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Isometric embedding into spaces of continuous functions

by

RAFAEL VILLA (Sevilla)

Abstract. We prove that some Banach spaces X have the property that every Banach space that can be isometrically embedded in X can be isometrically and linearly embedded in X. We do not know if this is a general property of Banach spaces. As a consequence we characterize for which ordinal numbers α, β there exists an isometric embedding between $\mathcal{C}_0(\alpha+1)$ and $\mathcal{C}_0(\beta+1)$.

Introduction. Let X and Y be two metric spaces. We say that a transformation F of the space X into Y is an *isometry* or an *isometric embedding* if $d_Y(Fx_1, Fx_2) = d_X(x_1, x_2)$.

In the sequel, we only consider linear spaces over the field $\mathbb{R}$. The letters $F, G, H, \ldots$ will denote arbitrary isometries and $S, T, U, V, \ldots$ linear isometries.

Mazur and Ulam [5] have proved that if X and Y are Banach spaces then each isometry F mapping X onto Y such that $F(0) = 0$ is a linear operator. If the isometry F maps X into Y, the linearity of F does not necessarily hold. Figiel [4] has proved in this case that there exists a unique continuous linear operator U of norm one mapping the closure of the linear hull of the set $F(X)$ into X and such that the superposition $U \circ F$ is the identity on X.

On the other hand, each isometry F mapping a Banach space into a strictly convex Banach space such that $F(0) = 0$ maps middle points to middle points, and hence it is linear.

In §1, we show that each isometry F mapping a Banach space into a space of continuous functions such that $F(0) = 0$ can be "linearized" in such a way that the isometric character is preserved.

In [1], p. 193, the linear dimension for Banach spaces is defined. Two Banach spaces are said to have the same linear dimension if each space is isomorphic to some subspace of the other. An (incomplete) dimensional

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classification of the spaces of all continuous functions defined on ordinal numbers is given in [2].

In the same way, changing isomorphisms to linear isometries, we say that two Banach spaces have the same metric linear dimension if each space is isometrically isomorphic to some subspace of the other.

In §2 we give a complete isometric classification of the spaces of continuous functions defined on ordinal numbers with respect to metric linear dimension. Actually, we show that two such spaces have the same metric linear dimension if and only if they are isometrically isomorphic.

In §3 we apply Theorem 1 to generalize the classification of §2 to non-linear isometries.

Let X be a Banach space. Let $S_r = \{x \in X : \|x\| = r\}$. A point $a \in S_r$ is a *smooth point* if there is only one continuous linear functional $f_a \in X^*$ of norm one such that $f_a(a) = r$.

Mazur proved (see [6], Prop. IX.4.3, p. 401) that the set of all smooth points in a separable Banach space X is a dense G_δ-set in X.

Let X and Y be two Banach spaces. We say that X has *not greater metric linear dimension* than Y, written $X \hookrightarrow Y$, if there is a subspace of Y isometrically isomorphic to X. If $X \hookrightarrow Y$ and $Y \hookrightarrow X$, we say that X and Y have *the same metric linear dimension*, written $X \cong Y$. By $X \not\hookrightarrow Y$ and $X \ncong Y$ we mean the opposites of $X \hookrightarrow Y$ and $X \cong Y$, resp.

In §2 and §3 we follow von Neumann's definition of an ordinal (see [8]). Greek letters will stand for arbitrary ordinal numbers, and the symbol ω will stand for the first infinite ordinal. An ordinal α is called a *prime component* if the condition $\alpha = \beta + \gamma$ implies $\alpha = \gamma$; equivalently, α is a prime component iff $\beta < \alpha$ and $\gamma < \alpha$ imply $\beta + \gamma < \alpha$, iff $\beta < \alpha$ implies $\beta + \alpha = \alpha$, or iff there exists a unique μ such that $\alpha = \omega^\mu$.

For any ordinal α let α' denote the greatest prime component $\leq \alpha$. Every ordinal α has a unique representation in the form $\alpha = \alpha' k + \gamma$ where $0 \leq k < \omega$ and $\gamma < \alpha'$.

Sets of ordinal numbers will always be assumed to be endowed with the order topology. Thus, an ordinal is compact iff it is a successor.

$\mathcal{C}(\alpha+1)$ denotes the Banach space of all continuous real functions defined on $\alpha+1$ with the supremum norm, and $\mathcal{C}_0(\alpha+1) = \{x \in \mathcal{C}(\alpha+1) : x(\alpha) = 0\}$.

1. Isometric embedding into spaces of continuous functions. Let K be a compact topological space and let $\mathcal{C}(K)$ be the space of all continuous functions defined on K. For every $t \in K$ let δ_t be the pointwise evaluation at t, i.e., $\delta_t(x) = x(t)$ for $x \in \mathcal{C}(K)$.

THEOREM 1. *Let K be a compact topological space and let $F : X \to \mathcal{C}(K)$ be an isometric embedding defined on a Banach space X, such that $F(0) = 0$. Then there is a nonempty and closed subset $L \subset K$ such that the superposition $Q \circ F : X \to \mathcal{C}(L)$ is a linear isometry, where Q denotes the mapping restriction to L.*

NOTE. The isometry F is not supposed to be linear or bijective. In the proof it is shown that we can take as L the set

$$L = \{t \in K : \delta_t \circ F \text{ is linear}\}.$$

Proof (of Theorem 1). Suppose first that X is separable.

Let $a \in X$ be a smooth point in the sphere $\{x \in X : \|x\| = \|a\|\}$. Let f_a be the functional of norm one such that $f_a(a) = \|a\|$. For every $n \in \mathbb{N}$, let $t_n \in K$ be such that $|(F(na) - F(-na))(t_n)| = \|F(na) - F(-na)\| = 2n\|a\|$. Thus, for every r with $|r| \leq n$,

$$\begin{aligned} 2n\|a\| = \|F(na) - F(-na)\| &= |(F(na) - F(-na))(t_n)| \\ &\leq |(F(na) - F(ra))(t_n)| + |(F(ra) - F(-na))(t_n)| \\ &\leq \|F(na) - F(ra)\| + \|F(ra) - F(-na)\| \\ &= |n - r| \cdot \|a\| + |n + r| \cdot \|a\| = 2n\|a\|, \end{aligned}$$

therefore, equality holds, and this implies $(n-r)\|a\| = |(F(na) - F(ra))(t_n)|$. Putting $r = 0$, we obtain $|F(na)(t_n)| = n\|a\|$. Let $\varepsilon_n \in \{-1, 1\}$ be such that $\varepsilon_n F(na)(t_n) = n\|a\|$. We claim that $\varepsilon_n F(ra)(t_n) = r\|a\|$ for $|r| \leq n$. Indeed,

$$\begin{aligned} \varepsilon_n(F(na)(t_n) - F(ra)(t_n)) &= n\|a\| - \varepsilon_n F(ra)(t_n) \\ &\geq n\|a\| - |F(ra)(t_n)| \\ &\geq n\|a\| - |r| \cdot \|a\| \geq 0, \end{aligned}$$

and therefore $\varepsilon_n(F(na) - F(ra))(t_n) = |(F(na) - F(ra))(t_n)| = (n - r)\|a\|$, which gives our claim.

By passing to a subsequence, we can assume that $\varepsilon_n = \varepsilon_a \in \{-1, 1\}$ for all n. The sequence $\{t_n\}$ has a cluster point $t_a \in K$. It is easy to check that $F(ra)(t_a) = \varepsilon_a r\|a\|$ for all $r \in \mathbb{R}$. This implies (see [6], Lemma IX.4.6, p. 405) that $f_a(x) = \varepsilon_a F(x)(t_a)$ for $x \in X$. Therefore, $f_a = \varepsilon_a \delta_{t_a} \circ F$ and consequently $\delta_{t_a} \circ F$ is linear. Consider the closed subset of K defined by

$$L = \{t \in K : \delta_t \circ F \text{ is linear}\}.$$

We have proved that for every smooth point $a \in X$ there exists $t_a \in L$ such that $|F(a)(t_a)| = \|a\|$. Since X is separable, the set of all smooth points is dense in X, and thus the superposition $Q \circ F : X \to \mathcal{C}(L)$ is a linear isometry, where Q is the restriction to L.

Now, suppose that X is not separable. We have just proved that for every separable subspace $Y \subset X$, the map $U_Y : Y \to \mathcal{C}(L(Y))$ is a linear isometry, where $L(Y) = \{t \in K : (\delta_t \circ F)_{|Y} \text{ is linear}\}$ and $U_Y(x) = F(x)_{|L(Y)}$. Note that if $Y_1, Y_2 \subset X$ are two separable subspaces, then the closure of their subspace sum, $\overline{Y_1+Y_2}$, is also separable, and $L(\overline{Y_1+Y_2}) \subseteq L(Y_1)\cap L(Y_2)$. Therefore, the family $\{L(Y) : Y \subset X,\ Y \text{ separable}\}$ has the finite intersection property, and then (see [3], Theorem 3.1.1, p. 166) the intersection of the family is nonvoid. Set $L = \{t \in K : \delta_t \circ F \text{ is linear}\}$. It is clear that

$$(*) \qquad L = \bigcap\{L(Y) : Y \subseteq X \text{ separable}\},$$

and so L is a nonvoid closed subset of K. Let $U_0 : X \to \mathcal{C}(L)$ be given by $U_0x = (Fx)_{|L}$. It is a linear operator. It remains to be proved that U_0 is an isometry.

Let $a \in X$ and consider the family (depending on a)

$$\mathcal{A} = \{Y \subset X : a \in Y,\ Y \text{ separable}\}.$$

The family $\mathcal{A}$ is a nonempty directed set. For all $Y \in \mathcal{A}$, we have $\|U_Y(a)\| = \|a\|$, and therefore there exists $t(Y) \in L(Y)$ such that $|(Ta)(t(Y))| = \|a\|$. We have a net $\{t(Y) : Y \in \mathcal{A}\}$ in the compact set K. Then (see [3], Theorem 3.1.23, p. 172), the net has a cluster point $t_0 \in K$, which means that for any neighbourhood V of t_0 and for any $Y_0 \in \mathcal{A}$, there exists $Y \in \mathcal{A}$ such that $Y \supseteq Y_0$ and $t(Y) \in V$. From this $|(Ta)(t_0)| = \|a\|$ follows. It remains to be proved that $t_0 \in L$. According to $(*)$, it is enough to prove that $t_0 \in L(Y)$ for all $Y \in \mathcal{A}$. If there existed $Y_0 \in \mathcal{A}$ such that $t_0 \notin L(Y_0)$, we could find a neighbourhood V of t_0 satisfying $V \cap L(Y_0) = \emptyset$. Since t_0 is a cluster point of the net $\{t(Y) : Y \in \mathcal{A}\}$, there would exist $Y \in \mathcal{A}$ such that $Y \supseteq Y_0$ and $t(Y) \in V$, contrary to the fact that $t(Y) \in L(Y) \subseteq L(Y_0)$. ∎

If the compact set K is metrizable, then using the simultaneous extension theorem due to Borsuk and Dugundji (see [7], Proposition 21.1.4, p. 365) we obtain a linear isometry into the original space $\mathcal{C}(K)$.

THEOREM 2. *Let K be a metrizable compact topological space and let $F : X \to \mathcal{C}(K)$ be an isometry defined on a Banach space X, such that $F(0) = 0$. Then there exists a linear isometry $T : X \to \mathcal{C}(K)$.*

NOTE. According to the Borsuk–Dugundji Theorem, we can assert that, for any $x \in X$,

$$(Tx)(K) \subset \mathrm{co}((Fx)(K)).$$

2. Metric linear dimension of the spaces of continuous functions.

Since $\mathcal{C}(\alpha+1) \cong \mathcal{C}_0(\alpha+2)$, it is enough to classify the spaces $\mathcal{C}_0(\alpha+1)$.

If $\alpha < \beta$, then there exists a linear isometry $T : \mathcal{C}_0(\alpha+1) \to \mathcal{C}_0(\beta+1)$, defined by $Tx = y$ where $y(\gamma) = x(\gamma)$ for $\gamma \le \alpha$ and $y(\gamma) = 0$ for $\gamma > \alpha$.

The following result provides a complete classification of the spaces $\mathcal{C}_0(\alpha+1)$ with respect to metric linear dimension.

THEOREM 3. *If $\alpha = \alpha' n + \gamma + 1$ and $\beta = \beta' m + \delta + 1$, with $\gamma < \alpha'$ and $\delta < \beta'$, then:*

(a) *If α is finite, then $\mathcal{C}_0(\alpha) \cong \mathcal{C}_0(\beta)$ if and only if $\alpha = \beta$.*

(b) *If α is infinite, then $\mathcal{C}_0(\alpha) \cong \mathcal{C}_0(\beta)$ if and only if the following three conditions are satisfied:* (1) $\alpha' = \beta'$; (2) $n = m$; *and* (3) $\gamma, \delta > 0$ *or* $\gamma = \delta = 0$.

The proof of Theorem 3 is based on the following lemmas.

LEMMA 1. (a) *$\mathcal{C}(\alpha+\beta) \cong \mathcal{C}(\alpha) \times \mathcal{C}(\beta)$ for arbitrary compact ordinals α and β, and so $\mathcal{C}(\sum_{i=1}^n \alpha_i) \cong \mathcal{C}(\sum_{i=1}^n \alpha_{\pi(i)})$ for arbitrary compact ordinals $\alpha_1, \ldots, \alpha_n$ and every permutation π of $\{1, \ldots, n\}$.*

(b) *If α is a prime component, $\gamma < \alpha$ and $k \in \mathbb{N}$ then $\mathcal{C}(\alpha k + \gamma + 1) \cong \mathcal{C}(\alpha k + 1)$.*

Proof. (a) The map $T : \mathcal{C}(\alpha+\beta) \to \mathcal{C}(\alpha) \times \mathcal{C}(\beta)$ defined by $T(z) = (x, y)$ where $x(t) = z(t)$ for $0 \le t < \alpha$ and $y(t) = z(\alpha + t)$ for $0 \le t < \beta$ is a one-to-one linear isometry.

(b) follows from (a) and the equality $\gamma + \alpha k = \alpha k$. ∎

LEMMA 2. *Let α be an ordinal. The following conditions are equivalent:*

(i) *If $\gamma < \alpha$ then $\mathcal{C}(\alpha+1) \not\hookrightarrow \mathcal{C}(\gamma+1)$.*

(ii) $\mathcal{C}(\alpha+1) \not\hookrightarrow \mathcal{C}_0(\alpha+1)$.

Proof. Suppose (i) holds and (ii) is false. Let

$$T : \mathcal{C}(\alpha+1) \to X \subset \mathcal{C}_0(\alpha+1)$$

be a one-to-one linear isometry. Let $U : X \to \mathcal{C}(\alpha+1)$ be the inverse map of T. We have $\|Ux\| = \|x\|$ for all $x \in X$.

Let $y_0 \in \mathcal{C}(\alpha+1)$ be the function identically equal to 1 and $x_0 = Ty_0$. Since $x_0 \in X \subset \mathcal{C}_0(\alpha+1)$, there exists $\gamma < \alpha$ such that $|x_0(t)| < 1/2$ for $t > \gamma$.

Let $P : \mathcal{C}(\alpha+1) \to \mathcal{C}(\gamma+1)$ be the "restriction" to $\gamma+1$, more exactly $Px = z$ where $z(t) = x(t)$ for $t \le \gamma$, and consider the superposition $P \circ T : \mathcal{C}(\alpha+1) \to \mathcal{C}(\gamma+1)$. According to (i), this map cannot be any isometry. Hence, there exists y_1 in $\mathcal{C}(\alpha+1)$ such that $\|y_1\| = 1$ and $\|P \circ T(y_1)\| < 1$. Put $x_1 = Ty_1 \in X \subset \mathcal{C}_0(\alpha+1)$, and take ξ such that $|y_1(\xi)| = 1$.

Let $\sigma = \mathrm{sgn}(y_1(\xi))$ and $z = x_0 + \sigma x_1 \in X$. We have

$$\|z\| = \|U(z)\| = \|y_0 + \sigma y_1\| \ge y_0(\xi) + \sigma y_1(\xi) = 2.$$

On the other hand, let s_0 be an ordinal such that $|z(s_0)| = \|z\|$. Now if $s_0 \le \gamma$ then $|x_1(s_0)| \le \|Px_1\| < 1$ and $|x_0(s_0)| \le 1$, and if $s_0 > \gamma$ then $|x_0(s_0)| < 1/2$ and $|x_1(s_0)| \le 1$. In both cases we have $\|z\| < 2$, which is impossible.

Now suppose (ii) holds. If there existed an ordinal number $\gamma < \alpha$ such that $\mathcal{C}(\alpha+1) \hookrightarrow \mathcal{C}(\gamma+1)$, then we would easily define the following linear isometric embedding:

$$\mathcal{C}(\alpha+1) \hookrightarrow \mathcal{C}(\gamma+1) \cong \mathcal{C}_0(\gamma+2) \hookrightarrow \mathcal{C}_0(\alpha+1),$$

contrary to (ii). ■

Now we are interested in proving that there is no linear isometry of $\mathcal{C}_0(\alpha(k+1)+1)$ into $\mathcal{C}(\alpha k+1)$ for any prime component α. For this purpose, we need the following notations.

Let α be a prime component, $\alpha \neq \omega$, $\beta \geq \alpha$ and let $T : \mathcal{C}_0(\alpha+1) \to \mathcal{C}(\beta+1)$ be a linear isometry.

For every open and closed set $A \subseteq \alpha$ we have $\chi_A \in \mathcal{C}_0(\alpha+1)$ and $\|\chi_A\| = 1$. Let

$$\langle A, T\rangle = \{\gamma \leq \beta : |T(\chi_A)(\gamma)| = 1\}.$$

Then $\langle A, T\rangle$ is a nonempty closed set. The following properties are easily checked:

(a) $A_1 \cap \ldots \cap A_p = \emptyset \Leftrightarrow \langle A_1, T\rangle \cap \ldots \cap \langle A_p, T\rangle = \emptyset$.
(b) $A \subset B \Rightarrow \langle A, T\rangle \subset \langle B, T\rangle$.

Let A be a subset of a topological space X. The *derived set* of A is the set $A^{(1)}$ of all accumulation points of A (it depends on both A and X). If ξ is an ordinal, we define the ξth derived set by transfinite induction:

$$A^{(0)} = A, \quad A^{(\xi+1)} = (A^{(\xi)})^{(1)}, \quad A^{(\lambda)} = \bigcap_{\xi<\lambda} A^{(\xi)}$$

if λ is a noncompact ordinal (see [7], p. 147).

Let $\eta > 1$ be such that $\alpha = \omega^\eta$. For any ξ with $1 \leq \xi < \eta$, let $A_\xi = [1, \omega^\xi]$.

LEMMA 3. *Let $\alpha = \omega^\eta$ and $\beta \geq \alpha$. For every ξ with $1 \leq \xi < \eta$ and linear isometry $T : \mathcal{C}_0(\alpha+1) \to \mathcal{C}(\beta+1)$, we have $\langle A_\xi, T\rangle^{(\xi)} \neq \emptyset$.*

NOTE. This lemma implies that for any $\lambda < \xi$, $\langle A_\xi, T\rangle \not\subset [0, \omega^\lambda]$.

Proof (of Lemma 3). The proof is by transfinite induction.

The case $\xi = 1$ follows from the fact that the set $\langle A_1, T\rangle$ is infinite, because it contains the sets $\langle \{n\}, T\rangle$ for every $n \in \mathbb{N}$, which are disjoint and nonempty.

Assume the lemma holds for the ordinal number ξ; we prove it for $\xi+1$. We have

$$A_{\xi+1} \supset \bigcup_{k=0}^{\infty} I_k,$$

where $I_k = [\omega^\xi k + 1, \omega^\xi (k+1)]$. The sets $\langle I_k, T\rangle$ are pairwise disjoint and closed, and so are the ξth derived sets. All of them are contained in $\langle A_{\xi+1}, T\rangle^{(\xi)}$.

Consider the continuous one-to-one map $\varphi_k : \alpha+1 \to \alpha+1$ defined by

$$\varphi_k(t) = \begin{cases} t & \text{if } t \notin I_k \cup A_\xi, \\ \omega^\xi k + t & \text{if } t \in A_\xi, \\ -\omega^\xi k + t & \text{if } t \in I_k, \end{cases}$$

and the isometry $T_k : \mathcal{C}_0(\alpha+1) \to \mathcal{C}(\beta+1)$ defined by $T_k x = T(x \circ \varphi_k)$. One can easily establish that $\langle I_k, T\rangle = \langle A_\xi, T_k\rangle$ and by the inductive hypothesis we have $\langle I_k, T\rangle^{(\xi)} \neq \emptyset$. It follows that $\langle A_{\xi+1}, T\rangle^{(\xi)}$ is infinite, and so it has an accumulation point.

Now assume the lemma holds for any $\xi < \lambda$, where λ is a noncompact ordinal; we prove it for λ. Since $\langle A_\lambda, T\rangle^{(\lambda)} = \bigcap_{\xi<\lambda} \langle A_\lambda, T\rangle^{(\xi)}$, what is left is to show that the family $\{\langle A_\lambda, T\rangle^{(\xi)} : \xi < \lambda\}$ has the finite intersection property. But if $\xi_1, \ldots, \xi_n < \lambda$ and $\xi = \max\{\xi_1, \ldots, \xi_n\}$ then $\langle A_\lambda, T\rangle^{(\xi_1)} \cap \ldots \cap \langle A_\lambda, T\rangle^{(\xi_n)} = \langle A_\lambda, T\rangle^{(\xi)}$ and this set contains $\langle A_\xi, T\rangle^{(\xi)}$, which is nonempty. ■

LEMMA 4. *If α is a prime component and $k \in \mathbb{N}$, then there is no linear isometry of $\mathcal{C}_0(\alpha(k+1)+1)$ into $\mathcal{C}(\alpha k+1)$.*

Proof. It is easy to check that $\mathcal{C}_0(\alpha(k+1)+1) \cong \mathcal{C}(\alpha+1)^k \times \mathcal{C}_0(\alpha+1)$ and $\mathcal{C}(\alpha k+1) \cong \mathcal{C}(\alpha+1)^k$. Suppose the assertion of the lemma is false, and let $T : \mathcal{C}(\alpha+1)^k \times \mathcal{C}_0(\alpha+1) \to \mathcal{C}(\alpha+1)^k$ be a linear isometry. For every p with $1 \leq p \leq k$, let $T_p : \mathcal{C}(\alpha+1) \to \mathcal{C}(\alpha+1)^k$ be the restriction to the pth coordinate, and let $T_{k+1} : \mathcal{C}_0(\alpha+1) \to \mathcal{C}(\alpha+1)^k$ be the restriction to the $(k+1)$th coordinate. All of them are linear isometries. Suppose $\alpha > \omega$ and let η be such that $\alpha = \omega^\eta$. For every ξ with $1 \leq \xi < \eta$, every p with $1 \leq p \leq k+1$ and every i with $1 \leq i \leq k$, let $x_i^{p,\xi} \in \mathcal{C}(\alpha+1)$ be the vector such that $T_p(\chi_{A_\xi}) = (x_1^{p,\xi}, \ldots, x_k^{p,\xi})$.

Fix p with $1 \leq p \leq k+1$. By Lemma 3, for every ξ with $1 \leq \xi < \eta$, there exist $\gamma > \omega^\xi$ and $i \in \{1, \ldots, k\}$ such that $|x_i^{p,\xi}(\gamma)| = 1$. Since i takes only a finite number of different values, there exists an $i = i(p) \in \{1, \ldots, k\}$, depending on p, such that for every $\xi < \eta$ there exists a $\gamma > \omega^\xi$ such that $|x_i^{p,\xi}(\gamma)| = 1$.

Let $y \equiv 1 \in \mathcal{C}(\alpha+1)$ and $B_\xi = (\alpha+1) \setminus A_\xi$. Set $z^p = (z_1^p, \ldots, z_k^p) = T_p(y) = T_p(\chi_{A_\xi}) + T_p(\chi_{B_\xi})$, where $z_j^p \in \mathcal{C}(\alpha+1)$. For every $\xi < \eta$ there exists $\gamma > \omega^\xi$ such that $|x_i^{p,\xi}(\gamma)| = 1$, and hence $(T_p(\chi_{B_\xi}))_j(\gamma) = 0$ for every $j \in \{1, \ldots, k\}$. From this, $|z_i^p(\gamma)| = 1$. As $z_i^p \in \mathcal{C}(\alpha+1)$ we have $|z_i^p(\alpha)| = 1$.

We consider two cases.

CASE I: The numbers $i(p)$ are different for each $1 \leq p \leq k$. Then there exists a p with $1 \leq p \leq k$ such that $i(p) = i(k+1)$. If we take $\varepsilon \in \{-1, 1\}$ and $\gamma_0 < \alpha$ in such a way that $\varepsilon z_i^p(\gamma) > 1/2$ for $\gamma > \gamma_0$, $\xi < \eta$ such that

$\omega^\xi > \gamma_0$, and $\gamma > \omega^\xi$ such that $x_i^{k+1,\xi}(\gamma) = \delta \in \{-1,1\}$, then we obtain

$$\begin{aligned} 1 &= \|T(0,\ldots,0,\overset{(p)}{\varepsilon y},0,\ldots,0,\delta\chi_{A_\xi})\| = \|\varepsilon z^p + \delta T_{k+1}(\chi_{A_\xi})\| \\ &\geq \varepsilon z_i^p(\gamma) + \delta x_i^{k+1,\xi}(\gamma) > 3/2, \end{aligned}$$

a contradiction.

CASE II: There exist p and q with $1 \le p < q \le k$ such that $i(p) = i(q)$. We have $z_i^p(\alpha) = \varepsilon \in \{-1,1\}$ and $z_i^q(\alpha) = \delta \in \{-1,1\}$, and hence

$$\begin{aligned} 1 &= \|T(0,\ldots,0,\overset{(p)}{\varepsilon y},0,\ldots,0,\overset{(q)}{\delta y},0,\ldots,0)\| = \|\varepsilon z^p + \delta z^q\| \\ &\geq \varepsilon z_i^p(\alpha) + \delta z_i^q(\alpha) = 2, \end{aligned}$$

which is impossible.

The same proof, using the sets $\{n\}$ for $n \in \mathbb{N}$ and A_ω, works for the case $\alpha = \omega$. ■

LEMMA 5. *If α is a prime component, $k \in \mathbb{N}$ and $\beta < \alpha k$, then* $\mathcal{C}_0(\alpha k+1) \not\hookrightarrow \mathcal{C}(\beta+1)$.

Proof. We may assume that $\beta \geq \omega$, since otherwise the result is trivial.

Put $\beta = \beta' n + \gamma$, and suppose $k = 1$. Hence $\beta < \alpha$ implies $\beta' < \alpha$. Therefore $\beta'\omega \le \alpha$ since α and β' are prime components. If the lemma were false, then we could easily define an isometric embedding $\mathcal{C}_0(\beta'(n+1)+1) \hookrightarrow \mathcal{C}_0(\alpha+1) \hookrightarrow \mathcal{C}(\beta+1) \hookrightarrow \mathcal{C}(\beta' n+1)$, which contradicts Lemma 4.

Now suppose $k > 1$. Hence $\beta < \alpha$ implies $\beta' < \alpha$, or $\beta' = \alpha$ and $n < k$. In the first case, if the assertion of the lemma were false, then we could find an isometric embedding $\mathcal{C}_0(\alpha k+1) \hookrightarrow \mathcal{C}(\beta+1) \hookrightarrow \mathcal{C}(\alpha(k-1)+1)$, contrary to Lemma 4. In the second case, if the lemma were false, we could define an isometric embedding $\mathcal{C}_0(\alpha k+1) \hookrightarrow \mathcal{C}(\beta+1) \hookrightarrow \mathcal{C}(\alpha(k-1)+\gamma+1) \hookrightarrow \mathcal{C}(\alpha(k-1)+1)$, which contradicts Lemma 4. ■

Proof of Theorem 3. (a) is trivial.

(b) By Lemma 1(a), we have an isometric embedding

$$\mathcal{C}_0(\alpha' n+\gamma+1) \hookrightarrow \mathcal{C}(\alpha' n+\gamma+1) \cong \mathcal{C}(\alpha' n+1) \cong \mathcal{C}_0(\alpha' n+2).$$

On the other hand, if $\beta < \alpha' n+1$ then, by Lemma 5, $\mathcal{C}_0(\alpha' n+1) \not\hookrightarrow \mathcal{C}(\beta)$, and hence $\mathcal{C}_0(\alpha' n+1) \not\hookrightarrow \mathcal{C}_0(\beta)$, and if $\beta > \alpha' n+1$ then, by Lemma 5, for every $\eta < \alpha' n$, we have $\mathcal{C}_0(\alpha' n+1) \not\hookrightarrow \mathcal{C}(\eta+1)$, and Lemma 2 shows that $\mathcal{C}(\alpha' n+1) \not\hookrightarrow \mathcal{C}_0(\alpha' n+1)$ and hence $\mathcal{C}_0(\beta) \not\hookrightarrow \mathcal{C}_0(\alpha' n+1)$. ■

3. Extension to the nonlinear case.

Theorem 1 allows one to characterize for which ordinal numbers α, β there exists an isometric embedding of $\mathcal{C}_0(\alpha+1)$ into $\mathcal{C}_0(\beta+1)$.

THEOREM 4. *Let α and β be arbitrary ordinals. There exists an isometric embedding of $\mathcal{C}_0(\alpha+1)$ into $\mathcal{C}_0(\beta+1)$ if and only if there exists a linear isometric embedding of $\mathcal{C}_0(\alpha+1)$ into $\mathcal{C}_0(\beta+1)$.*

Proof. Let $F : \mathcal{C}_0(\alpha+1) \to \mathcal{C}_0(\beta+1)$ be an isometric embedding. We can assume that $F(0) = 0$. By Theorem 1, there exists a closed subset L of $\beta+1$ such that $U : \mathcal{C}_0(\alpha+1) \to \mathcal{C}(L)$ is a linear isometric embedding, where $Ux = Fx_{|L}$. It is easily seen that there exists a linear isometry $\Phi : \mathcal{C}(L) \to \mathcal{C}(\beta+1)$ such that $(\Phi \circ U(x))(\beta) = 0$ and therefore $\Phi \circ U$ is a linear isometric embedding of $\mathcal{C}_0(\alpha+1)$ into $\mathcal{C}_0(\beta+1)$. ■

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Departamento de Análisis Matemático
Facultad de Matemáticas
Universidad de Sevilla
c/Tarfia, s/n
41012 Sevilla, Spain
E-mail: rvcaro@cica.es

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