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Coherent and strong expansions of spaces coincide

by

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Abstract. In the existing literature there are several constructions of the strong shape category of topological spaces. In the one due to Yu. T. Lisitsa and S. Mardešić [LM1-3] an essential role is played by coherent polyhedral (ANR) expansions of spaces. Such expansions always exist, because every space admits a polyhedral resolution, resolutions are strong expansions and strong expansions are always coherent. The purpose of this paper is to prove that conversely, every coherent polyhedral (ANR) expansion is a strong expansion. This result is obtained by showing that a mapping of a space into a system, which is coherently dominated by a strong expansion, is itself a strong expansion.

1. Introduction. Let $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, \mathrm{M})$ be inverse systems of topological spaces, indexed by cofinite directed ordered sets (every element has finitely many predecessors). A *mapping of systems*, shorter, a *mapping*, $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ consists of an increasing function $f : \mathrm{M} \to \Lambda$ and of mappings $f_\mu : X_{f(\mu)} \to Y_\mu$, $\mu \in \mathrm{M}$, such that

$$f_\mu p_{f(\mu)f(\mu')} = q_{\mu\mu'} f_{\mu'}, \qquad \mu \le \mu'. \tag{1}$$

The composition $\boldsymbol{h} = \boldsymbol{g}\boldsymbol{f}$ of mappings $\boldsymbol{f} = (f, f_\mu) : \boldsymbol{X} \to \boldsymbol{Y}$ and $\boldsymbol{g} = (g, g_\nu) : \boldsymbol{Y} \to \boldsymbol{Z} = (Z_\nu, r_{\nu\nu'}, \mathrm{N})$ is given by the function $h = fg$ and the mappings $h_\nu = g_\nu f_{g(\nu)}$. If $\Lambda = \mathrm{M}$ and the indexing function $f = \mathrm{id}$, we speak of a *level mapping*. In particular, the identity mapping $\boldsymbol{1} : \boldsymbol{X} \to \boldsymbol{X}$, given by the identity function $\mathrm{id} : \Lambda \to \Lambda$ and the identity mappings $f_\lambda = \mathrm{id} : X_\lambda \to X_\lambda$, is a level mapping. Inverse systems as objects and mappings as morphisms form a category, here denoted by inv-Top.

If $\boldsymbol{f}' = (f', f'_\mu) : \boldsymbol{X} \to \boldsymbol{Y}$ is a mapping and $f \ge f'$ is an increasing function, then f and the mappings $f_\mu = f'_\mu p_{f'(\mu)f(\mu)}$ form a mapping $\boldsymbol{f}$ to which we refer as the *shift* of $\boldsymbol{f}'$ by f. Two mappings $\boldsymbol{f}', \boldsymbol{f}'' : \boldsymbol{X} \to \boldsymbol{Y}$ are said to be *congruent*, $\boldsymbol{f}' \equiv \boldsymbol{f}''$, provided they have a common shift $\boldsymbol{f}$. Inverse

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systems and congruence classes of mappings form a category, denoted by pro-Top.

A rather simple (but not very satisfactory) homotopy category of inverse systems π(pro-Top) can be described as follows. Its objects are cofinite directed ordered sets. Its morphisms are homotopy classes $[\boldsymbol{f}]$ of mappings $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$, where two mappings $\boldsymbol{f}' = (f', f'_\mu)$ and $\boldsymbol{f}'' = (f'', f''_\mu)$ are considered homotopic, $\boldsymbol{f}' \simeq \boldsymbol{f}''$, if there exists a mapping of systems $\boldsymbol{F} = (F, F_\mu) : \boldsymbol{X} \times I \to \boldsymbol{Y}$ such that $F \geq f, f'$ and

$$F_\mu(x,0) = f_\mu p_{f(\mu)F(\mu)}(x), \qquad F_\mu(x,1) = f'_\mu p_{f'(\mu)F(\mu)}(x). \tag{2}$$

Here $\boldsymbol{X} \times I = (X_\lambda \times I, p_{\lambda\lambda'} \times 1, \Lambda)$. Composition of morphisms is well defined by the formula $[\boldsymbol{g}][\boldsymbol{f}] = [\boldsymbol{g}\boldsymbol{f}]$. Note that congruent mappings $\boldsymbol{f}' \equiv \boldsymbol{f}''$ always determine the same morphism of π(pro-Top).

In 1983 Yu. T. Lisitsa and S. Mardešić [LM1-3] defined the more subtle *coherent homotopy category* CH(Top) (then denoted by $\boldsymbol{CPHTop}$). Its objects are again cofinite directed ordered sets. Its morphisms are homotopy classes of *coherent mappings* $\boldsymbol{f} = (f, f_{\underline{\mu}}) : \boldsymbol{X} \to \boldsymbol{Y}$, where $f : \mathrm{M} \to \Lambda$ is an increasing function, $\underline{\mu} = (\mu_0, \ldots, \mu_n)$, $\mu_0 \leq \ldots \leq \mu_n$, are increasing multiindices of length $n \geq 0$ and $f_{\underline{\mu}} : X_{f(\mu_n)} \times \Delta^n \to Y_{\mu_0}$ are mappings satisfying the natural coherence conditions:

$$f_{\underline{\mu}}(x, d_j t) = \begin{cases} q_{\mu_0\mu_1} f_{d_0\underline{\mu}}(x,t), & j = 0, \\ f_{d_j\underline{\mu}}(x,t), & 0 < j < n, \\ f_{d_n\underline{\mu}}(p_{f(\mu_{n-1})f(\mu_n)}(x), t), & j = n, \end{cases} \tag{3}$$

$$f_{\underline{\mu}}(x, s_j t) = f_{s_j\underline{\mu}}(x,t), \qquad 0 \leq j \leq n. \tag{4}$$

Here $d_j : \Delta^{n-1} \to \Delta^n$ and $s_j : \Delta^{n+1} \to \Delta^n$, $0 \leq j \leq n$, denote the boundary and degeneracy operators between standard simplices. The corresponding operators on multiindices are defined by

$$d_j(\mu_0, \ldots, \mu_n) = (\mu_0, \ldots, \mu_{j-1}, \mu_{j+1}, \mu_n), \tag{5}$$

$$s_j(\mu_0, \ldots, \mu_n) = (\mu_0, \ldots, \mu_j, \mu_j, \ldots, \mu_n). \tag{6}$$

A (coherent) *homotopy* connecting coherent mappings $\boldsymbol{f}' = (f', f'_{\underline{\mu}}) : \boldsymbol{X} \to \boldsymbol{Y}$ and $\boldsymbol{f}'' = (f'', f''_{\underline{\mu}}) : \boldsymbol{X} \to \boldsymbol{Y}$ is a coherent mapping $\boldsymbol{F} = (F, F_{\underline{\mu}}) : \boldsymbol{X} \times I \to \boldsymbol{Y}$ such that $F \geq f', f''$ and

$$F_{\underline{\mu}}(x,0,t) = f'_{\underline{\mu}}(p_{f'(\mu_n)F(\mu_n)}(x), t), \qquad F_{\underline{\mu}}(x,1,t) = f''_{\underline{\mu}}(p_{f''(\mu_n)F(\mu_n)}(x), t).$$

Composition of coherent mappings is defined by a geometrically transparent explicit formula. Composition of their homotopy classes is defined by composing representatives (all details are given in [LM3]). Note that every mapping $\boldsymbol{f} = (f, f_\mu) : \boldsymbol{X} \to \boldsymbol{Y}$ determines a coherent mapping $\boldsymbol{f}' = (f', f'_{\underline{\mu}})$:

$\boldsymbol{X} \to \boldsymbol{Y}$, defined by putting $f' = f$ and $f'_{\underline{\mu}}(x,t) = f_{\mu_0} p_{f(\mu_0) f(\mu_n)}(x)$. We denote $\boldsymbol{f}'$ by $C(\boldsymbol{f})$ and refer to C as the *coherence operator*. It induces a functor $C : \text{pro-}\mathsf{Top} \to \mathrm{CH}(\mathsf{Top})$.

A mapping $\boldsymbol{p} = (p_\lambda) : X \to \boldsymbol{X}$ of a space into a system is said to be a *coherent expansion* of X provided it has the property that for every $\boldsymbol{HPol}$-system $\boldsymbol{Y}$, i.e., a system consisting of spaces having the homotopy type of polyhedra (or equivalently, of ANR's) and every coherent mapping $\boldsymbol{h} : X \to \boldsymbol{Y}$, there exists a coherent mapping $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ such that

$$\boldsymbol{h} \simeq \boldsymbol{f} C(\boldsymbol{p}). \tag{8}$$

Moreover, $\boldsymbol{f}$ is unique up to coherent homotopy. It was proved in [LM3] that every resolution $\boldsymbol{p} : X \to \boldsymbol{X}$ in the sense of [M1], [MS] is a coherent expansion of X.

On the other hand, a mapping $\boldsymbol{p} : X \to \boldsymbol{X}$ is said to be a *strong expansion* provided for every polyhedron P, the following two conditions (S1), (S2) are satisfied:

(S1) If $\phi : X \to P$ is a mapping, then there exist a $\lambda \in \Lambda$ and a mapping $\psi : X_\lambda \to P$ such that the mappings ϕ and ψp_λ are homotopic,

$$\phi \simeq \psi p_\lambda. \tag{9}$$

(S2) If $\lambda \in \Lambda$, $\psi_0, \psi_1 : X_\lambda \to P$ are mappings and $F : X \times I \to P$ is a homotopy which connects $\psi_0 p_\lambda$ and $\psi_1 p_\lambda$, then there exist a $\lambda' \geq \lambda$ and a homotopy $H : X_{\lambda'} \times I \to P$ which connects $\psi_0 p_{\lambda\lambda'}$ and $\psi_1 p_{\lambda\lambda'}$. Moreover, the homotopies $F, H(p_{\lambda'} \times 1) : X \times I \to P$ are connected by a homotopy $K : (X \times I) \times I \to P$, fixed on $X \times \partial I$, i.e.,

$$F \simeq H(p_{\lambda'} \times 1) \operatorname{rel} (X \times \partial I). \tag{10}$$

In the above definition one can replace polyhedra by spaces from the class $\boldsymbol{HPol}$ (see [M4]).

It was proved in [M3] that every resolution is a strong expansion, and in [M2] that every strong expansion is a coherent expansion. These two assertions together give a new proof of the fact that resolutions are coherent expansions. The first result of the present paper is the following converse of the second of the two assertions.

THEOREM 1. *If $\boldsymbol{p} : X \to \boldsymbol{X}$ is a coherent expansion and $\boldsymbol{X}$ consists of spaces from the class $\boldsymbol{HPol}$, i.e., spaces having the homotopy type of polyhedra, then $\boldsymbol{p}$ is a strong expansion.*

REMARK 1. B. Günther in a remark on p. 149 of [G] makes the stronger assertion that coherent expansions are always strong expansions. However, in his paper there is no indication of proof.

Consider two mappings $\boldsymbol{p} : X \to \boldsymbol{X}$, $\boldsymbol{q} : X \to \boldsymbol{Y}$ of the same space X. We will say that $\boldsymbol{p}$ is *coherently dominated* by $\boldsymbol{q}$ provided there exist coherent mappings $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ and $\boldsymbol{g} : \boldsymbol{Y} \to \boldsymbol{X}$ such that

$$\boldsymbol{f}C(\boldsymbol{p}) \simeq C(\boldsymbol{q}), \tag{11}$$

$$\boldsymbol{g}\boldsymbol{f} \simeq C(\mathbf{1}). \tag{12}$$

We will derive Theorem 1 from the next theorem, which is the main result of the present paper.

THEOREM 2. *If a mapping $\boldsymbol{p} : X \to \boldsymbol{X}$ is coherently dominated by a strong expansion $\boldsymbol{q} : X \to \boldsymbol{Y}$, then $\boldsymbol{p}$ itself is a strong expansion.*

Proof of Theorem 1. Let $\boldsymbol{p} : X \to \boldsymbol{X}$ be a coherent expansion, where $\boldsymbol{X}$ consists of spaces from the class $\boldsymbol{HPol}$. Choose a strong expansion $\boldsymbol{q} : X \to \boldsymbol{Y}$ such that $\boldsymbol{Y}$ also consists of spaces from $\boldsymbol{HPol}$. Since $\boldsymbol{p}$ is a coherent expansion, there exists a coherent mapping $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ such that (11) holds. Now use the fact that $\boldsymbol{q}$ is also a coherent expansion, because it is a strong expansion. Since $\boldsymbol{X}$ is an $\boldsymbol{HPol}$-system, we conclude that there exists a coherent mapping $\boldsymbol{g} : \boldsymbol{Y} \to \boldsymbol{X}$ such that

$$\boldsymbol{g}C(\boldsymbol{q}) \simeq C(\boldsymbol{p}), \tag{13}$$

and thus,

$$\boldsymbol{g}\boldsymbol{f}C(\boldsymbol{p}) \simeq C(\boldsymbol{p}). \tag{14}$$

Now the uniqueness property of the coherent expansion $\boldsymbol{p}$ implies (12). Consequently, $\boldsymbol{p}$ is coherently dominated by $\boldsymbol{q}$. Since $\boldsymbol{q}$ is a strong expansion, Theorem 2 yields the desired conclusion that also $\boldsymbol{p}$ is a strong expansion. ■

2. Some lemmas on π(pro-Top)

LEMMA 1. *Let $\boldsymbol{q}, \boldsymbol{q}' : X \to \boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, \mathrm{M})$ be two mappings which belong to the same class $[\boldsymbol{q}] = [\boldsymbol{q}'] \in \pi(\text{pro-}\mathsf{Top})$. If $\boldsymbol{q}$ is a strong expansion, then so is $\boldsymbol{q}'$.*

Proof. By assumption there exists a mapping $\boldsymbol{K} = (K_\mu) : X \times I \to \boldsymbol{Y}$ such that, for every $\mu \in \mathrm{M}$, K_μ connects q'_μ to q_μ, i.e.,

$$q'_\mu \simeq_{K_\mu} q_\mu. \tag{15}$$

Moreover,

$$q_{\mu\mu'} K_{\mu'} = K_\mu, \qquad \mu \le \mu'. \tag{16}$$

Now assume that $P \in \boldsymbol{HPol}$ and $\phi : X \to P$ is a mapping. By (S1) for $\boldsymbol{q}$, there exist a $\mu \in \mathrm{M}$ and a mapping $\psi : Y_\mu \to P$ such that $\phi \simeq \psi q_\mu$. By (15), $q_\mu \simeq q'_\mu$ and thus, $\phi \simeq \psi q'_\mu$. However, this is the desired condition (S1) for $\boldsymbol{q}'$.

Now assume that $\mu \in \mathrm{M}$, $\psi_0, \psi_1 : Y_\mu \to P$ are mappings and $F' : X \times I \to P$ is a homotopy such that

$$\psi_0 q'_\mu \simeq_{F'} \psi_1 q'_\mu. \tag{17}$$

Let $F : X \times I \to P$ be the homotopy obtained by juxtaposition of three homotopies according to the following formula:

$$F = \psi_0 K^-_\mu * F' * \psi_1 K_\mu, \tag{18}$$

where K^- denotes the opposite of the homotopy K, i.e., $K^-(x,t) = K(x, 1-t)$. The homotopy F is well defined and has the property that

$$\psi_0 q_\mu \simeq_F \psi_1 q_\mu. \tag{19}$$

Therefore, by condition (S2) for $\boldsymbol{q}$, there exist a $\mu' \geq \mu$ and a homotopy $H : Y_{\mu'} \times I \to P$ such that

$$\psi_0 q_{\mu\mu'} \simeq_H \psi_1 q_{\mu\mu'}. \tag{20}$$

Moreover,

$$F \simeq H(q_{\mu'} \times 1) \operatorname{rel}(X \times \partial I). \tag{21}$$

We shall prove that

$$F' \simeq H(q'_{\mu'} \times 1) \operatorname{rel}(X \times \partial I). \tag{22}$$

Clearly, equations (20) and (22) will establish the desired condition (S2) for $\boldsymbol{q}'$.

In order to prove (21), we define a homotopy $U : X \times I \times I \to P$ by putting

$$U(x,s,t) = H(K_{\mu'}(x,t), s). \tag{23}$$

Note that, by (15),

$$U(x,s,0) = H(q'_{\mu'}(x), s), \quad U(x,s,1) = H(q_{\mu'}(x), s). \tag{24}$$

Moreover, by (20) and (16),

$$U(x,0,t) = \psi_0 q_{\mu\mu'} K_{\mu'}(x,t) = \psi_0 K_\mu(x,t), \tag{25}$$

$$U(x,1,t) = \psi_1 q_{\mu\mu'} K_{\mu'}(x,t) = \psi_1 K_\mu(x,t). \tag{26}$$

Let $V : X \times I \times I \to P$ be a homotopy which realizes (21). Using U and V, we will now define a homotopy $W : X \times I \times I \to P$ which realizes (22). Divide the square $I \times I$ into two rectangles as shown in Fig. 1. Since V is a homotopy $\operatorname{rel}(X \times \partial I)$ which connects F to $H(q_{\mu'} \times 1)$ and F is of the form (18), we can use V to fill up the lower rectangle as indicated in the figure. Then we use U^- to fill up the upper rectangle (observe the orientation of the upper rectangle in the figure).

In this way we obtain a homotopy $W' : X \times I \times I \to P$ such that

$$W'|X \times I \times 0 = F', \quad W'|X \times I \times 1 = H(q'_{\mu'} \times 1). \tag{27}$$

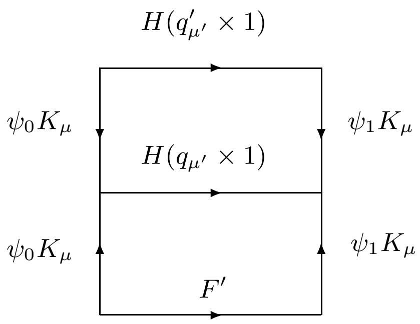

Fig. 1

Moreover,

(28) $$W'|X \times 0 \times I = \psi_0(K_\mu * K_\mu^-), \qquad W'|X \times 1 \times I = \psi_1(K_\mu * K_\mu^-).$$

Therefore, one can identify the left sides of the two rectangles and also their right sides. One obtains a mapping $W : X \times D^2 \to P$, where D^2 is a disc. If S^- and S^+ denote the lower and upper halves of the boundary ∂D^2, then $W|X \times S^-$ coincides with F', while $W|X \times S^+$ coincides with $H(q'_{\mu'} \times 1)$. Consequently, W can be viewed as the desired homotopy rel $(X \times \partial I)$. ∎

Lemma 2. *Let* $\boldsymbol{p} : X \to \boldsymbol{X}$, $\boldsymbol{q} : X \to \boldsymbol{Y}$ *and* $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ *be mappings such that*

(29) $$[\boldsymbol{f}][\boldsymbol{p}] = [\boldsymbol{q}]$$

in $\pi(\text{pro-}\mathsf{Top})$. *Moreover, let* $g : \Lambda \to \mathrm{M}$ *be an increasing function and let* $g_\lambda : Y_{g(\lambda)} \to X_\lambda$ *be mappings having the property that every* $\lambda \in \Lambda$ *admits a* $\lambda^* \geq \lambda, fg(\lambda)$ *such that*

(30) $$p_{\lambda\lambda^*} \simeq g_\lambda f_{g(\lambda)} p_{fg(\lambda)\lambda^*}.$$

Then the assumption that $\boldsymbol{q}$ *is a strong expansion implies that also* $\boldsymbol{p}$ *is a strong expansion.*

Proof. It suffices to prove that the assertion holds when (29) is replaced by the stronger assumption

(31) $$\boldsymbol{f}\boldsymbol{p} = \boldsymbol{q}.$$

Indeed, if $\boldsymbol{q}$ satisfies (29), then $\boldsymbol{q}' = \boldsymbol{f}\boldsymbol{p}$ satisfies $[\boldsymbol{q}] = [\boldsymbol{q}'] \in \pi(\text{pro-}\mathsf{Top})$. Therefore, by Lemma 1, $\boldsymbol{q}'$ is also a strong expansion. However, $\boldsymbol{q}'$ satisfies (31). Hence, the weaker version of Lemma 2 implies that $\boldsymbol{p}$ is a strong expansion.

We now prove the assertion of Lemma 2 assuming (31). For a mapping $\phi : X \to P \in \boldsymbol{HPol}$, property (S1) for $\boldsymbol{q}$ yields a $\mu \in \mathrm{M}$ and a mapping $\psi' : Y_\mu \to P$ such that $\psi' q_\mu \simeq \phi$. However, by (31), $q_\mu = f_\mu p_{f(\mu)}$ and thus,

$\lambda = f(\mu)$ and the mapping $\psi = \psi' f_\mu : X_\lambda \to P$ satisfy $\psi p_\lambda \simeq \phi$, which is the desired property (S1) for $\boldsymbol{p}$.

To establish (S2), let $\psi_0, \psi_1 : X_\lambda \to P$, $\lambda \in \Lambda$, be mappings and let $F : X \times I \to P$ be a homotopy such that

$$\psi_0 p_\lambda \simeq_F \psi_1 p_\lambda. \tag{32}$$

Choose a $\lambda^* \geq \lambda, fg(\lambda)$ and a homotopy $K_\lambda : X_{\lambda^*} \times I \to P$ which realizes (30). Since $q_{g(\lambda)} = f_{g(\lambda)} p_{fg(\lambda)}$, one sees that $\psi_0 K_\lambda^-(p_{\lambda^*} \times 1)$ is a homotopy which connects $\psi_0 g_\lambda q_{g(\lambda)}$ to $\psi_0 p_\lambda$. Similarly, $\psi_1 K_\lambda(p_{\lambda^*} \times 1)$ is a homotopy which connects $\psi_1 p_\lambda$ to $\psi_1 g_\lambda q_{g(\lambda)}$. Therefore,

$$F' = \psi_0 K_\lambda^-(p_{\lambda^*} \times 1) * F * \psi_1 K_\lambda(p_{\lambda^*} \times 1) \tag{33}$$

is a well-defined homotopy $F' : X \times I \to P$ which connects $\psi_0 g_\lambda q_{g(\lambda)}$ to $\psi_1 g_\lambda q_{g(\lambda)}$. Consequently, $\psi_0' = \psi_0 g_\lambda$ and $\psi_1' = \psi_1 g_\lambda$ are mappings $Y_{g(\lambda)} \to P$ such that

$$\psi_0' q_{g(\lambda)} \simeq_{F'} \psi_1' q_{g(\lambda)}. \tag{34}$$

Using property (S2) for $\boldsymbol{q}$, we conclude that there exist an index $\mu' \geq g(\lambda)$ and a homotopy $H' : Y_{\mu'} \times I \to P$ such that

$$\psi_0' q_{g(\lambda)\mu'} \simeq_{H'} \psi_1' q_{g(\lambda)\mu'}. \tag{35}$$

Moreover,

$$F' \simeq H'(q_{\mu'} \times 1) \operatorname{rel} (X \times \partial I). \tag{36}$$

Now choose a $\lambda' \geq \lambda^*, f(\mu')$. Note that $H'(f_{\mu'} p_{f(\mu')\lambda'} \times 1) : X_{\lambda'} \times I \to P$ is a homotopy which connects the mapping $\psi_0' q_{g(\lambda)\mu'} f_{\mu'} p_{f(\mu')\lambda'} = \psi_0' f_{g(\lambda)} p_{fg(\lambda)\lambda'}$ to $\psi_1' f_{g(\lambda)} p_{fg(\lambda)\lambda'}$. Since K_λ realizes (30), we conclude that

$$H = \psi_0 K_\lambda(p_{\lambda^*\lambda'} \times 1) * H'(f_{\mu'} p_{f(\mu')\lambda'} \times 1) * \psi_1 K_\lambda^-(p_{\lambda^*\lambda'} \times 1) \tag{37}$$

is a well-defined homotopy $H : X_{\lambda'} \times I \to P$ such that

$$\psi_0 p_{\lambda\lambda'} \simeq_H \psi_1 p_{\lambda\lambda'}. \tag{38}$$

Hence, to complete the proof of Lemma 2, it suffices to prove that

$$F \simeq H(p_{\lambda'} \times 1) \operatorname{rel} (X \times \partial I). \tag{39}$$

Choose a homotopy U which realizes (36). Clearly, it can be viewed as a mapping $U : X \times D^2 \to P$ such that $U|X \times S^- = F'$, while $U|X \times S^+ = H'(q_{\mu'} \times 1)$. By (33), $U|X \times S^-$ is the juxtaposition of three homotopies, defined on three consecutive arcs S_l^-, S_c^-, S_r^-. Now view the boundary ∂D^2 as divided into two arcs A^-, A^+. The arc $A^- = S_c^-$, while A^+ consists of the arcs S_l^-, S^+ and S_r^-, where S_l^- and S_r^- are taken with opposite orientations. Clearly, $U|X \times A^-$ can be viewed as F, while $U|X \times A^+$ can be viewed as the juxtaposition of homotopies which form $H(p_{\lambda'} \times 1)$ following (37). Consequently, U can be viewed as a homotopy realizing (39). ■

Remark 2. If $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{f}$ are as in Lemma 2 and $\boldsymbol{g} = (g, g_\lambda) : \boldsymbol{Y} \to \boldsymbol{X}$ is a mapping such that $[\boldsymbol{g}][\boldsymbol{f}] = [\mathbf{1}]$ in $\pi(\text{pro-}\mathsf{Top})$, then all the assumptions of Lemma 2 are satisfied. Therefore, if $\boldsymbol{q}$ is a strong expansion, so is $\boldsymbol{p}$.

Remark 3. If $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{f}$ are as in Lemma 2, $\boldsymbol{f} = (f_\lambda)$ is a level homotopy equivalence and $g_\lambda : Y_\lambda \to X_\lambda$ are homotopy inverses of f_λ, $\lambda \in \Lambda$, then all the assumptions of Lemma 2 are satisfied. Therefore, if $\boldsymbol{q}$ is a strong expansion, so is $\boldsymbol{p}$.

3. A lemma on level homotopy equivalences. A level mapping $\boldsymbol{f} = (f_\lambda) : \boldsymbol{X} \to \boldsymbol{Y}$ is called a *level homotopy equivalence* provided every mapping $f_\lambda : X_\lambda \to Y_\lambda$, $\lambda \in \Lambda$, has a homotopy inverse $g_\lambda : Y_\lambda \to X_\lambda$. The following lemma plays an important role in the proof of Theorem 2.

Lemma 3. *Let $\boldsymbol{p} : X \to \boldsymbol{X}$ be a strong expansion and let $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ be a level homotopy equivalence. Then $\boldsymbol{q} = \boldsymbol{f}\boldsymbol{p} : X \to \boldsymbol{Y}$ is also a strong expansion.*

Proof. If $\phi : X \to P \in \boldsymbol{HPol}$ is a mapping, then there exist a $\lambda \in \Lambda$ and a mapping $\psi' : X_\lambda \to P$ such that $\psi' p_\lambda \simeq \phi$. Since $1 \simeq g_\lambda f_\lambda$ and $q_\lambda = f_\lambda p_\lambda$, the mapping $\psi = \psi' g_\lambda : Y_\lambda \to P$ has the property that $\psi q_\lambda = \psi' g_\lambda f_\lambda p_\lambda \simeq \psi' p_\lambda \simeq \phi$, which establishes property (S1) for $\boldsymbol{q}$.

To prove property (S2) for $\boldsymbol{q}$, consider mappings $\psi_0, \psi_1 : Y_\lambda \to P$ and a homotopy $F : X \times I \to P$ such that

$$\psi_0 q_\lambda \simeq_F \psi_1 q_\lambda. \tag{40}$$

Note that (40) implies

$$\psi'_0 p_\lambda \simeq_F \psi'_1 p_\lambda, \tag{41}$$

where $\psi'_0 = \psi_0 f_\lambda$ and $\psi'_1 = \psi_1 f_\lambda$. Therefore, by assumption on $\boldsymbol{p}$, there exist a $\lambda' \ge \lambda$ and a homotopy $H' : X_{\lambda'} \times I \to P$ such that

$$\psi'_0 p_{\lambda\lambda'} \simeq_{H'} \psi'_1 p_{\lambda\lambda'}. \tag{42}$$

Moreover,

$$F \simeq H'(p_{\lambda'} \times 1) \operatorname{rel}(X \times \partial I). \tag{43}$$

To continue the proof we need a lemma due to R. M. Vogt [V]. It asserts that for a homotopy equivalence $f : X \to Y$ with a homotopy inverse $g : Y \to X$ and for a homotopy $K : X \times I \to X$ which connects id to gf, there exists a homotopy $L : Y \times I \to Y$ which connects id to fg and is such that $L(f \times 1) \simeq fK \operatorname{rel}(X \times \partial I)$. Applying this lemma, for every $\lambda \in \Lambda$, we define homotopies K_λ, L_λ such that

$$\mathrm{id} \simeq_{K_\lambda} g_\lambda f_\lambda, \quad \mathrm{id} \simeq_{L_\lambda} f_\lambda g_\lambda. \tag{44}$$

Moreover,

$$L_\lambda(f_\lambda \times 1) \simeq f_\lambda K_\lambda \ \mathrm{rel}\,(X \times \partial I). \tag{45}$$

Now note that the homotopy $H'(g_{\lambda'} \times 1) : Y_{\lambda'} \times I \to P$ connects $\psi_0 f_\lambda p_{\lambda\lambda'} g_{\lambda'} = \psi_0 q_{\lambda\lambda'} f_{\lambda'} g_{\lambda'}$ to $\psi_1 f_\lambda p_{\lambda\lambda'} g_{\lambda'} = \psi_1 q_{\lambda\lambda'} f_{\lambda'} g_{\lambda'}$. Therefore, the formula

$$H = \psi_0 q_{\lambda\lambda'} L_{\lambda'} * H'(g_{\lambda'} \times 1) * \psi_1 q_{\lambda\lambda'} L^-_{\lambda'} \tag{46}$$

yields a well-defined homotopy $H : Y_{\lambda'} \times I \to P$ which connects $\psi_0 q_{\lambda\lambda'}$ to $\psi_1 q_{\lambda\lambda'}$. To complete the proof of Lemma 3, it remains to prove that

$$F \simeq H(q_{\lambda'} \times 1) \ \mathrm{rel}\,(X \times \partial I). \tag{47}$$

We first define a homotopy $U : X \times I \times I \to P$ by putting

$$U(x, s, t) = H'(K_{\lambda'}(p_{\lambda'}(x), t), s). \tag{48}$$

Note that

$$U(x, s, 0) = H'(p_{\lambda'}(x), s), \tag{49}$$

$$U(x, s, 1) = H'(g_{\lambda'} q_{\lambda'}(x), s), \tag{50}$$

$$U(x, 0, t) = \psi_0 q_{\lambda\lambda'} f_{\lambda'} K_{\lambda'}(p_{\lambda'}(x), t), \tag{51}$$

$$U(x, 1, t) = \psi_1 q_{\lambda\lambda'} f_{\lambda'} K_{\lambda'}(p_{\lambda'}(x), t). \tag{52}$$

In order to define a homotopy $W : X \times I \times I \to P$ which realizes (47), we first define a mapping $W' : X \times D \to P$, where D is the polygon, described by Fig. 2. It consists of four rectangles, denoted by D_l^+, D_c^+, D_r^+, and D^-.

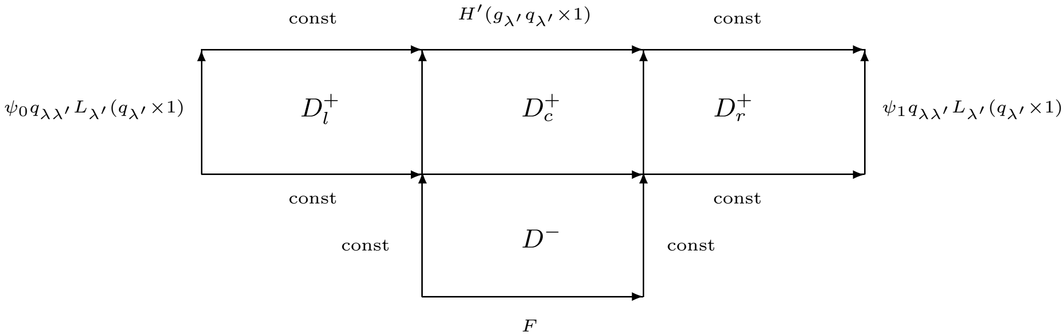

Fig. 2

By definition, $W'|X \times D_c^+$ is given by the homotopy U, while $W'|X \times D^-$ is given by a homotopy V which realizes (43). Note that (49) insures that the two definitions of W' on $X \times (D_c^+ \cap D^-)$ coincide. We define $W'|D_l^+$ using the homotopy $\psi_0 q_{\lambda\lambda'} T_{\lambda'}(p_{\lambda'} \times 1)$, where $T_\lambda : X_\lambda \times I \times I \to Y_\lambda$ is a homotopy which realizes (45). More precisely,

$$T_\lambda(x, 0, t) = L_\lambda(f_\lambda(x), t), \tag{53}$$

$$T_\lambda(x,1,t) = f_\lambda K_\lambda(x,t). \tag{54}$$

Moreover, $T_\lambda(x,s,0)$ and $T_\lambda(x,s,1)$ do not depend on s. Note that (51) and (54) insure that the two definitions of W' on $X \times (D_l^+ \cap D_c^+)$ coincide.

Similarly, we define $W'|D_r^+$, using the homotopy $\psi_1 q_{\lambda\lambda'} T_{\lambda'}^-(p_{\lambda'} \times 1)$. Note that on each of the horizontal sides of the rectangles D_l^+ and D_r^+ and on the vertical sides of D_-, W' assumes constant values. Therefore, by collapsing each of these sides to a point, one obtains a mapping $W : X \times D^2 \to P$. Thereby, $W|S^-$ coincides with F, while $W|S^+$ coincides with the juxtaposition of the following three homotopies: $\psi_0 q_{\lambda\lambda'} L_{\lambda'}(q_{\lambda'} \times 1)$, $H'(g_{\lambda'} q_{\lambda'} \times 1)$, $\psi_1 q_{\lambda\lambda'} L_{\lambda'}^-(q_{\lambda'} \times 1)$. However, according to (46), this is just the homotopy $H(q_{\lambda'} \times 1)$. Hence, W can be viewed as a homotopy realizing (47). ∎

4. Proof of Theorem 2. In the proof of Theorem 2 we use two functors $\tau : \text{inv-}\mathsf{Top} \to \text{inv-}\mathsf{Top}$, $\boldsymbol{T} : \mathrm{CH}(\mathsf{Top}) \to \pi(\text{pro-}\mathsf{Top})$ and a natural transformation $\phi_{\boldsymbol{X}} : \boldsymbol{X} \to \tau(\boldsymbol{X})$ between the identity functor on inv-Top and the functor τ ([M5], Theorems 5 and 6) (also see [T]). We will also need the following facts.

(i) If a system $\boldsymbol{X}$ is indexed by Λ then the system $\tau(\boldsymbol{X}) = \boldsymbol{T}(\boldsymbol{X})$ is also indexed by Λ. Moreover, if $\boldsymbol{X}$ is a single space X, then $\tau(X) = X$ and $\phi_X = \mathrm{id}$ (see [M5]).

(ii) For every system $\boldsymbol{X}$, $\phi_{\boldsymbol{X}}$ is a level homotopy equivalence (see [M5], Theorem 6).

(iii) For every mapping $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$, $[\tau(\boldsymbol{f})] = \boldsymbol{T}[C(\boldsymbol{f})]$ in $\pi(\text{pro-}\mathsf{Top})$ ([M5], Lemma 13).

Proof of Theorem 2. Let $\boldsymbol{p} : X \to \boldsymbol{X}$ be a mapping coherently dominated by a strong expansion $\boldsymbol{q} : X \to \boldsymbol{Y}$. Then there exist coherent mappings $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ and $\boldsymbol{g} : \boldsymbol{Y} \to \boldsymbol{X}$ such that (11) and (12) hold. Applying the functor $\boldsymbol{T}$ we conclude that

$$\boldsymbol{T}[\boldsymbol{f}]\boldsymbol{T}[C(\boldsymbol{p})] = \boldsymbol{T}[C(\boldsymbol{q})], \tag{55}$$

$$\boldsymbol{T}[\boldsymbol{g}]\boldsymbol{T}[\boldsymbol{f}] = [\mathbf{1}]. \tag{56}$$

Note that $\boldsymbol{T}[\boldsymbol{f}]$ is the class in $\pi(\text{pro-}\mathsf{Top})$ of a mapping $\boldsymbol{X} \to \boldsymbol{Y}$, which we denote by $\boldsymbol{T}(\boldsymbol{f})$. Hence, $\boldsymbol{T}[\boldsymbol{f}] = [\boldsymbol{T}(\boldsymbol{f})]$. Similarly, there is a mapping $\boldsymbol{T}(\boldsymbol{g}) : \boldsymbol{Y} \to \boldsymbol{X}$ such that $\boldsymbol{T}[\boldsymbol{g}] = [\boldsymbol{T}(\boldsymbol{g})]$. Moreover, by (ii), $\boldsymbol{T}[C(\boldsymbol{p})] = [\tau(\boldsymbol{p})]$ and $\boldsymbol{T}[C(\boldsymbol{q})] = [\tau(\boldsymbol{q})]$. Consequently, (55) and (56) become

$$[\boldsymbol{T}(\boldsymbol{f})][\tau(\boldsymbol{p})] = [\tau(\boldsymbol{q})], \tag{57}$$

respectively,

$$[\boldsymbol{T}(\boldsymbol{g})][\boldsymbol{T}(\boldsymbol{f})] = [\mathbf{1}]. \tag{58}$$

On the other hand, by the naturality of ϕ, the following diagram commutes:

$$\begin{tikzcd} \boldsymbol{Y} \arrow[d, "\phi_{\boldsymbol{Y}}"'] & X \arrow[l, "\boldsymbol{q}"'] \arrow[d, "\phi_X = \mathrm{id}"] \\ \tau(\boldsymbol{Y}) & X \arrow[l, "\tau(\boldsymbol{q})"] \end{tikzcd} \tag{59}$$

In other words,

$$\phi_{\boldsymbol{Y}} \boldsymbol{q} = \tau(\boldsymbol{q}). \tag{60}$$

Since $\boldsymbol{q}$ is a strong expansion, $\tau(\boldsymbol{q})$ is a mapping and $\phi_{\boldsymbol{Y}}$ is a level homotopy equivalence. Therefore, Lemma 3 applies and yields the conclusion that $\tau(\boldsymbol{q})$ is also a strong expansion. We now apply Lemma 2 to the mappings $\tau(\boldsymbol{p}) : X \to \tau(\boldsymbol{X})$, $\tau(\boldsymbol{q}) : X \to \tau(\boldsymbol{Y})$, $\boldsymbol{T}(\boldsymbol{f}) : \tau(\boldsymbol{X}) \to \tau(\boldsymbol{Y})$, $\boldsymbol{T}(\boldsymbol{g}) : \tau(\boldsymbol{Y}) \to \tau(\boldsymbol{X})$ and conclude that $\tau(\boldsymbol{p}) : X \to \tau(\boldsymbol{X})$ is a strong expansion. Note that conditions (57) and (58) insure that the assumptions of Lemma 2 are satisfied (see Remark 2).

Now note that the analogue of (60) for $\boldsymbol{p}$ has the form

$$\phi_{\boldsymbol{X}} \boldsymbol{p} = \tau(\boldsymbol{p}). \tag{61}$$

Since $\phi_{\boldsymbol{X}} : \boldsymbol{X} \to \tau(\boldsymbol{X})$ is a level homotopy equivalence, one can apply Lemma 2 to $\boldsymbol{p}$, $\tau(\boldsymbol{p})$ and $\phi_{\boldsymbol{X}}$ (see Remark 3) and conclude that $\boldsymbol{p}$ is indeed a strong expansion. ■

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