N:=n+\sum_{j=1}^k u_j$.

Choose an $m\in\mathbb{N}$, and let $V\in\mathcal{Z}_m$. By (2.1) we obtain

$$N\mu_p(V)=NF_{p,0,N}(V)\quad\text{and}\quad nq_j\mu_{p_j}(V)=nq_jF_{p_j,0,nq_j}(V)$$

for $j\in\{1,\ldots,k\}$. If we use $\sum_{j=1}^k nq_j=n$ and (2.5) this implies

$$\Big|\sum_{j=1}^k nq_j\mu_{p_j}(V)-N\mu_p(V)\Big|\le\sum_{j=1}^k(u_j+m)\le k(u+m).\tag{2.6}$$

Since $n\le N\le n+ku$ we get $|N\mu_p(V)-n\mu_p(V)|\le ku$. Therefore (2.2) and (2.6) give

$$\Big|\sum_{j=1}^{k} nq_j F_{x_j,0,l_j}(V) - n\mu_p(V)\Big| \le n((1-q_1)a_V + q_1 b_V) + 2ku + km.$$

Dividing by n and using (2.4) we obtain (2.3). ∎

We will need the following special case of Lemma 1.

LEMMA 2. *Let $T:[0,1]\to[0,1]$ be a topologically transitive piecewise monotonic map with $h_{\mathrm{top}}(T)>0$. Suppose that $x\in[0,1]$, $k\in\mathbb{N}$ and $L_1,\ldots,L_k\in\mathbb{N}$ with $L_0:=0<L_1<\ldots<L:=L_k$. Assume that for every $m\in\mathbb{N}$ and every $V\in\mathcal{Z}_m$ there exist $a_V>0$ and $B_V>0$ with the following property*: *for every $j\in\{1,\ldots,k\}$ there exists a periodic point $p_j\in[0,1]$ such that*

$$\begin{aligned}(2.7)\qquad &|F_{x,L_{j-1},L_j}(V)-\mu_{p_j}(V)|<a_V \quad \text{for } 2\le j\le k, \quad \text{and}\\ &|F_{x,0,L_1}(V)-\mu_{p_1}(V)|<\frac{B_V+m}{L_1}\end{aligned}$$

for every $m\in\mathbb{N}$ and every $V\in\mathcal{Z}_m$. Then there exists a periodic point $p\in[0,1]$ such that

$$(2.8)\qquad |F_{x,0,L}(V)-\mu_p(V)|<a_V+\frac{B_V+2m}{L}$$

for every $m\in\mathbb{N}$ and every $V\in\mathcal{Z}_m$.

Proof. If $j\in\{1,\ldots,k\}$, then define $x_j:=T^{L_{j-1}}x$, $l_j:=L_j-L_{j-1}$ and $q_j:=l_j/L$. By (2.1) we have

$$F_{x,0,L}(U)=\sum_{j=0}^{k} q_j F_{x_j,0,l_j}(U)$$

for every $U\subseteq[0,1]$. Now apply Lemma 1 with $b_V:=(B_V+m)/L_1$ and $\eta:=1/L$, and use $1-q_1\le 1$. ∎

Consider a topologically transitive piecewise monotonic map $T:[0,1]\to[0,1]$ with $h_{\mathrm{top}}(T)>0$, and let $\mathcal{D}'$ be the maximal irreducible subset of $(\mathcal{D},\to)$ such that every $x\in[0,1]$ is represented by an infinite path in $\mathcal{D}'$. By Theorem 10 in [4] there exists an $n_1\in\mathbb{N}$ such that for every $x\in[0,1]$ there exists an infinite path $D_0\to D_1\to D_2\to\ldots$ in $\mathcal{D}'$ with $D_0\in\mathcal{D}_{n_1}$ which represents x. There exist $n_2,n_3\in\mathbb{N}$ with $n_2\ge n_1$ such that for every $C\in\mathcal{D}_{n_2}$ and every $D\in\mathcal{D}'\cap\mathcal{D}_{n_1}$ there exists a path $D_0\to D_1\to\ldots\to D_n$ of length $n+1<n_3$ in $\mathcal{D}$ with $D_0=C$ and $D_n=D$. If $s\in\mathbb{R}$, then let $\mathcal{R}(s)$ be the set of all $C\in\mathcal{D}$ such that for every $D\in\mathcal{D}'\cap\mathcal{D}_{n_1}$ there exists a path $D_0\to D_1\to\ldots\to D_n$ of length $n+1<s$ in $\mathcal{D}$ with $D_0=C$ and $D_n=D$.

For $n \in \mathbb{N}$ with $n \geq n_3^2$ define

$$\gamma(n) := \max\{r \in \mathbb{N} : \mathcal{D}_r \subseteq \mathcal{R}(\sqrt{n})\}, \tag{2.9}$$

where we set $\gamma(n) := \infty$ if $\mathcal{D} \subseteq \mathcal{R}(\sqrt{n})$. Obviously $\gamma(n) \geq n_2$ if $n \geq n_3^2$, $n \leq n'$ implies $\gamma(n) \leq \gamma(n')$, and $\lim_{n\to\infty} \gamma(n) = \infty$.

LEMMA 3. *Let $T : [0,1] \to [0,1]$ be a topologically transitive piecewise monotonic map with $h_{\text{top}}(T) > 0$. Suppose that $l, n, r \in \mathbb{N}$ with $n \geq n_3^2$. Assume that $x \in [0,1]$ is represented by an infinite path $D_0 \to D_1 \to D_2 \to \ldots$ in $(\mathcal{D}, \to)$ with $D_r \in \mathcal{D}' \cap \mathcal{D}_{n_1}$, and suppose that D_{l-1} has a successor in $\mathcal{D}_{\gamma(n)}$. Then there exists a periodic point $p \in [0,1]$ such that*

$$|F_{x,0,l}(V) - \mu_p(V)| < \frac{2\sqrt{n} + 2r + m}{l} \tag{2.10}$$

for every $m \in \mathbb{N}$ and every $V \in \mathcal{Z}_m$.

Proof. If $l \leq r$, then let α be a periodic path of length $u < \sqrt{n}$. For $r < l$ set $\alpha_0 := D_r \to D_{r+1} \to \ldots \to D_{l-1}$. As $D_{l-1} \in \mathcal{D}_{\gamma(n)}$ the definition of $\gamma(n)$ gives the existence of a path υ of length $u < \sqrt{n}$ in $(\mathcal{D}, \to)$ with $\alpha_0 \to \upsilon \to \alpha_0$. Define $\alpha := \alpha_0 \to \upsilon$. Then α represents a periodic point $p \in [0,1]$ (this is also true in the case $l \leq r$). We get

$$l|F_{x,0,l}(V) - \mu_p(V)| \leq |lF_{x,0,l}(V) - (l - r + u)\mu_p(V)| + |u - r|.$$

Since $|lF_{x,0,l}(V) - (l - r + u)\mu_p(V)| \leq u + r + m$ and $u < \sqrt{n}$ we obtain

$$l|F_{x,0,l}(V) - \mu_p(V)| < 2\sqrt{n} + 2r + m. \tag{2.11}$$

An analogous calculation proves (2.11) also in the case $l \leq r$. Dividing (2.11) by l gives (2.10). ∎

Now we are able to prove the main result of this section.

THEOREM 1. *Let $T : [0,1] \to [0,1]$ be a topologically transitive piecewise monotonic map with $h_{\text{top}}(T) > 0$. Fix $n_0 \in \mathbb{N}$ and $d(m) > 0$ for $m \in \mathbb{N}$. Suppose that for every essential critical orbit $(T^n a)_{n\in\mathbb{N}}$ and every $j \in \mathbb{N}$ with $r_j^a > n_0$ there exists an $l \in \{0, 1, \ldots, j-1\}$ and a periodic point $p_{a,j} \in [0,1]$ with*

$$|F_{a,R_l^a,R_j^a}(Z) - \mu_{p_{a,j}}(Z)| < \frac{d(m)}{R_j^a - R_l^a} \tag{2.12}$$

for every $m \in \mathbb{N}$ and for every $Z \in \mathcal{Z}_m$. Then the periodic orbit measures are dense in $M([0,1], T)$.

Proof. Let $U \subseteq M([0,1], T)$ be nonempty and open with respect to the weak star topology. Then there exists a $\mu \in M([0,1], T)$, an $\varepsilon > 0$, a $K \in \mathbb{N}$, and continuous functions $f_1, \ldots, f_K : [0,1] \to \mathbb{R}$ with

$$\Big\{\widetilde{\mu} : \Big| \int_{[0,1]} f_t \, d\widetilde{\mu} - \int_{[0,1]} f_t \, d\mu \Big| < \varepsilon \text{ for } t = 1, \ldots, K\Big\} \subseteq U. \tag{2.13}$$

Set

$$c := \max_{t=1,\ldots,K} \|f_t\|_\infty. \tag{2.14}$$

There exists an $r \in \mathbb{N}$, and for $j \in \{1, \ldots, r\}$ there exists an ergodic $\mu_j \in M([0,1],T)$ and a $q_j \in \mathbb{Q}$ with $q_j \geq 0$ such that $\sum_{j=1}^r q_j = 1$ and

$$\max_{t=1,\ldots,K} \Big| \sum_{j=1}^r q_j \int_{[0,1]} f_t \, d\mu_j - \int_{[0,1]} f_t \, d\mu \Big| < \frac{\varepsilon}{5}. \tag{2.15}$$

As T is topologically transitive, $\mathcal{Z}$ is a generator, and therefore there exists an $m \in \mathbb{N}$ with

$$\max_{t=1,\ldots,K} \sup_{Z \in \mathcal{Z}_m} \sup_{x,y \in Z} |f_t(x) - f_t(y)| < \frac{\varepsilon}{5}. \tag{2.16}$$

Fix this m for the rest of this proof. Now choose a $\delta > 0$ such that

$$2c\delta \operatorname{card} \mathcal{Z}_m < \frac{\varepsilon}{5}. \tag{2.17}$$

Since μ_j is ergodic, there exists an $N \in \mathbb{N}$ and there exist $x_1, \ldots, x_r \in [0,1]$ such that

$$\Big| \frac{1}{n} \sum_{s=0}^{n-1} f_t(T^s x_j) - \int_{[0,1]} f_t \, d\mu_j \Big| < \frac{\varepsilon}{5} \tag{2.18}$$

for every $j \in \{1, \ldots, r\}$, for every $t \in \{1, \ldots, K\}$, and for every $n \geq N$.

Fix a $j \in \{1, \ldots, r\}$. Then x_j is represented by an infinite path $D_0 \to D_1 \to D_2 \to \ldots$ in $\mathcal{D}'$ with $D_0 \in \mathcal{D}' \cap \mathcal{D}_{n_1}$. We claim that there exists an $l_j \in \mathbb{N}$ with $l_j \geq N$ and a periodic point $p_j \in [0,1]$ with

$$\max_{Z \in \mathcal{Z}_m} |F_{x_j,0,l_j}(Z) - \mu_{p_j}(Z)| < \delta. \tag{2.19}$$

If there exists an $n \in \mathbb{N}$ with $\gamma(n) = \infty$, then choose an $l_j \geq N$ with $(2\sqrt{n} + m)/l_j < \delta$. In this case Lemma 3 implies (2.19).

It remains to prove (2.19) in the case $\gamma(n) < \infty$ for every $n \in \mathbb{N}$. As

$$\lim_{n\to\infty} \gamma(n) = \infty,$$

we can choose an $R \in \mathbb{N}$ with $R \geq N$, $R \geq n_3^2$, $\gamma(R) \geq n_3^2$ and $\gamma(\gamma(R)) > n_0$ such that

$$\frac{1}{\gamma(\gamma(R))} d(m) + \frac{2}{\sqrt{\gamma(R)}} + \frac{2N_1 + 3m}{\gamma(R)} + \frac{2}{\sqrt{R}} + \frac{2m}{R} < \delta. \tag{2.20}$$

This R may be chosen in such a way that for every $C \in \mathcal{D}' \cap \mathcal{D}_{n_1}$ there exists a path $C_0 \to C_1 \to \ldots \to C_n$ of length $n+1 < \gamma(R)$ in $\mathcal{D}$ such that $C_0 = C$, C_n has at least two different successors, and every successor of C_n is an element of $\mathcal{D}_{\gamma(R)}$. Furthermore we may assume

$$\frac{2k(a) + 2n(a)}{\gamma(R)} < \frac{1}{\sqrt{\gamma(R)}}$$

for every critical point a with $T^{k(a)+n(a)}a = T^{k(a)}a$.

Now let a be an essential critical point, and let $u \in \mathbb{N}$ with $R_u^a > \gamma(R)$. Using (2.12) we find by induction that there exist $L_0 = 0 < L_1 < \ldots \ldots < L_k = R_u^a$ with $L_v - L_{v-1} > \gamma(\gamma(R))$ for $v = 2, \ldots, k$, and there exist periodic points $P_{a,2}, \ldots, P_{a,k}$ such that

$$|F_{a,L_{v-1},L_v}(Z) - \mu_{P_{a,v}}(Z)| < \frac{d(m)}{L_v - L_{v-1}} \le \frac{d(m)}{\gamma(\gamma(R))} \tag{2.21}$$

for every $Z \in \mathcal{Z}_m$ and every $v \in \{2, \ldots, k\}$. Furthermore, either $C_{L_1-1}^a$ has a successor in $\mathcal{D}_{\gamma(\gamma(R))}$, or $L_v - L_{v-1} > \gamma(\gamma(R))$ and (2.21) hold also for $v = 1$. In the first case Lemma 3 gives the existence of a periodic point $P_{a,1}$ with

$$|F_{a,0,L_1}(Z) - \mu_{P_{a,1}}(Z)| < \frac{2\sqrt{\gamma(R)} + 2N_1 + m}{L_1}$$

for every $Z \in \mathcal{Z}_m$. Applying Lemma 2 with $a_Z := d(m)/\gamma(\gamma(R))$ and $B_Z := 2\sqrt{\gamma(R)} + 2N_1$ we get the existence of a periodic point $p_{a,u}$ with

$$|F_{a,0,R_u^a}(Z) - \mu_{p_{a,u}}(Z)| < \frac{d(m)}{\gamma(\gamma(R))} + \frac{2}{\sqrt{\gamma(R)}} + \frac{2N_1 + 2m}{\gamma(R)} \tag{2.22}$$

for every $Z \in \mathcal{Z}_m$. If we set

$$a_Z := b_Z := \frac{d(m)}{\gamma(\gamma(R))}, \qquad \eta := \frac{2}{m\sqrt{\gamma(R)}} + \frac{2N_1 + 2m}{m\gamma(R)},$$
$$q_v := \frac{L_v - L_{v-1}}{R_u^a} \qquad \text{for } v = 1, \ldots, k,$$

Lemma 1 implies that (2.22) remains also true in the second case. Finally, (2.22) is trivial by the choice of R if a is a critical point with $T^{k(a)+n(a)}a = T^{k(a)}a$. Therefore (2.22) holds for every critical point a.

Choose an $l_j > R$ such that D_{l_j-1} has at least two different successors. By the choice of R we see by induction that there exist $L_0 = 0 < L_1 < \ldots \ldots < L_k = l_j$ such that $L_v - L_{v-1} > \gamma(R)$ for $v = 2, \ldots, k$, D_{L_1-1} has a successor in $\mathcal{D}_{\gamma(R)}$, D_{L_v-1} has at least two different successors for $v = 1, \ldots, k$, and $D_{L_{v-1}+i}$ has only one successor in $\mathcal{D}$ for $v = 2, \ldots, k$ and $i = 0, 1, \ldots, L_v - L_{v-1} - 2$. Hence for every $v \in \{2, \ldots, k\}$ there exists a critical point a_v and a $u_v \in \mathbb{N}$ with $R_{u_v}^{a_v} = L_v - L_{v-1}$ and $D_{L_{v-1}+i} \subseteq C_i^{a_v}$

for $i = 0, 1, \ldots, L_v - L_{v-1} - 1$. By (2.1) this gives

$$|F_{x_j, L_{v-1}, L_v}(Z) - F_{a_v, 0, R_{u_v}^{a_v}}(Z)| < \frac{m}{\gamma(R)} \tag{2.23}$$

for every $Z \in \mathcal{Z}_m$ and every $v \in \{2, \ldots, k\}$. Moreover, Lemma 3 implies the existence of a periodic point P_j with

$$|F_{x_j, 0, L_1}(Z) - \mu_{P_j}(Z)| < \frac{2\sqrt{R} + m}{L_1} \tag{2.24}$$

for every $Z \in \mathcal{Z}_m$. For $Z \in \mathcal{Z}_m$ set

$$a_Z := \frac{d(m)}{\gamma(\gamma(R))} + \frac{2}{\sqrt{\gamma(R)}} + \frac{2N_1 + 3m}{\gamma(R)} \quad \text{and} \quad B_Z := 2\sqrt{R}.$$

Then by (2.22)–(2.24) and Lemma 2 we find out that there exists a periodic point p_j with

$$|F_{x_j, 0, l_j}(Z) - \mu_{p_j}(Z)| < \frac{d(m)}{\gamma(\gamma(R))} + \frac{2}{\sqrt{\gamma(R)}} + \frac{2N_1 + 3m}{\gamma(R)} + \frac{2}{\sqrt{R}} + \frac{2m}{R}$$

for every $Z \in \mathcal{Z}_m$. Therefore (2.20) implies (2.19), completing the proof of the claim.

Using (2.19) and applying Lemma 1 with $a_Z := b_Z := \delta$ and $\eta := \delta/m$ we obtain the existence of a periodic point $p \in [0, 1]$ with

$$\max_{Z \in \mathcal{Z}_m} \Big| \sum_{j=1}^{r} q_j F_{x_j, 0, l_j}(Z) - \mu_p(Z) \Big| < 2\delta. \tag{2.25}$$

For every $Z \in \mathcal{Z}_m$ choose an $x_Z \in Z$, and for $t \in \{1, \ldots, K\}$ define $f_t(Z) := f_t(x_Z)$. Fix a $t \in \{1, \ldots, K\}$. Then

$$\begin{aligned}
&\Big| \sum_{j=1}^{r} q_j \int_{[0,1]} f_t \, d\mu_j - \int_{[0,1]} f_t \, d\mu_p \Big| \\
&\quad \le \sum_{j=1}^{r} q_j \Big| \int_{[0,1]} f_t \, d\mu_j - \frac{1}{l_j} \sum_{s=0}^{l_j - 1} f_t(T^s x_j) \Big| \\
&\qquad + \sum_{j=1}^{r} q_j \frac{1}{l_j} \sum_{s=0}^{l_j - 1} \sum_{Z \in \mathcal{Z}_m} |f_t 1_Z(T^s x_j) - f_t(Z) 1_Z(T^s x_j)| \\
&\qquad + \sum_{Z \in \mathcal{Z}_m} |f_t(Z)| \Big| \sum_{j=1}^{r} q_j F_{x_j, 0, l_j}(Z) - \mu_p(Z) \Big| + \sum_{Z \in \mathcal{Z}_m} \int_Z |f_t(Z) - f_t| \, d\mu_p.
\end{aligned}$$

By (2.18) the first sum on the right hand side is smaller than $\varepsilon/5$ and by

(2.16) the fourth sum is smaller than $\varepsilon/5$ as well. Again using (2.16) we get

$$\sum_{Z\in\mathcal{Z}_m} |f_t 1_Z(T^s x_j) - f_t(Z)1_Z(T^s x_j)| < \frac{\varepsilon}{5},$$

and therefore also the second sum is smaller than $\varepsilon/5$. We deduce by (2.14) and (2.25) that

$$\sum_{Z\in\mathcal{Z}_m} |f_t(Z)| \Big| \sum_{j=1}^{r} q_j F_{x_j,0,l_j}(Z) - \mu_p(Z)\Big| \le 2c\delta \operatorname{card} \mathcal{Z}_m.$$

Hence (2.15) and (2.17) give $|\int_{[0,1]} f_t\,d\mu_p - \int_{[0,1]} f_t\,d\mu| < \varepsilon$, and therefore (2.13) implies $\mu_p \in U$. ■

3. Transformations with two monotonic pieces. In this section we investigate transformations with two monotonic pieces. We show that the periodic orbit measures are dense in $M([0,1],T)$ if $T:[0,1]\to[0,1]$ is a topologically transitive transformation with two monotonic pieces which has positive topological entropy. By Theorem 1 it suffices to prove that T satisfies the assumptions of that theorem.

Let $T:[0,1]\to[0,1]$ be a transformation with two monotonic pieces. Observe that T has exactly two critical points and every $D\in\mathcal{D}$ has at most two successors, where $(\mathcal{D},\to)$ denotes the Markov diagram of T. Now we describe some more details of the Markov diagram of T. The proof of these details is by easy calculations.

Suppose that x is a critical point, that $u\in\mathbb{N}$ with $u>1$ and that $R^x_{u+1} \ne \infty$. Let b be the critical point with $C^x_{R^x_{u-1}+k} \subseteq C^b_k$ for $k\in\{0,1,\dots,r^x_u-1\}$ and $b\ne T^{R^x_{u-1}}x$, and assume $r^x_u > R^b_1$. Then there exists a $w\in\mathbb{N}$ with $w>1$ and $R^b_w = r^x_u$. Assume that y is the critical point with $C^b_{R^b_{w-1}+k} \subseteq C^y_k$ for $k\in\{0,1,\dots,r^b_w-1\}$ and $y\ne T^{R^b_{w-1}}b$. Therefore there exists a $v\in\mathbb{N}$ with $R^y_v = r^b_w$, and we have $C^x_{R^x_u} \subseteq C^y_{R^y_v}$ and $R^y_v < R^x_u$. Hence $R^y_{v+1}\ne\infty$ and there exists a critical point a with $a\notin\{T^{R^x_u}x, T^{R^y_v}y\}$, $C^x_{R^x_{u+1}-1}\to C^a_{r^x_{u+1}}$ and $C^y_{R^y_{v+1}-1}\to C^a_{r^y_{v+1}}$. Furthermore, $(r^y_{v+n})_{n\ge1} \le (r^x_{u+n})_{n\ge1}$ in the lexicographical order, where we set $r^z_k := \infty$ if $R^z_l := \infty$ for an $l\le k$.

THEOREM 2. *Let $T:[0,1]\to[0,1]$ be a transformation with two monotonic pieces which is topologically transitive and satisfies $h_{\text{top}}(T)>0$. Then the periodic orbit measures are dense in $M([0,1],T)$.*

Proof. As T is topologically transitive and $h_{\text{top}}(T)>0$ there exists a maximal irreducible $\mathcal{D}'\subseteq\mathcal{D}$ such that every $x\in[0,1]$ is represented by an infinite path in $\mathcal{D}'$. Now choose an $n_0\in\mathbb{N}$ with $n_0>R^x_2$ and $C^x_{n_0}\in\mathcal{D}'$ for every critical point x. Let a be an essential critical point, and let $j\in\mathbb{N}$

with $r_j^a > n_0$. Then $j > 2$. For $n \in \mathbb{N}_0$ set $A_n := C_n^a$, $B_n := C_n^b$, $R_n := R_n^a$, $S_n := R_n^b$, and for $n \in \mathbb{N}$ set $r_n := r_n^a$ and $s_n := r_n^b$, where b is the critical point with $b \neq a$. By Theorem 1 it suffices to show that there exists an $l \in \{0, 1, \ldots, j-1\}$ and a periodic point p such that

$$|F_{a,R_l,R_j}(Z) - \mu_p(Z)| < \frac{m}{R_j - R_l} \tag{3.1}$$

for every $m \in \mathbb{N}$ and every $Z \in \mathcal{Z}_m$. In order to prove (3.1) we consider different cases.

Case 1: There exists a $u < j$ with $A_{R_j-1} \to A_{R_u}$. Consider the periodic path

$$\alpha := A_{R_u} \to A_{R_u+1} \to \ldots \to A_{R_j-1},$$

and let p be the periodic point represented by α. Since $T^k p \in A_{R_u+k}$ for $k \in \{0, 1, \ldots, R_j - R_u - 1\}$ we obtain (3.1) with $l := u$.

From now on we assume that Case 1 does not hold. Therefore there exists a $u \in \mathbb{N}$ with $u > 2$ and

$$A_{R_j-1} \to B_{S_u}. \tag{3.2}$$

Case 2: There is a $v \le j$ with $A_{R_{j+1}-1} \to A_{R_v}$. In this case consider the periodic path

$$\alpha := A_{R_j} \to A_{R_j+1} \to \ldots \to A_{R_{j+1}-1} \to A_{R_v} \to A_{R_v+1} \to \ldots \to A_{R_j-1},$$

and let p be the periodic point represented by α. Since $A_{R_j+k} \subseteq A_k$ for $k \in \{0, 1, \ldots, r_{j+1}-1\}$ and $r_{j+1} = R_v$ we get $T^k p \in A_k$ for $k \in \{0, 1, \ldots, R_j - 1\}$. Hence (3.1) holds with $l := 0$.

In the rest of this proof we assume that Case 2 does not hold. Therefore $A_{R_j} \subseteq B_0$ and there exists a $v_1 \in \mathbb{N}$ with

$$A_{R_{j+1}-1} \to B_{S_{v_1}}. \tag{3.3}$$

Using (3.2) we obtain $B_{S_u} \subseteq A_0$ and hence there exists a $v_2 \in \mathbb{N}$ with

$$B_{S_{u+1}-1} \to A_{R_{v_2}}. \tag{3.4}$$

In order to continue the proof we need the following lemma.

Lemma 4. *Assume that* (3.2)–(3.4) *hold. If*

$$(r_j, r_j, r_j, \ldots) \le (r_{j+1}, r_{j+2}, r_{j+3}, \ldots) \quad \textit{and}$$
$$(R_j, R_j, R_j, \ldots) \le (s_{u+1}, s_{u+2}, s_{u+3}, \ldots)$$

in the lexicographical order, then the set $\mathcal{C} := \{A_n, B_k : n \ge R_j,\ k \ge S_u\}$ *has no successors in* $\mathcal{D} \setminus \mathcal{C}$.

Proof. Set

$$\varrho_n := (r_{n+1}, r_{n+2}, r_{n+3}, \ldots), \quad \sigma_k := (s_{k+1}, s_{k+2}, s_{k+3}, \ldots),$$
$$\varrho' := (r_j, r_j, r_j, \ldots) \quad \text{and} \quad \sigma' := (R_j, R_j, R_j, \ldots).$$

To prove the result it suffices to show that $A_{R_n} \subseteq B_0$ and $\varrho' \leq \varrho_n$ in the lexicographical order for all $n \geq j$, and $B_{S_k} \subseteq A_0$ and $\sigma' \leq \sigma_k$ in the lexicographical order for all $k \geq u$. We prove this by induction. Assume that $q \in \mathbb{N}$, $A_{R_n} \subseteq B_0$ and $\varrho' \leq \varrho_n$ in the lexicographical order for all $n \geq j$ with $R_n < q$, and $B_{S_k} \subseteq A_0$ and $\sigma' \leq \sigma_k$ in the lexicographical order for all $k \geq u$ with $S_k < q$. For $q = R_j + 1$ this is an easy consequence of our assumption. Suppose therefore $q > R_j + 1$. First assume $R_n = q$. If $r_n = r_j$, then $A_{R_n - 1} \to B_{S_u}$ and $A_{R_n} \subseteq B_0$. As $\varrho' \leq \varrho_{n-1}$ we get $\varrho' \leq \varrho_n$ in the lexicographical order. Otherwise $r_n > r_j$, and hence there exists a $k > u$ with $S_k < q$ and $A_{R_n - 1} \to B_{S_k}$. As $S_k < q$ we get $B_{S_k} \subseteq A_0$ and $\varrho_w \leq \varrho_n$ in the lexicographical order for a $w \in \{j, j+1, \ldots, n-1\}$. Therefore $A_{R_n} \subseteq B_0$. Since $\varrho' \leq \varrho_w$ we get $\varrho' \leq \varrho_n$ in the lexicographical order. An analogous proof shows $B_{S_k} \subseteq A_0$ and $\sigma' \leq \sigma_k$ in the lexicographical order if $S_k = q$. ∎

We continue with the proof of Theorem 2. As $R_j > n_0$, the set $\mathcal{C}$ in Lemma 4 does not contain A_{n_0}. But since $A_{n_0} \in \mathcal{D}'$ and $\mathcal{D}'$ is irreducible, the assumption of Lemma 4 cannot hold. Hence $(r_{j+1}, r_{j+2}, r_{j+3}, \ldots) < (r_j, r_j, r_j, \ldots)$ in the lexicographical order or $(s_{u+1}, s_{u+2}, s_{u+3}, \ldots) < (R_j, R_j, R_j, \ldots)$ in the lexicographical order.

Case 3: There exists an $n \geq 1$ with $s_{u+q} = R_j$ for $q \in \{1, \ldots, n-1\}$ and $s_{u+n} < R_j$. Consider the periodic path

$$\begin{aligned}\alpha := B_{S_{u+n-1}} &\to B_{S_{u+n-1}+1} \to \ldots \to B_{S_{u+n}-1} \to A_{s_{u+n}} \\ &\to A_{s_{u+n}+1} \to \ldots \to A_{R_j} \to B_{S_u} \to B_{S_u+1} \to \ldots \to B_{S_{u+n-1}-1},\end{aligned}$$

and let p be the periodic point represented by α. Then $B_{S_{u+n-1}+k} \subseteq A_k$ for $k \in \{0, 1, \ldots, s_{u+n} - 1\}$. Furthermore, $B_{S_{u+q-1}+k} \subseteq A_k$ and $s_{u+q} = R_j$ for $k \in \{0, 1, \ldots, R_j - 1\}$ and $q \in \{1, \ldots, n-1\}$. Therefore $T^{qR_j + k}p$ and A_k are contained in the same element of $\mathcal{Z}$ for $q \in \{0, 1, \ldots, n-1\}$ and $k \in \{0, 1, \ldots, R_j - 1\}$. Hence (3.1) holds with $l := 0$.

From now on we suppose that Case 3 does not hold. Therefore Lemma 4 implies that there exists an $n \geq 1$ with $r_{j+q} = r_j$ for $q \in \{1, \ldots, n-1\}$ and $r_{j+n} < r_j$.

Case 4: There exists a $t \in \mathbb{N}$ with $B_{S_u - 1} \to A_{R_t}$. Obviously, $t < j$. By (3.4) we get $A_{R_t} \subseteq B_0$ and $A_{R_{t+1}-1} \to B_{r_{t+1}}$. As $A_{R_{j+n-1}} \subseteq A_{R_t}$ we have $r_{t+1} \leq r_{j+n} < r_j$. Then

$$\alpha := A_{R_t} \to A_{R_t+1} \to \ldots \to A_{R_{t+1}-1} \to B_{r_{t+1}} \to B_{r_{t+1}+1} \to \ldots \to B_{S_u - 1}$$

is a periodic path. Let p be the periodic point represented by α. We have $A_{R_t+k} \subseteq B_k$ for $k \in \{0, 1, \ldots, r_{t+1} - 1\}$, $A_{R_{j-1}+k} \subseteq B_k$ for $k \in \{0, 1, \ldots, r_j - 1\}$ and $r_j = S_u$. Hence $T^k p$ and $A_{R_{j-1}+k}$ are contained in the same element of $\mathcal{Z}$ for $k \in \{0, 1, \ldots, r_j - 1\}$. Therefore (3.1) holds with $l := j - 1$.

Case 5: Finally, we suppose that Case 4 does not hold either. Then there exists a $t < u$ with $B_{S_u - 1} \to B_{S_t}$. Using (3.4) we get $B_{S_{t+1}-1} \to B_{s_{t+1}}$ and $s_{t+1} \le S_t$. In this case consider the periodic path

$$\alpha := B_{S_t} \to B_{S_t+1} \to \ldots \to B_{S_{t+1}-1} \to B_{s_{t+1}} \to B_{s_{t+1}-1} \to \ldots \to B_{S_u-1},$$

and let p be the periodic point represented by α. Since $B_{S_t+k} \subseteq B_k$ for $k \in \{0, 1, \ldots, s_{t+1} - 1\}$, $A_{R_{j-1}+k} \subseteq B_k$ for $k \in \{0, 1, \ldots, r_j - 1\}$ and $r_j = S_u$, we deduce that $T^k p$ and $A_{R_{j-1}+k}$ are contained in the same element of $\mathcal{Z}$ for $k \in \{0, 1, \ldots, r_j - 1\}$. Hence (3.1) holds with $l := j - 1$. ∎

References

[1] A. Blokh, *The 'spectral' decomposition for one-dimensional maps*, in: Dynam. Report. 4, C. K. R. T. Jones, V. Kirchgraber and H.-O. Walther (eds.), Springer, Berlin, 1995, 1–59.

[2] R. Bowen, *Periodic points and measures for axiom-A-diffeomorphisms*, Trans. Amer. Math. Soc. 154 (1971), 377–397.

[3] M. Denker, C. Grillenberger and K. Sigmund, *Ergodic Theory on Compact Spaces*, Lecture Notes in Math. 527, Springer, Berlin, 1976.

[4] F. Hofbauer, *Piecewise invertible dynamical systems*, Probab. Theory Related Fields 72 (1986), 359–386.

[5] —, *Generic properties of invariant measures for simple piecewise monotonic transformations*, Israel J. Math. 59 (1987), 64–80.

[6] —, *Hausdorff dimension and pressure for piecewise monotonic maps of the interval*, J. London Math. Soc. 47 (1993), 142–156.

[7] —, *Local dimension for piecewise monotonic maps on the interval*, Ergodic Theory Dynam. Systems 15 (1995), 1119–1142.

[8] F. Hofbauer and M. Urbański, *Fractal properties of invariant subsets for piecewise monotonic maps of the interval*, Trans. Amer. Math. Soc. 343 (1994), 659–673.

[9] P. Raith, *Continuity of the Hausdorff dimension for piecewise monotonic maps*, Israel J. Math. 80 (1992), 97–133.

[10] P. Walters, *An Introduction to Ergodic Theory*, Grad. Texts in Math. 79, Springer, New York, 1982.

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