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Dugundji extenders and retracts on generalized ordered spaces

by

Gary Gruenhage (Auburn, Ala.),
Yasunao Hattori (Shimane) and **Haruto Ohta** (Shizuoka)

Abstract. For a subspace A of a space X, a linear extender $\varphi : C(A) \to C(X)$ is called an L_{ch}-extender (resp. L_{cch}-extender) if $\varphi(f)[X]$ is included in the convex hull (resp. closed convex hull) of $f[A]$ for each $f \in C(A)$. Consider the following conditions (i)–(vii) for a closed subset A of a GO-space X: (i) A is a retract of X; (ii) A is a retract of the union of A and all clopen convex components of $X \setminus A$; (iii) there is a continuous L_{ch}-extender $\varphi : C(A \times Y) \to C(X \times Y)$, with respect to both the compact-open topology and the pointwise convergence topology, for each space Y; (iv) $A \times Y$ is C^*-embedded in $X \times Y$ for each space Y; (v) there is a continuous linear extender $\varphi : C^*_{\mathrm{k}}(A) \to C_{\mathrm{p}}(X)$; (vi) there is an L_{ch}-extender $\varphi : C(A) \to C(X)$; and (vii) there is an L_{cch}-extender $\varphi : C(A) \to C(X)$. We prove that these conditions are related as follows: (i)⇒(ii)⇔(iii)⇔(iv)⇔(v)⇒(vi)⇒(vii). If A is paracompact and the cellularity of A is nonmeasurable, then (ii)–(vii) are equivalent. If there is no connected subset of X which meets distinct convex components of A, then (ii) implies (i). We show that van Douwen's example of a separable GO-space satisfies none of the above conditions, which answers questions of Heath–Lutzer [9], van Douwen [1] and Hattori [8].

1. Introduction. For a topological space X, let $C(X)$ be the linear space of real-valued continuous functions on X and $C^*(X)$ the subspace of bounded functions of $C(X)$. Let A be a subspace of X. A map $\varphi : C(A) \to C(X)$ is called an *extender* if $\varphi(f)$ is an extension of f for each $f \in C(A)$. An extender $\varphi : C(A) \to C(X)$ is called an L_{ch}-*extender* (resp. L_{cch}-*extender*) if φ is a linear map and $\varphi(f)[X]$ is included in the convex hull (resp. closed

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convex hull) of $f[A]$ for each $f \in C(A)$. The notions of an $L_{\rm ch}$-extender and an $L_{\rm cch}$-extender from $C^*(A)$ to $C^*(X)$ are analogously defined. An $L_{\rm ch}$-extender is an $L_{\rm cch}$-extender and, by the definition, an $L_{\rm cch}$-extender is continuous with respect to the uniform convergence topology. We refer to these extenders generically as *Dugundji extenders*. A *generalized ordered space* (= GO-space) is a triple $(X, \leq, \tau)$, where $(X, \leq)$ is a linearly ordered set and where τ is a topology on X such that τ is finer than the order topology and has a base consisting of convex sets. It is known that X is a GO-space if and only if it is a subspace of a linearly ordered topological space (= LOTS) (cf. [12]).

Let A be a closed subspace of a GO-space X. The purpose of this paper is to consider the problems when there is a Dugundji extender $\varphi : C(A) \to C(X)$ and when there is a Dugundji extender $\psi : C(A \times Y) \to C(X \times Y)$ for each space Y. In Sections 2 and 3, we prove the results stated in the abstract. What the results say is that if either there is an $L_{\rm cch}$-extender $\varphi : C(A) \to C(X)$ or $A \times Y$ is C^*-embedded in $X \times Y$, i.e., there is an extender $\psi : C^*(A \times Y) \to C^*(X \times Y)$, for each space Y, then A is close to being a retract of X. Heath–Lutzer [9] asked:

(a) If A is a closed subspace of a perfectly normal GO-space X, is there an $L_{\rm cch}$-extender $\varphi : C(A) \to C(X)$?

(b) What if X is assumed to be a LOTS?

Recently, Hattori [8] also asked:

(c) If A is a closed subspace of a perfectly normal GO-space X, is $A \times Y$ C^*-embedded in $X \times Y$ for each space Y?

By applying our results to van Douwen's example, we answer the questions (a), (b), (c) and that of van Douwen [1, Remark IV.5.2] (cf. [14, Question 134]) all negatively. In Section 4, we consider the monotone extension property as well as the Dugundji extension property of perfectly normal GO-spaces.

As usual, $\mathbb{R}$ denotes the set of reals, $\mathbb{Q}$ the set of rationals, $\mathbb{Z}$ the set of integers and $\mathbb{N} = \{n \in \mathbb{Z} : n > 0\}$. For a space X, $C_{\rm k}(X)$ (resp. $C_{\rm p}(X)$) denotes the space $C(X)$ with the compact open topology (resp. pointwise convergence topology) and $C^*_{\rm k}(X)$ (resp. $C^*_{\rm p}(X)$) the subspace of bounded functions. Let $(X, \leq)$ be a linearly ordered set. For $a, b \in X$ with $a < b$, we write $(a, b] = \{x \in X : a < x \leq b\}$, $(-\infty, b] = \{x \in X : x \leq b\}$, and define $[a, b)$, $[a, +\infty)$, (a, b) and $[a, b]$ analogously. For $A, B \subseteq X$ we also write $A < x$ to mean that $a < x$ for each $a \in A$; and correspondingly, $x < A$ and $A < B$ for brevity. A subset A of X is called *convex* if $[a, b] \subseteq A$ for each $a, b \in A$ with $a < b$. For maps $f : A \to Y$ and $g : B \to Y$ with $f|_{A\cap B} = g|_{A\cap B}$, the combination $h = f \bigtriangledown g$ is the map from $A \cup B$ to Y

defined by $h|_A = f$ and $h_B = g$. Other terms and symbols will be used as in [4].

2. Dugundji extenders and retracts. In this section, we state without proof our main theorem, which shows the relationship between the existence of Dugundji extenders and the existence of a retraction. The theorem will be proved in the next section. We use the following notation throughout the paper.

Notation. Let A be a closed subspace of a GO-space X. Let $\mathcal{U}_A$ denote the family of all convex components of $X\setminus A$. For $S \subseteq X$, let $l(S) = \max\{x \in X : x < S\}$ and $r(S) = \min\{x \in X : x > S\}$ if they exist. Note that if $x = l(U)$ or $x = r(U)$ for $U \in \mathcal{U}_A$, then $x \in A$. Let $\mathcal{U}_{A,1} = \{U \in \mathcal{U}_A : U$ has exactly one of $l(U)$ and $r(U)\}$, $\mathcal{U}_{A,2} = \{U \in \mathcal{U}_A : U$ has both $l(U)$ and $r(U)\}$ and $\mathcal{U}_{A,0} = \mathcal{U}_A \setminus (\mathcal{U}_{A,1} \cup \mathcal{U}_{A,2})$. For $i = 0, 1, 2$, we define $U_{A,i} = \bigcup\{U : U \in \mathcal{U}_{A,i}\}$ and consider the subspace $X_{A,i} = A \cup U_{A,i}$ of X. Then each $X_{A,i}$ is closed in X and $X_{A,i} \cap X_{A,j} = A$ for $i \neq j$. For example, if M is the Michael line, i.e., the space obtained from the LOTS $\mathbb{R}$ by making each point of $\mathbb{R} \setminus \mathbb{Q}$ isolated, then $M_{\mathbb{Q},0} = M$ and $M_{\mathbb{Q},1} = M_{\mathbb{Q},2} = \mathbb{Q}$.

Note that each $U \in \mathcal{U}_{A,0}$ is clopen in X, so $X_{A,0}$ is the union of A and some of the clopen convex components of $X \setminus A$. It is easy to check that A is a retract of $X_{A,0}$ if and only if A is a retract of the union of A and all clopen convex components of $X \setminus A$ (a retraction $f : X_{A,0} \to A$ can be extended by declaring $f(x)$ to be $l(U)$ or $r(U)$ whenever x is in a clopen convex component U of $X \setminus A$ which is not in $\mathcal{U}_{A,0}$).

For a closed subset A of a GO-space X, we say that A *separates* X if the closed interval $[l(U), r(U)]$ is disconnected for each $U \in \mathcal{U}_{A,2}$. If X is totally disconnected, then every closed subset separates X.

THEOREM 1. *Let A be a closed subspace of a GO-space X. Consider the following conditions* (1)–(10):

(1) *A is a retract of X.*

(2) *A is a retract of $X_{A,0}$.*

(2′) *A is a retract of the union of A and all clopen convex components of $X \setminus A$.*

(3) *There is a continuous L_{ch}-extender $\varphi : C(A \times Y) \to C(X \times Y)$, with respect to both the compact-open topology and the pointwise convergence topology*, *for each space Y.*

(4) *$A \times Y$ is C^*-embedded in $X \times Y$ for each space Y.*

(5) *There is a continuous linear extender $\varphi : C_{\mathrm{k}}(A) \to C_{\mathrm{k}}(X)$.*

(6) *There is a continuous linear extender $\varphi : C_{\mathrm{p}}(A) \to C_{\mathrm{p}}(X)$.*

(7) *There is a continuous linear extender $\varphi : C^*_{\mathrm{k}}(A) \to C^*_{\mathrm{k}}(X)$.*

(8) *There is a continuous linear extender $\varphi : C^*_{\mathrm{p}}(A) \to C^*_{\mathrm{p}}(X)$.*

(9) *There is an* L_{ch}*-extender* $\varphi : C(A) \to C(X)$.

(10) *There is an* L_{cch}*-extender* $\varphi : C(A) \to C(X)$.

These conditions are related as follows: (1)⇒(2)⇔(i)⇒(9)⇒(10) *for each* $i \in \{2', 3, 4, \ldots, 8\}$. *If* A *is paracompact and the cellularity of* A *is nonmeasurable*, *then* (2)–(10) *are equivalent. If* A *separates* X, *then* (2) *implies* (1).

In Section 3, we give examples showing that (9) does not imply (2) without the assumption on A. We do not know if (10) implies (9) in general (see Section 4).

Remark 1. Since the closed subspace $\mathbb{Q}$ of the Michael line M is not a retract, the pair $(\mathbb{Q}, M)$ satisfies none of the conditions of Theorem 1. (Morita [15] proved that $(\mathbb{Q}, M)$ does not satisfy (4), Heath–Lutzer–Zenor [11] proved that $(\mathbb{Q}, M)$ does not satisfy (7) and (8), and Heath–Lutzer [9] proved that $(\mathbb{Q}, M)$ does not satisfy (10).)

Now, we consider the following additional conditions (9^*) and (10^*):

(9^*) *There is an* L_{ch}*-extender* $\varphi : C^*(A) \to C^*(X)$.

(10^*) *There is an* L_{cch}*-extender* $\varphi : C^*(A) \to C^*(X)$.

Clearly, (9^*) implies (10^*) and, since an L_{cch}-extender $\varphi : C(A) \to C(X)$ carries a bounded function to a bounded function, (i) implies (i^*) for each $i = 9, 10$. In [1] van Douwen proved that the pair $(\mathbb{Q}, M)$ does not satisfy (9^*) for the Michael line M, while Heath–Lutzer [9] proved that a closed subspace A of a GO-space X always satisfies (10^*). We refer to the latter statement as *Heath–Lutzer's extension theorem.* We now show that (10^*) implies the following condition:

(11) *There is a continuous linear extender* $\varphi : C_{\mathrm{u}}(A) \to C_{\mathrm{u}}(X)$,

where $C_{\mathrm{u}}(E)$ denotes the space $C(E)$ with the uniform convergence topology. Let $\varphi : C^*(A) \to C^*(X)$ be an L_{cch}-extender. Since $C^*(A)$ is a linear subspace of $C(A)$, there is a Hamel base B of $C(A)$ such that $B \cap C^*(A)$ is a Hamel base of $C^*(A)$. For each $h \in B \setminus C^*(A)$, h extends to $\overline{h} \in C(X)$, because X is normal. For each $f \in C(A)$, f can be written as a linear combination $f = \sum_{h \in F} \alpha(h)h$, where F is a finite subset of B and $\alpha(h) \in \mathbb{R}$ for each $h \in F$. Define $\psi(f) = \sum_{h \in F \cap C^*(A)} \alpha(h)\varphi(h) + \sum_{h \in F \setminus C^*(A)} \alpha(h)\overline{h}$. Then $\psi : C(A) \to C(X)$ is a linear extender. For each $f, g \in C(A)$, if $\|f - g\| < \varepsilon$, then $f - g \in C^*(A)$ so that linearity of ψ and the fact that ψ extends φ yields $\|\psi(f) - \psi(g)\| = \|\psi(f - g)\| = \|\varphi(f - g)\| = \|\varphi(f) - \varphi(g)\| \le \varepsilon$. Hence, ψ is continuous with respect to the uniform convergence topology.

In [1], van Douwen gave an example of a 0-dimensional, separable, GO-space S with a closed subspace F which is not a retract (Example IV.5.1) and asked whether for each closed subspace A of S there is an L_{cch}-extender $\varphi : C(A) \to C(S)$. It is known that a separable GO-space is perfectly normal and Lindelöf. Hence, it follows from Theorem 1 that the space S gives a negative answer to van Douwen's question and questions (a) and (c) stated in the introduction. Moreover, since S embeds as a closed subspace in a separable LOTS, S also answers question (b) negatively. Below we give an example which is essentially the same as S but is easier to describe.

EXAMPLE 1. *There exists a* 0-*dimensional*, *separable*, *GO-space* X *with a closed subspace* A *which is not a retract of* X, *and hence satifies none of the conditions* (1)–(10) *of Theorem* 1.

Proof. Let $L = (\mathbb{P}\times\{0,1\})\cup(\mathbb{Q}\times\{0\})$, where $\mathbb{P}=\mathbb{R}\setminus\mathbb{Q}$, and consider the lexicographic order on L. Let X be the space obtained from the LOTS L by making each point of $\mathbb{Q}\times\{0\}$ isolated and let $A=\mathbb{P}\times\{0,1\}$. Then X is a 0-dimensional, separable, GO-space and A is closed. We show that A is not a retract of X. Suppose that there is a retraction $r : X \to A$. Let $\pi : X \to \mathbb{R}$ be the projection. Let $Q_1 = \{q \in \mathbb{Q} : \pi(r(\langle q,0\rangle)) > q\}$ and $Q_2 = \{q \in \mathbb{Q} : \pi(r(\langle q,0\rangle)) < q\}$. Then Q_1 or Q_2 is dense in some open interval I of the LOTS $\mathbb{R}$. Now, we assume that Q_1 is dense in I. Then we can find a sequence $\{q_k : k\in\mathbb{N}\}\subseteq Q_1$ and $p\in\mathbb{P}$ such that $q_k < p < \pi(r(\langle q_k,0\rangle))$ for each $k\in\mathbb{N}$ and $\sup_{k\in\mathbb{N}} q_k = p$ in $\mathbb{R}$. Indeed, let $q_1 \in Q_1\cap I$ be arbitrary. Given q_k, let $m_k = \frac{1}{2}(q_k+\pi(r(\langle q_k,0\rangle)))$ and choose $q_{k+1}\in I\cap Q_1$ in such a way that $q_k < q_{k+1} < \min\{m_j : 1\le j\le k\}$. Because q_k is bounded, $p=\sup\{q_k : k\ge 1\}$ exists in $\mathbb{R}$. Choose irrational numbers $y_k \in (q_k, q_{k+1})$. Then $\pi(r(\langle y_k,0\rangle)) = y_k$ so that continuity yields $p = \lim y_k = \lim \pi(r(\langle y_k,0\rangle)) = \pi(r(\langle p,0\rangle))$. Thus $p\in\mathbb{P}$. Observe that $q_k < p \le m_k < \pi(r(\langle q_k,0\rangle))$ for each k. The sequence $\{\langle q_k,0\rangle : k\in\mathbb{N}\}$ converges to $\langle p,0\rangle$ in X, but $r(\langle q_k,0\rangle) > \langle p,1\rangle > \langle p,0\rangle = r(\langle p,0\rangle)$ for each $k\in\mathbb{N}$. This contradicts the continuity of r. Hence, A is not a retract of X.

For the benefit of the reader who may be particularly interested in our solution to Heath and Lutzer's questions (a) and (b), we give here a short direct proof (i.e., without appealing to Theorem 1) that there is no $L_{\rm cch}$-extender $\varphi : C(A)\to C(X)$. Suppose such a φ exists. Note that if $g\in C(A)$ and $g\ge h$ pointwise in A, then $\varphi(g)\ge\varphi(h)$ pointwise in X. For each rational q and $n\in\omega$, let $A_{q,n} = \{\langle x,i\rangle \in \mathbb{P}\times\{0,1\} : |x-q| < 1/2^n\}$. Each $A_{q,n}$ is clopen in A so that the characteristic function $\chi_{q,n}$ of $A_{q,n}$ belongs to $C(A)$. We claim that, for each q, there must be some $k_q\in\omega$ such that $\varphi(\chi_{q,k_q})(\langle q,0\rangle) = 0$. If not, we can choose $k_n\in\omega$ such that $\varphi(k_n\chi_{q,n})(\langle q,0\rangle)\ge n$. Then $f=\sum_{n\in\omega} k_n\chi_{q,n}\in C(A)$ and yet $\varphi(f)(\langle q,0\rangle)\ge\varphi(k_n\chi_{q,n})(\langle q,0\rangle)\ge n$ for each n, a contradiction.

Now let $\chi^-(q,n_q)$ be the characteristic function of $\{x\in\mathbb{P}\times\{0,1\} : \pi(x) < q-1/2^{n_q}\}$ and define $\chi^+(q,n_q)$ similarly. Then $\chi^-(q,n_q) + \chi(q,n_q) + \chi^+(q,n_q)$ is the constant 1, and $\varphi(\chi(q,n_q))(\langle q,0\rangle) = 0$, so either $\varphi(\chi^-(q,n_q))(\langle q,0\rangle)\ge 1/2$ or $\varphi(\chi^+(q,n_q))(\langle q,0\rangle)\ge 1/2$. We may assume without loss of generality that the set $Q^+ = \{q \in \mathbb{Q} : \varphi(\chi^+(q, n_q))(\langle q, 0\rangle) \geq 1/2\}$ is dense in some interval. Then there exists $q(i) \in Q^+$ and an irrational α such that $q(i) \to \alpha$ and $q(i) < \alpha < q(i)+1/2^{n_{q(i)}}$. Let $\chi^+(\alpha)$ be the characteristic function of $\{x \in \mathbb{P} \times \{0,1\} : x = \langle \alpha, 1\rangle$ or $\pi(x) > \alpha\}$. Then $\chi^+(\alpha) \geq \chi^+(q(i), n_{q(i)})$ for each i, so $\varphi(\chi^+(\alpha))(\langle q(i), 0\rangle) \geq \varphi(\chi^+(q(i), n_{q(i)})(\langle q(i), 0\rangle) \geq 1/2$ for all i. But $\langle q(i), 0\rangle \to \langle \alpha, 0\rangle$ and $\chi^+(\alpha)(\langle \alpha, 0\rangle) = 0$, contradicting the continuity of $\varphi(\chi^+(\alpha))$. ∎

Remark 2. The space X in Example 1 embeds as a retract in the separable LOTS $L_0 = (\mathbb{P} \times \{0,1\}) \cup (\mathbb{Q} \times \mathbb{Z})$ with the lexicographic order. The pair (A, L_0) also satisfies none of the conditions (1)–(10) in Theorem 1.

Recall from [16] that a subspace B of a space E is π-*embedded* in E if $B \times Y$ is C^*-embedded in $E \times Y$ for each space Y. By Theorem 1, the closed subspace A of the space X in Example 1 is not π-embedded in X. Let T be the space obtained from the LOTS $\mathbb{R}$ by making each point of $\mathbb{Q}$ isolated. Then T is a separable metrizable space and the projection $\pi : X \to T$ is a perfect map. Hence, X is a perfectly normal, Lindelöf, M-space and A is Čech-complete but not π-embedded. This gives a simple answer to [16, Problems 14 and 17], which have been solved by Waśko [18]. The Michael line M witnesses that A is not π-embedded in X. In fact, the function $f \in C^*(A \times M)$ defined by

$$f(\langle\langle x, i\rangle, y\rangle) = \begin{cases} 1 & \text{if } x > y \text{ or } (x = y \text{ and } i = 1), \\ 0 & \text{if } x < y \text{ or } (x = y \text{ and } i = 0) \end{cases}$$

does not extend continuously to $X \times M$.

We conclude this section with some corollaries of Theorem 1.

Corollary 1. *Let A be a closed subspace of a locally compact GO-space X. Then the pair (A, X) satisfies conditions* (2)–(10) *in Theorem* 1. *Moreover*, *A is a retract of X if and only if A separates X.*

Proof. Since X is locally compact, $\mathcal{U}_{A,0}$ is discrete in X. Thus, $U_{A,0}$ is open and closed in X, which implies that A is a retract of $X_{A,0}$. Hence, the statements follow from Theorem 1. ∎

Corollary 2. *Every closed subspace A of a GO-space X whose underlying set is well ordered is a retract of X.*

Proof. Since the underlying set of X is well ordered, $\mathcal{U}_{A,0} = \emptyset$ and X is totally disconnected. Hence, this follows from Theorem 1. ∎

Remark 3. In [1] van Douwen proved that every closed subspace of a totally disconnected, locally compact, GO-space is a retract. Heath–Lutzer–Zenor [11] proved that every closed subspace of a GO-space whose underlying set is well ordered satisfies conditions (7) and (8) in Theorem 1.

3. Proof of Theorem 1 and examples. First, we prove Theorem 1. Let X be a GO-space and A a closed subspace of X. Then the implications (1)⇒(2), (3)⇒(j) for $j \in \{4,5,\dots,9\}$ and (9)⇒(10) are obviously true. As stated before Theorem 2, (2) is equivalent to (2′). We temporarily say that A is *πL-embedded* in S, where $A \subseteq S \subseteq X$, if there is an L_{ch}-extender $\varphi : C(A\times Y) \to C(S\times Y)$ which is continuous with respect to both the compact-open topology and the pointwise convergence topology, for each space Y. The following lemma sharpens Heath–Lutzer [9, Lemma 3.7], which says that there is an L_{cch}-extender $\varphi : C(A) \to C(X)$ in case $\mathcal{U}_{A,0} = \emptyset$.

LEMMA 1. *The subspace A is a retract of $X_{A,1}$ and is πL-embedded in $X_{A,2}$. If A separates X, then A is a retract of $X_{A,1} \cup X_{A,2}$.*

Proof. For each $U \in \mathcal{U}_{A,1}$, there is exactly one of $l(U)$ and $r(U)$. We denote it by x_U. Then we get a retraction $r : X_{A,1} \to A$ by letting $r(a) = a$ for each $a \in A$ and $r(u) = x_U$ for each $u \in U$ with $U \in \mathcal{U}_{A,1}$.

We show that A is πL-embedded in $X_{A,2}$. Let $U \in \mathcal{U}_{A,2}$. If U is a singleton, let k_U be the constant function on U with the value 0. If $|U| \ge 2$, then we choose $s(U), t(U) \in U$ such that $s(U) < t(U)$. Then there exists a continuous function $k_U : U \to [0,1]$ such that $k_U(x) = 0$ for each $x \le s(U)$ and $k_U(x) = 1$ for each $x \ge t(U)$. Let Y be a space and let $T = X_{A,2}\times Y$. For each $f \in C(A\times Y)$, define a function $\varphi(f) : T \to \mathbb{R}$ by $\varphi(f)|_{A\times Y} = f$ and $\varphi(f)(\langle u,y\rangle) = (1-k_U(u))\cdot f(\langle l(U),y\rangle) + k_U(u)\cdot f(\langle r(U),y\rangle)$ for $\langle u,y\rangle \in U\times Y$ with $U \in \mathcal{U}_{A,2}$. Then

$$\begin{aligned}(3.1)\quad \min\{f(\langle l(U),y\rangle), f(\langle r(U),y\rangle)\} &\le \varphi(f)(\langle u,y\rangle)\\ &\le \max\{f(\langle l(U),y\rangle), f(\langle r(U),y\rangle)\}\end{aligned}$$

for each $\langle u,y\rangle \in U\times Y$ with $U \in \mathcal{U}_{A,2}$. This implies that $\varphi(f)$ is continuous and $\varphi : C(A\times Y) \to C(T)$ is an L_{ch}-extender. Since the continuity of φ with respect to the pointwise convergence topology is obvious, we show that φ is continuous with respect to the compact-open topology. To do this, we define a map $\psi : T \to 2^{A\times Y}$ as follows. For each $p \in A\times Y$ we define $\psi(p) = \{p\}$. Let $\langle u,y\rangle \in (X_{A,2}\setminus A)\times Y$. Then there is $U \in \mathcal{U}_{A,2}$ such that $u \in U$. If $U = \{u\}$, then we define $\psi(\langle u,y\rangle) = \{\langle l(U),y\rangle\}$. Suppose that $|U| \ge 2$. We define $\psi(\langle u,y\rangle) = \{\langle l(U),y\rangle\}$ if $u \le s(U)$, $\psi(\langle u,y\rangle) = \{\langle r(U),y\rangle\}$ if $u \ge t(U)$, and $\psi(\langle u,y\rangle) = \{\langle l(U),y\rangle, \langle r(U),y\rangle\}$ if $s(U) < u < t(U)$. It is easily checked that ψ is upper semicontinuous, i.e., for each open set V in $A\times Y$, the set $\{p \in T : \psi(p) \subseteq V\}$ is open in T. Now, it is enough to show that φ is continuous at $\mathbf{0} \in C_{\mathrm{k}}(A\times Y)$. Let K be a compact set of T and $\varepsilon > 0$. Since ψ is upper semicontinuous, it follows from [13, Corollary 9.6] that $K_0 = \bigcup_{p\in K}\psi(p)$ is compact. If $f \in C_{\mathrm{k}}(A\times Y)$ and $f[K_0] \subseteq (-\varepsilon,+\varepsilon)$, then $\varphi(f)[K] \subseteq (-\varepsilon,+\varepsilon)$ by (3.1). Hence, φ is continuous at $\mathbf{0}$ with respect to the compact-open topology.

Finally, we assume that A separates X. Then there is a retraction r_U from $[l(U), r(U)]$ to $\{l(U), r(U)\}$ for each $U \in \mathcal{U}_{A,2}$. Define a map $r_2 : X_{A,2} \to A$ by $r_2|_A = \mathrm{id}_A$ and $r_2|_U = r_U$ for each $U \in \mathcal{U}_{A,2}$. Then r_2 is a retraction. On the other hand, there is a retraction $r_1 : X_{A,1} \to A$ as we have proved above. By [4, Proposition 2.1.13], the combination $r_1 \bigtriangledown r_2$ is a retraction from $X_{A,1} \cup X_{A,2}$ to A. ■

(I) We prove that (2) implies (3). Let Y be a space. By Lemma 1, A is πL-embedded in $X_{A,1}$ and in $X_{A,2}$, and by (2), A is also πL-embedded in $X_{A,0}$. Hence, for each $i = 0, 1, 2$, there is an L_{ch}-extender $\varphi_i : C(A \times Y) \to C(X_{A,i} \times Y)$ which is continuous with respect to both the compact open topology and the pointwise convergence topology. Define an extender $\varphi : C(A \times Y) \to C(X \times Y)$ by $\varphi(f) = \varphi_0(f) \bigtriangledown \varphi_1(f) \bigtriangledown \varphi_2(f)$ for $f \in C(A \times Y)$. Then φ is an L_{ch}-extender which is continuous with respect to both the compact-open topology and the pointwise convergence topology. ■

(II) We prove that (2) implies (1) if A separates X. By (2), there is a retraction $r_0 : X_{A,0} \to A$. Since A separates X, there is a retraction $r : X_{A,1} \cup X_{A,2} \to A$ by Lemma 1. Then the combination $r_0 \bigtriangledown r : X \to A$ is a retraction. ■

We shall establish some conventions which will be used in the rest of the proof. It remains to show that A is a retract of $X_{A,0} = A \cup U_{A,0}$ (i.e., condition (2)) when the pair (A, X) satisfies condition (i) for $i \in \{4, 5, 6, 7, 8\}$, or for $i = 10$ if also A is paracompact and the cellularity of A is nonmeasurable. Obviously, if (A, X) satisfies condition (i) for $i \in \{4, 5, 6, 7, 8, 10\}$, so does the pair (A, S) for every subspace S with $A \subseteq S \subseteq X$. Thus, we may assume without losing generality that $X = X_{A,0}$. Moreover, if we choose a point $x_U \in U$ for each $U \in \mathcal{U}_{A,0}$, then it is easily checked that $A \cup \{x_U : U \in \mathcal{U}_{A,0}\}$ is a retract of $X_{A,0}$. Hence, it suffices to show that A is a retract of $A \cup \{x_U : U \in \mathcal{U}_{A,0}\}$. This means that we may assume that each element of $\mathcal{U}_{A,0}$ is a singleton. That is, we assume that

$$(*) \qquad X = X_{A,0} \text{ and each element of } \mathcal{U}_{A,0} \text{ is singleton.}$$

Further, we then denote the family of convex components of A in X by $\mathcal{A}$. By the assumptions, $X \setminus A$ is discrete and for each $u, u' \in X \setminus A$ with $u < u'$, there is $B \in \mathcal{A}$ such that $u < B < u'$. Let $\mathcal{Z} = \mathcal{A} \cup (X \setminus A)$. Then we can regard $\mathcal{Z}$ naturally as a linearly ordered set. For $B \in \mathcal{A}$ and $u \in X \setminus A$, we write $u = B^+$ to mean that u is an immediate successor of B in $\mathcal{Z}$; and analogously, $B = u^+$. Let τ denote the topology of X. Fix a point $a_0 \in A$ and a point $a_B \in B$ for each $B \in \mathcal{A}$.

(III) We prove that (4) implies (2). Let $\mathcal{B} = \mathcal{A} \setminus \{B \in \mathcal{A} : |B| = 1$ and $(-\infty, B] \notin \tau$ and $[B, +\infty) \notin \tau\}$, where $(-\infty, B] = \{x \in X : (\exists b \in B)(x \le b)\}$ and $[B, +\infty) = \{x \in X : (\exists b \in B)(x \ge b)\}$. Note that $\mathcal{B}$ may be empty.

Let Y be the space obtained from the subspace $\mathcal{B} \cup (X \setminus A)$ of the LOTS $\mathcal{Z}$ by making each point of $\mathcal{B}$ isolated, i.e., Y has the topology generated by a base $\{\{B\} : B \in \mathcal{B}\} \cup \{(C, D) \cap Y : C < D \text{ and } C, D \in \mathcal{A}\}$.

For each $B \in \mathcal{A}$ with $|B| \geq 2$, fix $x_B, y_B \in B$ with $x_B < y_B$ and choose $f_B \in C(B)$ such that $f_B(x) = 0$ for each $x \leq x_B$, $f_B(x) = 1$ for each $x \geq y_B$, and $0 \leq f_B(x) \leq 1$ for each $x \in B$. We define $f \in C(A \times Y)$ as follows: Let $\langle a, y\rangle \in A \times Y$. If $a \notin y \in \mathcal{B}$ or $y \in X \setminus A$, define $f(\langle a, y\rangle) = 0$ if $a < y$, and $f(\langle a, y\rangle) = 1$ if $a > y$. If $y = B \in \mathcal{B}$ and $a \in B$, then we distinguish three cases: If $(-\infty, B] \in \tau$, let $f(\langle a, y\rangle) = 0$. If $(-\infty, B] \notin \tau$ and $[B, +\infty) \in \tau$, let $f(\langle a, y\rangle) = 1$. If $(-\infty, B] \notin \tau$ and $[B, +\infty) \notin \tau$, then $|B| \geq 2$ by the definition of $\mathcal{B}$. Define $f(\langle a, y\rangle) = f_B(a)$. Then it is easily checked that f is continuous. By (4), f extends to $g \in C(X \times Y)$.

We define a retraction $r : X \to A$ as follows: Define $r(a) = a$ for each $a \in A$. Let $u \in X \setminus A$. First, if $u = \max \mathcal{Z}$ or $u = \min \mathcal{Z}$, let $r(u) = a_0$. Next, we assume that $u \neq \max \mathcal{Z}$ and $u \neq \min \mathcal{Z}$. If u has an immediate predecessor B in $\mathcal{Z}$, let $r(u) = a_B$. If u has no immediate predecessor but has an immediate successor B' in $\mathcal{Z}$, let $r(u) = a_{B'}$. Finally, assume that u is not as above. Then, by the continuity of g, there are $C, D \in \mathcal{A}$, with $C < u < D$, such that $|g(\langle u, u\rangle) - g(\langle u, y\rangle)| < 1/4$ for each $y \in Y$ with $C \leq y \leq D$. Define $r(u) = a_C$ if $g(\langle u, u\rangle) < 1/2$, and $r(u) = a_D$ if $g(\langle u, u\rangle) \geq 1/2$. By the definition, the following (3.2) and (3.3) hold for each $u \in X \setminus A$:

(3.2) If $r(u) \in B < u$ and $u \neq B^+$ in $\mathcal{Z}$, then
$$g(\langle u, y\rangle) < 3/4 \quad \text{for each } y \in Y \text{ with } B \leq y \leq u.$$

(3.3) If $r(u) \in B > u$ and $B \neq u^+$ in $\mathcal{Z}$, then
$$g(\langle u, y\rangle) > 1/4 \quad \text{for each } y \in Y \text{ with } u \leq y \leq B.$$

It suffices to show that r is continuous at each point of A. Suppose that r is not continuous at $p \in A$. Then there exist a convex neighborhood H of p in X and $S \subseteq H \setminus A$ such that $p \in \mathrm{cl}_X S$ and $r[S] \cap H = \emptyset$. Put $S_1 = \{u \in S : u < p \text{ and } r(u) < H\}$, $S_2 = \{u \in S : u < p \text{ and } r(u) > H\}$, $S_3 = \{u \in S : u > p \text{ and } r(u) < H\}$ and $S_4 = \{u \in S : u > p \text{ and } r(u) > H\}$. Since $S = S_1 \cup S_2 \cup S_3 \cup S_4$, either $p \in \mathrm{cl}_X(S_1 \cup S_2)$ or $p \in \mathrm{cl}_X(S_3 \cup S_4)$. Now, we only show that a contradiction occurs in the former case, since the latter case can be proved similarly. Choose $B \in \mathcal{A}$ with $p \in B$. Since $S_1 \cup S_2 < p$ and $p \in \mathrm{cl}_X(S_1 \cup S_2)$, $p = \min B$, $[p, +\infty) \notin \tau$ and B has no immediate predecessor in $\mathcal{Z}$. We consider three cases:

Case 1: $p \in \mathrm{cl}_X S_1$. Since $[p, +\infty) \notin \tau$, there is $v \in H \setminus A$ with $v < p$. We may assume that $v < S_1 < p$. For each $u \in S_1$, since $r(u) < v < u$, it follows from (3.2) that $g(\langle u, v\rangle) < 3/4$. Since $p \in \mathrm{cl}_X S_1$, this implies that $g(\langle p, v\rangle) \leq 3/4$, but $f(\langle p, v\rangle) = 1$ since $p > v$. This contradicts the fact that g is an extension of f.

CASE 2: $p \in \mathrm{cl}_X S_2$ and $B \in Y$. For each $u \in S_2$, $u < p < r(u)$ and $B \neq u^+$ in $\mathcal{Z}$. Hence, it follows from (3.3) that $g(\langle u, B\rangle) > 1/4$ for each $u \in S_2$. Since $p \in \mathrm{cl}_X S_2$, this implies that $g(\langle p, B\rangle) \geq 1/4$, but $f(\langle p, B\rangle) = 0$, because $p \in B \in \mathcal{B}$, $[p, +\infty) \notin \tau$ and $p = \min B$. This is a contradiction.

CASE 3: $p \in \mathrm{cl}_X S_2$ and $B \notin Y$. Then $B = \{p\}$ and $(-\infty, p] \notin \tau$ by the definition of Y. Thus, there is $w \in H \setminus A$ with $w > p$. Since $r(u) > w > u$ for each $u \in S_2$, $g(\langle p, w\rangle) \geq 1/4$ by the similar argument to Case 1, but $f(\langle p, w\rangle) = 0$ since $p < w$. This is a contradiction. Hence, the proof of (4)⇒(2) is complete. ∎

(IV) We prove that (i) implies (2) for each $i = 5, 6, 7, 8$. Recall our simplifying assumptions $(*)$. If one of the conditions (5), (6), (7) and (8) holds, then there is a continuous linear extender $\varphi : C^*_{\mathrm{k}}(A) \to C_{\mathrm{p}}(X)$. Let I be the closed set $\{x \in X : \varphi(\mathbf{1}_A)(x) \leq 1/2\}$, where $\mathbf{1}_A$ is a constant function on A taking the value 1. Since $I \subseteq X \setminus A$ and $X \setminus A$ is discrete, I is open and closed in X. Let $u \in X \setminus (A \cup I)$. Since φ is continuous, there is a compact set K_u of A such that $\varphi(f)(u) > 1/2$ for each $f \in C^*_{\mathrm{k}}(A)$ with $f[K_u] = \{1\}$. Let $K_{u,1} = K_u \cap (-\infty, u)$ and $K_{u,2} = K_u \cap (u, +\infty)$. We may assume that both $K_{u,1}$ and $K_{u,2}$ are nonempty unless $u = \min \mathcal{Z}$ or $u = \max \mathcal{Z}$. Let $C_{u,1} = \{f \in C^*_{\mathrm{k}}(A) : f[K_{u,1}] = \{1\}$ and $f[(u, +\infty) \cap A] = \{0\}\}$ and $C_{u,2} = \{f \in C^*_{\mathrm{k}}(A) : f[K_{u,2}] = \{1\}$ and $f[(-\infty, u) \cap A] = \{0\}\}$.

We define a retraction $r : X \to A$. Define $r(a) = a$ for each $a \in A$. For $u \in X \setminus A$, we define $r(u)$ as follows: First, if $u = \min \mathcal{Z}$ or $u = \max \mathcal{Z}$, let $r(u) = a_0$. Next, we assume that $u \neq \min \mathcal{Z}$ and $u \neq \max \mathcal{Z}$. If $u \in I$, let $r(u) = a_0$. If $u \in X \setminus (A \cup I)$, then $\varphi(\mathbf{1}_A)(u) > 1/2$. Now, suppose that there exist $f_1 \in C_{u,1}$ with $\varphi(f_1)(u) \leq 1/4$ and $f_2 \in C_{u,2}$ with $\varphi(f_2)(u) \leq 1/4$. Then $\varphi(f_1 + f_2)(u) = \varphi(f_1)(u) + \varphi(f_2)(u) \leq 1/2$. Since $(f_1 + f_2)[K_u] = \{1\}$, this contradicts the definition of K_u. Hence, either $\varphi(f)(u) > 1/4$ for each $f \in C_{u,1}$ or $\varphi(f)(u) > 1/4$ for each $f \in C_{u,2}$. In the former case, define $r(u) = \max K_{u,1}$, and otherwise, define $r(u) = \min K_{u,2}$. It suffices to show that r is continuous at each point of A. Suppose that r is not continuous at $p \in A$. Then there exist a convex neighborhood H of p in X and $S \subseteq H \setminus A$ such that $p \in \mathrm{cl}_X S$ and $r[S] \cap H = \emptyset$. Since I is a closed set missing A, we may assume that $S \cap I = \emptyset$. Let S_i, $i = 1, 2, 3, 4$, be the same as in the proof of (4)⇒(2). Then $p \in \mathrm{cl}_X S_i$ for some i. Now, we only show that a contradiction occurs when $p \in \mathrm{cl}_X S_1$ or $p \in \mathrm{cl}_X S_2$, since the other cases can be proved similarly. First, assume that $p \in \mathrm{cl}_X S_1$. Since $S_1 < p$ and $p \in \mathrm{cl}_X S_1$, there is $v \in H \setminus A$ with $v < p$. We may assume that $v < S_1 < p$. For each $u \in S_1$, since $r(u) < u$,

$$\varphi(f)(u) > 1/4 \quad \text{for each } f \in C_{u,1}. \tag{3.4}$$

Define $g \in C^*_{\mathrm{k}}(A)$ by $g(x) = 1$ for each $x \in (-\infty, v) \cap A$ and $g(x) = 0$ for each $x \in (v, +\infty) \cap A$. Then, for each $u \in S_1$, we have $g \in C_{u,1}$, because $\max K_{u,1} = r(u) < v < u$. Hence, it follows from (3.4) that $\varphi(g)(u) > 1/4$ for each $u \in S_1$. Since $p \in \mathrm{cl}_X S_1$, this implies that $\varphi(g)(p) \geq 1/4$, but $g(p) = 0$ since $p > v$. This is a contradiction. Next, assume that $p \in \mathrm{cl}_X S_2$. For each $u \in S_2$, since $r(u) > u$,

$$\varphi(f)(u) > 1/4 \quad \text{for each } f \in C_{u,2}. \tag{3.5}$$

There is $h \in C^*_{\mathrm{k}}(A)$ such that $h(x) = 0$ for each $x \in (-\infty, p] \cap A$ and $h(x) = 1$ for each $x \in ([p, +\infty) \setminus H) \cap A$. Then, for each $u \in S_2$, we have $h \in C_{u,2}$, because $\min K_{u,2} = r(u) > p > u$. Hence, it follows from (3.5) that $\varphi(h)(u) > 1/4$ for each $u \in S_2$. Since $p \in \mathrm{cl}_X S_2$, this implies that $\varphi(h)(p) \geq 1/4$, but $h(p) = 0$ by the definition. This is a contradiction. ∎

(V) Finally, we prove that (10) implies (2) if A is paracompact and the cellularity of A is nonmeasurable. Let $\varphi : C(A) \to C(X)$ be an L_{cch}-extender. We define a retraction $r : X \to A$. Define $r(a) = a$ for each $a \in A$. For $u \in X \setminus A$, we define $r(u)$ as follows: First, if $u = \min \mathcal{Z}$ or $u = \max \mathcal{Z}$, let $r(u) = a_0$. Next, we assume that $u \neq \min \mathcal{Z}$ and $u \neq \max \mathcal{Z}$. If u has an immediate predecessor B in $\mathcal{Z}$, let $r(u) = a_B$. If u has no immediate predecessor but has an immediate successor B' in $\mathcal{Z}$, let $r(u) = a_{B'}$. Finally, assume that u is not as above. For an open and closed set D in A, define $e_D \in C(A)$ by $e_D(a) = 1$ if $a \in D$, and $e_D(a) = 0$ otherwise.

Claim. *There exist $u_0, u_1 \in X \setminus A$ such that $u_0 < u < u_1$ and $\varphi(e_D)(u) = 0$, where $D = (u_0, u_1) \cap A$.*

Proof. Suppose that the claim fails. Then either $\varphi(e_{(v,u)\cap A})(u) > 0$ for each $v \in X \setminus A$ with $v < u$ or $\varphi(e_{(u,w)\cap A})(u) > 0$ for each $w \in X \setminus A$ with $w > u$. We only consider the former case, since the proof for the latter case is similar. Since A is paracompact, there is a regular infinite cardinal κ and an increasing κ-sequence $s : \kappa \to A$ such that $u = \sup s[\kappa]$ and $s[\kappa]$ is discrete closed in A. By the assumption, κ is nonmeasurable. Since u has no immediate predecessor in $\mathcal{Z}$, by passing to a subsequence if necessary, we may assume that for each $\alpha < \kappa$, there is $y_\alpha \in X \setminus A$ with $s(\alpha) < y_\alpha < s(\alpha + 1)$. Then the set $Y = \{y_\alpha : \alpha < \kappa\}$ is discrete closed in X because the point u is isolated. For each $\alpha < \kappa$, let $I_\alpha = \bigcup\{B \in \mathcal{A} : y_\beta < B < y_\alpha$ for each $\beta < \alpha\}$. Since Y is discrete closed, each I_α is open and closed in A and $\{I_\alpha : \alpha < \kappa\}$ is a partition of $A \cap (-\infty, u)$. For each $E \subseteq \kappa$, let $f_E = \sum_{\alpha \in E} e_{I_\alpha}$, and let $\mathcal{E} = \{E \subseteq \kappa : \varphi(f_E)(u) > 0\}$. Observe that if $E \in \mathcal{E}$ and $E \subseteq F$, then $F \in \mathcal{E}$, and if $E_1 \cup E_2 \in \mathcal{E}$, then $E_1 \in \mathcal{E}$ or $E_2 \in \mathcal{E}$.

Now, suppose that there is an infinite, point-finite subfamily $\{E_n : n \in \mathbb{N}\}$ of $\mathcal{E}$. For each $n \in \mathbb{N}$, choose $k_n > 0$ with $\varphi(k_n f_{E_n})(u) \geq n$, and let $f = \sum_{n \in \mathbb{N}} k_n f_{E_n}$. Then $f \in C(A)$, because all but finitely many f_{E_n} vanish on each I_α. For each $n \in \mathbb{N}$, since $f \geq k_n f_{E_n}$, $\varphi(f)(u) \geq \varphi(k_n f_{E_n})(u) \geq n$, which is impossible. Hence, $\mathcal{E}$ includes no infinite, point-finite subfamily. It follows that there is $\alpha_0 < \kappa$ such that $\{\alpha\} \notin \mathcal{E}$ for each $\alpha > \alpha_0$. Put $E_0 = \{\alpha : \alpha_0 < \alpha < \kappa\}$; then $E_0 \in \mathcal{E}$ by our assumption that $\varphi(e_{(v,u)\cap A}(u)) > 0$ for each $v \in X - A$ with $v < u$. If for each $E \in \mathcal{E}$ with $E \subseteq E_0$, there is $E' \subseteq E$ such that $E' \in \mathcal{E}$ and $E \setminus E' \in \mathcal{E}$, then we can find an infinite, disjoint subfamily of $\mathcal{E}$. Since it is impossible, there is $F \in \mathcal{E}$, with $F \subseteq E_0$, such that for each $F' \subseteq F$, either $F' \notin \mathcal{E}$ or $F \setminus F' \notin \mathcal{E}$. Then the family $\mathcal{F} = \{E \in \mathcal{E} : E \subseteq F\}$ is a free ultrafilter on the set F. Indeed, we can show that $\mathcal{F}$ is closed under finite intersections as follows. First note that because $E_1 \cup E_2 \in \mathcal{E}$ implies that E_1 or E_2 belongs to $\mathcal{E}$, it follows that if $F' \subset F$ then exactly one of the sets F' and $F - F'$ fails to belong to $\mathcal{E}$. Now let $E_i \in \mathcal{F}$ for $i = 1, 2$. Then $F - E_i \notin \mathcal{E}$ so that $(F - E_1) \cup (F - E_2) \notin \mathcal{E}$. Thus $F - (E_1 \cap E_2) \notin \mathcal{E}$, so that $E_1 \cap E_2 \in \mathcal{E}$. Hence $E_1 \cap E_2 \in \mathcal{F}$. Since κ is nonmeasurable, $\mathcal{F}$ cannot have the countable intersection property, i.e., there is $\{F_n : n \in \mathbb{N}\} \subseteq \mathcal{F}$ such that $F_{n+1} \subseteq F_n$ for each n and $\bigcap_{n\in\mathbb{N}} F_n = \emptyset$. Since $\{F_n : n \in \mathbb{N}\}$ is point-finite, this is a contradiction. ∎

We finish the definition of $r(u)$. Let $D = (u_1, u_2) \cap A$ be as in the Claim, i.e., $\varphi(e_D)(u) = 0$. Since u has neither an immediate predecessor nor an immediate successor in $\mathcal{Z}$, there are $B, C \in \mathcal{A}$ with $u_1 < B < u < C < u_2$. If we put $D_{u,1} = (-\infty, u_1) \cap A$ and $D_{u,2} = (u_2, +\infty) \cap A$, then $e_D + e_{D_{u,1}} + e_{D_{u,2}} = \mathbf{1}_A$. Since $\varphi(e_D)(u) + \varphi(e_{D_{u,1}})(u) + \varphi(e_{D_{u,2}})(u) = 1$, either $\varphi(e_{D_{u,1}})(u) \geq 1/2$ or $\varphi(e_{D_{u,2}})(u) \geq 1/2$. In the former case, define $r(u) = a_B$, otherwise define $r(u) = a_C$.

We show that r is continuous at each point of A. Let $p \in A$ and H a convex neighborhood of p in X. Then there is $g \in C(A)$ such that $g(p) = 0$, $g[A \setminus H] = \{1\}$ and $0 \leq g(a) \leq 1$ for each $a \in A$. Let $G = H \cap \{x \in X : \varphi(g)(x) < 1/2\} \setminus M$, where $M = \{u : u = \min(H \setminus A) \text{ or } u = \max(H \setminus A)\}$; of course, M may be empty. Then G is a neighborhood of p in X such that $G \cap A \subseteq H$. To show that $r[G] \subseteq H$, let $u \in G \setminus A$. If u has an immediate predecessor or an immediate successor in $\mathcal{Z}$, then $r(u) \in H$, because $u \notin M$. Suppose that u has neither. If $r(u) < H$, then $\varphi(e_{D_{u,1}})(u) \geq 1/2$ and $e_{D_{u,1}} \leq g$. If $r(u) > H$, then $\varphi(e_{D_{u,2}})(u) \geq 1/2$ and $e_{D_{u,2}} \leq g$. In each case, $\varphi(g)(u) \geq 1/2$, which contradicts the fact that $u \in G$. Hence, $r[G] \subseteq H$, which completes the proof of Theorem 1. ∎

We give examples showing that the implication (9)⇒(2) need not be true without the assumptions on A. The first one shows that paracompactness of A is necessary to prove (9)⇒(2). Let X be a linearly ordered set and x a point of X with no immediate predecessor. Then there exists a unique regular cardinal κ such that there is an increasing κ-sequence $s : \kappa \to (-\infty, x)$ with $x = \sup s[\kappa]$. We call κ the *left cofinality* of x and write $\kappa = \operatorname{lcf}(x)$. Similarly we define the *right cofinality* $\operatorname{rcf}(x)$ of x using a decreasing κ-sequence.

EXAMPLE 2. *There exists a* 0*-dimensional*, *countably compact*, *GO-space* X *such that for every closed subspace* A, *there is an* L_{ch}*-extender* $\varphi : C(A) \to C(X)$, *but some closed subspace is not a retract.*

Proof. Let Q be an η_1-set, i.e., a linearly ordered set Q such that for each pair of subsets $C, D \subseteq Q$ with $|C| < \omega_1$, $|D| < \omega_1$ and $C < D$, there is $x \in Q$ with $C < x < D$ (for details on η_1-sets, see [7, Chapter 13]). Let R be the Dedekind completion of Q and X the space obtained from the LOTS R by making each point of Q isolated. For each countable set $C \subseteq Q$, there are $x, y \in Q$ such that $\emptyset < x < C < y < \emptyset$ by the definition of an η_1-set. Hence, R has neither a countable cofinal subset nor a countable coinitial subset. Moreover, $\mathrm{lcf}(x) \geq \omega_1$ and $\mathrm{rcf}(x) \geq \omega_1$ for each $x \in Q$. Hence, X is countably compact. Let A be a closed subspace of X. Since $C(A) = C^*(A)$, there is an L_{cch}-extender $\varphi : C(A) \to C(X)$ by Heath–Lutzer's extension theorem (cf. Remark 1).

Now, suppose that φ is not an L_{ch}-extender. Then there are $f \in C(A)$ and $x \in X$ such that $\varphi(f)(x) \in \mathrm{cl}_{\mathbb{R}} f[A] \setminus f[A]$. If we define $g(a) = |f(a) - \varphi(f)(x)|^{-1}$ for each $a \in A$, then g is continuous and unbounded, which contradicts countable compactness of A. Hence, φ is an L_{ch}-extender.

We show that the closed subspace $B = X \setminus Q$ is not a retract of X. Suppose that there is a retraction $r : X \to B$. Let $Q_1 = \{q \in Q : r(q) > q\}$ and $Q_2 = \{q \in Q : r(q) < q\}$. Then Q_1 or Q_2 is dense in some open interval I of the LOTS R. Now, we assume that Q_1 is dense in I. Then we can inductively define $q_n \in Q_1$ so as to satisfy $q_{n-1} < q_n < \min\{r(q_1), \ldots, r(q_{n-1})\}$ for each $n > 1$. Let $p = \sup_{n\in\mathbb{N}} q_n$. Since Q is an η_1-set, $p \in B$. Thus, $p = \lim q_n$ in X, but there is $x \in Q$ with $p < x < \inf_{n\in\mathbb{N}} r(q_n)$, because Q is an η_1-set. This contradicts the continuity of r. Hence, B is not a retract of X. ∎

The next example shows that the assumption that the cellularity of A is nonmeasurable is necessary to prove (9)⇒(2).

EXAMPLE 3. *If there exists a measurable cardinal*, *then there exists a* 0*-dimensional*, *hereditarily paracompact*, *GO-space* X *with a closed subspace* A *which has an* L_{ch}*-extender* $\varphi : C(A) \to C(X)$ *but is not a retract.*

Proof. Let κ be the first measurable cardinal. Let $L = \mathbb{Z}^\kappa$ be the LOTS with the lexicographic order and let $A = \{x \in L : (\exists \alpha < \kappa)(\forall \beta > \alpha)(x(\beta) = 0)\}$. Then it is easily checked that A is dense in L, $|A| = \kappa$ and $\mathrm{lcf}(x) = \mathrm{rcf}(x) = \kappa$ for each $x \in L$. Let X be the space obtained from L by making each point of $L \setminus A$ isolated.

First, suppose that X has a nonparacompact subspace. Then it follows from [4, Theorem 2.3] that for some uncountable regular cardinal τ, some stationary set T of τ is homeomorphic to a subspace of X. By the proof of Theorem 1, we may assume that the embedding $h : T \to X$ is monotone increasing or monotone decreasing. Since $|A| = \kappa$ and each point of $X \setminus A$ is isolated, $\tau \le \kappa$. Since $\mathrm{lcf}(x) = \mathrm{rcf}(x) = \kappa$ for each $x \in X$, X cannot contain any limit point of $h[T]$, which is a contradiction. Hence, X is hereditarily paracompact.

Next, we show that there is an L_{ch}-extender $\varphi : C(A) \to C(X)$. Let $f \in C(A)$ and $u \in X \setminus A$. Since $\mathrm{lcf}(u) = \kappa$, there is an increasing κ-sequence $s : \kappa \to X$ such that $u = \sup s[\kappa]$. Since A is dense in the LOTS L, we may assume that $s[\kappa] \subseteq A$. Put $D = s[\kappa]$. Since $|D|$ is measurable, there is a free κ-complete ultrafilter p on D. Then f takes a constant value r_u on some element of p. For each $x < u$, $\{q \in D : q > x\} \in p$, because $|\{q \in D : q \le x\}| < \kappa$. This implies that $\liminf_{x<u} f(x) \le r_u \le \limsup_{x<u} f(x)$. Define $\varphi(f)$ by $\varphi(f)|_A = f$ and $\varphi(f)(u) = r_u$ for each $u \in X \setminus A$. Then $\varphi : C(A) \to C(X)$ is an L_{ch}-extender.

Finally, we show that A is not a retract of X. The order topology of L is identical with the $<\kappa$-box topology. Hence, it is easily proved that L is κ^+-Baire, i.e., L cannot be the union of κ nowhere dense subsets. Now, suppose that there is a retraction $r : X \to A$. Since L is κ^+-Baire, there is $p \in A$ such that $r^{-1}(p)$ is dense in some open interval I in L. Choose $q \in A \cap I$ with $q \ne p$. Then $q \in \mathrm{cl}_L\, r^{-1}(p)$, and hence $q \in \mathrm{cl}_X\, r^{-1}(p) = r^{-1}(p)$ by the definition of the topology of X. Thus $q = r(q) = p$. This contradicts the choice of q. Hence, A is not a retract of X. ∎

The space X in Example 3 is not perfectly normal. We do not know whether the implication (10)⇒(2) holds for every closed subspace of a perfectly normal GO-space assuming no cellularity conditions.

4. Perfectly normal GO-spaces. In this section, we consider extension properties of perfectly normal GO-spaces. For $f, g \in C(X)$, we write $f \le g$ if $f(x) \le g(x)$ for each $x \in X$. For a subset $I \subseteq \mathbb{R}$, a map $\varphi : C(X, I) \to C(Y, I)$ is said to be *monotone* if for each $f, g \in C(X, I)$, $\varphi(f) \le \varphi(g)$ whenever $f \le g$. For a subspace $A \subseteq X$, we call an extender $\varphi : C(A, I) \to C(X, I)$ an M_{ch}-*extender* (resp. M_{cch}-*extender*) if it is monotone and $\varphi(f)[X]$ is included in the convex hull (resp. closed convex hull) of $f[A]$ for each $f \in C(A, I)$. Every L_{cch}-extender is an M_{cch}-extender and every L_{ch}-extender is an M_{ch}-extender. Recall that a *zero-set* of a space X is a set of the form $h^{-1}(0)$ for some $h \in C(X)$.

Theorem 2. *The following hold for a zero-set A of a space X.*

(1) *If there exists an L_{cch}-extender from $C(A)$ to $C(X)$, then there exists an L_{ch}-extender from $C(A)$ to $C(X)$.*

(2) *If there exists an L_{cch}-extender from $C^*(A)$ to $C^*(X)$, then there exists an L_{ch}-extender from $C^*(A)$ to $C^*(X)$.*

(3) *If there exists an $M_{\rm cch}$-extender from $C^*(A)$ to $C^*(X)$, then there exists an $M_{\rm ch}$-extender from $C(A)$ to $C(X)$.*

Proof. We may assume that A is nonempty. Fix a point $a_0 \in A$. Since A is a zero-set, there is $h \in C(X)$ such that $h^{-1}(0) = A$ and $0 \le h(x) \le 1$ for each $x \in X$. Let $\varphi : C(A) \to C(X)$ be an $L_{\rm cch}$-extender. For each $f \in C(A)$, define $\theta(f) \in C(X)$ by $\theta(f)(x) = (1-h(x))\cdot\varphi(f)(x)+h(x)\cdot f(a_0)$ for $x \in X$. Then $\theta : C(A) \to C(X)$ is an $L_{\rm ch}$-extender. The second statement can be proved similarly. To prove the third statement, let $\psi^* : C^*(A) \to C^*(X)$ be an $M_{\rm cch}$-extender and let $I = (-1,1) \subseteq \mathbb{R}$. For each $f \in C(A,I)$, define $\psi(f) \in C^*(X)$ by $\psi(f)(x) = (1 - h(x)) \cdot \psi^*(f)(x) + h(x) \cdot f(a_0)$ for $x \in X$. Then $\psi(f) \in C(X,I)$ and $\psi : C(A,I) \to C(X,I)$ is an $M_{\rm ch}$-extender. Consider the function $g : \mathbb{R} \to I$ defined by $g(x) = x/(1 + |x|)$ for $x \in \mathbb{R}$. Define a monotone map $\mu_1 : C(A) \to C(A,I)$ by $\mu_1(f) = g \circ f$ for $f \in C(A)$ and a monotone map $\mu_2 : C(X,I) \to C(X)$ by $\mu_2(f) = g^{-1} \circ f$ for $f \in C(X,I)$. Then $\mu_2 \circ \psi \circ \mu_1$ is an $M_{\rm ch}$-extender from $C(A)$ to $C(X)$. ■

Statement (1) of Theorem 2 shows that the converse of the implication (9)⇒(10) in Theorem 1 holds for a zero-set A of a GO-space X. By Heath–Lutzer's extension theorem and Theorem 2, we have the following corollary:

COROLLARY 3. *Let X be a perfectly normal GO-space. Then there exists an $L_{\rm ch}$-extender from $C^*(A)$ to $C^*(X)$ for every closed subspace A of X.*

REMARK 4. In [1, Remark IV.5.2], van Douwen asked if there is an $L_{\rm ch}$-extender $\varphi : C^*(A) \to C^*(S)$ for every closed subspace A of the GO-space S quoted before Example 1. Since S is perfectly normal, Corollary 2 answers the question positively. (The question also appears in [14, Question 134], but is misquoted mixing up the space S with the Sorgenfrey line.) For the Michael line M, it is known that there is neither an $L_{\rm ch}$-extender from $C^*(\mathbb{Q})$ to $C^*(M)$ nor a monotone extender from $C(\mathbb{Q})$ to $C(M)$ (cf. van Douwen [1] and Stares–Vaughan [17]).

Heath–Lutzer–Zenor [10] proved that every GO-space is monotonically normal and for every closed subspace A of a monotonically normal space X, there exists a monotone extender $\varphi : C(A,[0,1]) \to C(X,[0,1])$. In [1, Theorem 2.1(23b)], van Douwen proved that if there is a monotone extender from $C(A,[0,1])$ to $C(X,[0,1])$, then there is an $M_{\rm cch}$-extender from $C^*(A)$ to $C^*(X)$. Hence, we have the following corollary by Theorem 2:

COROLLARY 4. *Let X be a perfectly normal, monotonically normal space. Then there exists an $M_{\rm ch}$-extender from $C(A)$ to $C(X)$ for every closed subspace A of X.*

As we have shown in Section 2, there exists a perfectly normal GO-space X with a closed subspace A which satisfies none of conditions (1)–(10) in

Theorem 1. Finally, we give a sufficient condition for A to satisfy those conditions for a closed subspace A of a perfectly normal GO-space X. We need some definitions. For a GO-space $X = (X, \le, \tau)$, let $E(X) = \{x \in X : [x, +\infty) \in \tau \text{ or } (-\infty, x] \in \tau\}$. Let $\lambda(\le)$ be the order topology on $(X, \le)$. For $S \subseteq X$, let $\mathrm{cl}_\lambda S$ denote the closure of S in $(X, \lambda(\le))$ and $\mathrm{cl}_\tau S$ the closure of S in $(X, \le, \tau)$. For $a, b \in X$, if there is no $x \in X$ with $a < x < b$, we write $a = b^-$ and $b = a^+$.

Definition. Let $X = (X, \le, \tau)$ be a GO-space and A a closed subspace. Recall that $U_{A,0} = \bigcup\{U : U \in \mathcal{U}_{A,0}\}$. For $x \in A$, we write $U_{A,0}(<x) = U_{A,0} \cap (-\infty, x)$ and $U_{A,0}(>x) = U_{A,0} \cap (x, +\infty)$. Observe that a point $x \in A$ is in the boundary of A in $X_{A,0}$ if and only if either $x \in \mathrm{cl}_\tau U_{A,0}(<x)$ or $x \in \mathrm{cl}_\tau U_{A,0}(>x)$. A point $x \in A$ is a *singular point* of A if x satisfies one of the following conditions (i) and (ii):

(i) $x \in \mathrm{cl}_\lambda U_{A,0}(<x) \cap \mathrm{cl}_\lambda U_{A,0}(>x)$ and either $x \in \mathrm{cl}_\tau U_{A,0}(<x) \setminus \mathrm{cl}_\tau U_{A,0}(>x)$ or $x \in \mathrm{cl}_\tau U_{A,0}(>x) \setminus \mathrm{cl}_\tau U_{A,0}(<x)$.

(ii) $x \in \{a, b\}$, where $a = b^-$ in X, $a \in \mathrm{cl}_\tau U_{A,0}(<a)$ and $b \in \mathrm{cl}_\tau U_{A,0}(>b)$.

The set of all singular points of A is denoted by $S(A)$.

For example, consider the Cantor set K as a closed subspace of the Sorgenfrey line $\mathbb{S}$. Let K' be the subset of K consisting of all end-points. Let $\mathbb{S}' = \mathbb{S} \setminus K'$ and $A = \mathbb{S}' \cap K$. Then A is a closed subset of $\mathbb{S}'$ and all points in A are singular points of A satisfying condition (i).

On the other hand, in the space X in Example 1, all points of A are singular points of A satisfying condition (ii). Hence, $S(A) = A$.

Theorem 3. *Let X be a perfectly normal GO-space and A a closed subspace such that $S(A)$ is σ-discrete in X. Then A is a retract of $X_{A,0}$, and hence, A satisfies conditions* (2)–(10) *in Theorem* 1.

Proof. As in the proof of $(i) \Rightarrow (2)$ in Theorem 1, we may assume that each element of $\mathcal{U}_{A,0}$ is a singleton, i.e., $U_{A,0}$ is a discrete subspace. Since X is perfectly normal, $U_{A,0}$ is σ-discrete in X. Let Z be the boundary of A in $X_{A,0}$ and let $Y = Z \cup U_{A,0}$, i.e., Y is the closure of $U_{A,0}$ in $X_{A,0}$.

We now show that Y is metrizable. If we prove it, then it follows from [3, Lemma] that Z is a retract of Y, which immediately implies that A is a retract of $X_{A,0}$. We need the following theorem by Faber [6].

Faber's Theorem. *Let S be a GO-space. Then S is perfectly normal if and only if every disjoint family of convex open sets in S is σ-discrete in S. Further, S is metrizable if and only if S has a σ-discrete dense subset D such that $E(S) \subseteq D$.*

We continue the proof of Theorem 3. Since Y is closed in $X_{A,0}$ and $X_{A,0}$ is closed in X, Y is closed in X. Let $\mathcal{V}$ be the family of all convex components of $X \setminus Y$ and put $B = \{x \in Z : (\exists V \in \mathcal{V})(x = l(V) \text{ or } x = r(V))\}$. Then, by Faber's theorem, $\mathcal{V}$ is σ-discrete in X, and hence, so is the set B. Let $C = \{x \in Z : (\exists u \in U_{A,0})(x = u^- \text{ or } x = u^+)\}$. Since $U_{A,0}$ is σ-discrete in X, so is the set C. Let $D = S(A) \cup B \cup C \cup U_{A,0}$. By the assumption, it follows that D is also σ-discrete in X. Finally, let $P = \{x \in Z : x \in \mathrm{cl}_\tau U_{A,0}(<x) \cap \mathrm{cl}_\tau U_{A,0}(>x)\}$ and consider the subspace $Q = D \cup P$ of X. Then, since $U_{A,0} \subseteq D$, D is dense in Q and $E(Q) \subseteq D$. Hence, it follows from Faber's theorem that Q is metrizable. We show that $Y \subseteq Q$. Since $U_{A,0} \subseteq Q$, it is enough to show that $Z \subseteq Q$. Let $x \in Z$. Then, by the definition of Z, either $x \in \mathrm{cl}_\tau U_{A,0}(<x)$ or $x \in \mathrm{cl}_\tau U_{A,0}(>x)$. If $x \in \mathrm{cl}_\tau U_{A,0}(<x) \cap \mathrm{cl}_\tau U_{A,0}(>x)$, then $x \in P \subseteq Q$.

Now, we assume that $x \in \mathrm{cl}_\tau U_{A,0}(<x) \setminus \mathrm{cl}_\tau U_{A,0}(>x)$. We consider two cases:

Case 1: x has no immediate successor in X. If $x = l(V)$ for some $V \in \mathcal{V}$, then $x \in B \subseteq Q$. If $x \neq l(V)$ for each $V \in \mathcal{V}$, then $x = \inf (Y \cap (x, +\infty))$, and hence, $x \in \mathrm{cl}_\lambda U_{A,0}(>x)$. Since $x \in \mathrm{cl}_\tau U_{A,0}(<x)$, $x \in S(A) \subseteq Q$.

Case 2: x has an immediate successor x^+ in X. If $x^+ \notin Y$, then $x^+ \in V$ for some $V \in \mathcal{V}$. Since $x = l(V)$, $x \in B \subseteq Q$. If $x^+ \in U_{A,0}$, then $x \in C \subseteq Q$. If $x^+ \in Z$, then $x^+ \in \mathrm{cl}_\tau U_{A,0}(>x)$. Since $x \in \mathrm{cl}_\tau U_{A,0}(<x)$, $x \in S(A) \subseteq Q$.

Thus, $x \in Q$. If $x \in \mathrm{cl}_\tau U_{A,0}(>x) \setminus \mathrm{cl}_\tau U_{A,0}(<x)$, we can prove that $x \in Q$ similarly. Hence, $Y \subseteq Q$, which implies that Y is metrizable. ■

For a closed subspace A of a GO-space X, $S(A) \subseteq \partial A \cap E(X)$, where ∂A is the boundary of A in X. Hence, we have the following corollary from Theorem 3:

Corollary 5. *Let X be a perfectly normal GO-space and A a closed subspace of X such that $\partial A \cap E(X)$ is σ-discrete in X. Then A satisfies conditions* (2)–(10) *in Theorem* 1.

Remark 5. The set $S(A)$ need not be σ-discrete in X even if A is a retract of a separable GO-space X. In fact, let $\mathbb{S}'$ and A be as defined before Theorem 3. Since the Sorgenfrey line $\mathbb{S}$ is hereditarily retractifiable (cf. van Douwen [1], [2]), A is a retract of $\mathbb{S}'$, but, as we remarked before Theorem 3, $S(A)$ is not σ-discrete.

Now, let $S_2(A)$ be the set of all singular points of A satisfying condition (ii) in the Definition. For the closed set A in the space of Example 1, $S_2(A) = S(A) = A$ is not σ-discrete. We do not know whether Theorem 3 remains true if "$S(A)$" is replaced by "$S_2(A)$".

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Department of Mathematics
Auburn University
Auburn, Alabama 36849
U.S.A.
E-mail: gruengf@mail.auburn.edu

Department of Mathematics
and Computer Science
Shimane University
Matsue, Shimane
690 Japan
E-mail: hattori@math.shimane-u.ac.jp

Faculty of Education
Shizuoka University
Ohya, Shizuoka
422 Japan
E-mail: h-ohta@ed.shizuoka.ac.jp

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