According to Proposition 2.1 of [FöKr96], there exists a large number $\mu \in \mathbb{C}$ such that $\text{min.ind}(T + \mu F - \lambda) = 0$ for all $\lambda \in \Omega$. We have $(\mu F)^2 = 0$, because $R(\mu F) = L \subseteq W = N(\mu F)$. Finally, $\dim R(\mu F) = \dim L = \text{Max}$. Hence the operator μF has the desired properties. ■

References

[Bo88] Z. Boulmaarouf, *The Laffey–West decomposition*, Proc. Roy. Irish Acad. Sect. A 88 (1988), 125–131.

[Bo90] —, *Décomposition de Laffey–West globale*, thèse, l'Université de Nice, 1990.

[FöJa92] K.-H. Förster and K. Jahn, *Extremal compressions and perturbations of bounded operators*, ibid. 92 (1992), 289–296.

[FöKr96] K.-H. Förster and M. Krause, *On extremal perturbations of semi-Fredholm operators*, Integral Equations Operator Theory 26 (1996), 125–135.

[Is87] G. G. Islamov, *Control of the spectrum of a dynamical system*, Differential Equations 23 (1987), 872–875.

[Ka58] T. Kato, *Perturbation theory for nullity, deficiency and other quantities of linear operators*, J. Analyse Math. 6 (1958), 261–322.

[Ka66] —, *Perturbation Theory for Linear Operators*, Springer, New York, 1966.

[Kr95] T. Kröncke, *Extremale Störungen von Fredholmoperatoren*, Diplomarbeit, Technische Universität Berlin, 1995.

[LaWe82] T. J. Laffey and T. T. West, *Fredholm commutators*, Proc. Roy. Irish Acad. Sect. A 87 (1987), 137–146.

[MaSe78] A. S. Markus and A. A. Sementsul, *Operators which weakly perturb the spectrum*, Sibirsk. Mat. Zh. 19 (1978), 646–653 (in Russian).

[Mb93] M. Mbekhta, *Semi-Fredholm perturbations and commutators*, Math. Proc. Cambridge Philos. Soc. 113 (1993), 173–177.

[Ó S88] M. Ó Searcóid, *Economical finite rank perturbations of semi-Fredholm operators*, Math. Z. 198 (1988), 431–434.

[Sł86] Z. Słodkowski, *Operators with closed ranges in spaces of analytic vector-valued functions*, J. Funct. Anal. 69 (1986), 155–177.

[We90] T. T. West, *Removing the jump-Kato's decomposition*, Rocky Mountain J. Math. 20 (1990), 603–612.

[Ze92] J. Zemánek, *An analytic Laffey–West decomposition*, Proc. Roy. Irish Acad. Sect. A 92 (1992), 101–106.

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On regularization in superreflexive Banach spaces by infimal convolution formulas

by

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Abstract. We present here a new method for approximating functions defined on superreflexive Banach spaces by differentiable functions with α-Hölder derivatives (for some $0 < \alpha \le 1$). The smooth approximation is given by means of an explicit formula enjoying good properties from the minimization point of view. For instance, for any function f which is bounded below and uniformly continuous on bounded sets this formula gives a sequence of Δ-convex $C^{1,\alpha}$ functions converging to f uniformly on bounded sets and preserving the infimum and the set of minimizers of f. The techniques we develop are based on the use of *extended inf-convolution* formulas and convexity properties such as the preservation of smoothness for the convex envelope of certain differentiable functions.

0. Introduction and preliminaries. This paper introduces an explicit regularization procedure for functions defined on superreflexive Banach spaces. For any function f bounded below and l.s.c. (resp. uniformly continuous on bounded sets) on a superreflexive Banach space X we give by means of a "standard" formula a sequence of $C^{1,\alpha}$-smooth functions converging pointwise (resp. uniformly on bounded sets) to f (where $0 < \alpha \le 1$ only depends on X). Under some additional conditions, the convergence of the sequence of approximate functions is uniform on the whole space X. Moreover, the approximate functions preserve the infimum and the set of minimizers of f. These features cannot be easily obtained from regularization methods like the *smooth partition of unity* techniques (for a detailed study of this topic we refer to Chapter VIII.3 of [DGZ], the references therein and [Fr]) or other results that only ensure the existence of smooth approximants (for instance, see [DFH]).

In Hilbert spaces, our work is closely linked with the *Lasry–Lions approximation method* (introduced in [LL] and subsequently studied by several

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authors, such as [AA]) and its more general version given by T. Strömberg in [St$_2$]. Actually, we improve the results of [St$_2$] in the superreflexive case by providing the best uniformly smooth approximation possible for this setting. Nonetheless, we want to remark that the approximate functions explained herein cannot be reduced to those of Strömberg (or Lasry–Lions approximants in Hilbert spaces); we refer to the remark after Proposition 8 for a more precise explanation. Our approach to smooth regularization in non-Hilbert spaces comes from two main facts: the density of the linear span of the convex functions (studied in [C]) and the smoothness of the convex envelope of a "somehow" smooth function. In this direction, we also present more general versions of certain results in [GR] for infinite-dimensional Banach spaces.

This paper is organized in the following way. Our main result, Theorem 1, and several corollaries are explained in Section 1. The proof of Theorem 1 is given in Section 4 with the tools provided by Sections 2 and 3. Section 2 deals with the existence of approximants for a given function f using some results on extended inf-convolution formulas. Section 3 develops a procedure for regularizing certain Δ-convex approximate functions. This procedure is based on the smoothness of the convex envelope of certain "somehow" smooth functions.

NOTATION. In what follows, X denotes a Banach space and $\|\cdot\|$ an equivalent norm on X. We denote by B_X the unit closed ball of X under the norm $\|\cdot\|$ and by $B_X(r)$ the closed ball of radius $r>0$. A function $f: X \to \mathbb{R}\cup\{+\infty\}$ is called *proper* if $f \not\equiv +\infty$; $\mathfrak{S}_{\inf}(f)$ is the (possibly empty) set $\{x \in X : f(x) = \inf f\}$. We will deal with the following types of convergence in the set of lower semicontinuous (for short, l.s.c.) functions on X: pointwise, compact, uniform on bounded sets and uniform on X, abbreviated respectively by $\tau_{\rm p}$, $\tau_{\rm K}$, $\tau_{\rm b}$ and $\tau_{\rm u}$.

A function defined on X is called *Δ-convex* if it can be expressed as the difference of two continuous convex functions. The *convex envelope* $\operatorname{co} f$ of a function $f: X \to \mathbb{R}\cup\{+\infty\}$ is defined as the greatest proper convex l.s.c. function below f (if there exists a convex minorant of f). The explicit value of $\operatorname{co} f$ at $x\in X$ is

$$(1)\qquad \inf_{n\in\mathbb{N}}\Big\{\sum_{i=1}^{n}\lambda_i f(x_i) : x=\sum_{i=1}^{n}\lambda_i x_i,\ \sum_{i=1}^{n}\lambda_i=1,\ (x_i,\lambda_i)_{i=1}^{n}\subset X\times\mathbb{R}_+\Big\}.$$

Unless otherwise stated, differentiability will be understood in the *Fréchet sense.* The following notation is used throughout. $C^{1,\mathrm{u}}(X)$ (respectively $C^{1,\mathrm{u}}_B(X)$) stands for the set of differentiable functions defined on X with uniformly continuous derivative (resp. with derivative uniformly continuous on bounded sets). Similarly, $C^{1,\alpha}(X)$ (resp. $C^{1,\alpha}_B(X)$) stands for the class of functions on X having α-Hölder continuous derivative (resp. having derivative α-Hölder continuous on bounded sets) $(0<\alpha\le 1)$.

1. The main result.

We begin by stating the main result of this work.

THEOREM 1. *Let $p>1$, X be a Banach space and $\|\cdot\|$ be an equivalent norm on X which is locally uniformly convex and uniformly smooth. For any proper lower semicontinuous function $f: X\to\mathbb{R}\cup\{+\infty\}$ bounded below, consider the sequence of Δ-convex functions given by*

$$\Delta^p_n f := \operatorname{co} g^p_n - 2^{p-1}n\|\cdot\|^p \qquad (n\in\mathbb{N}),$$

where

$$g^p_n(x) := \inf_{y\in X}\{f(y) + 2^{p-1}n\|x\|^p + 2^{p-1}n\|y\|^p - n\|x+y\|^p\} + 2^{p-1}n\|x\|^p.$$

Then

(i) *For all n, $\inf f \le \Delta^p_n f \le f$ and $\mathfrak{S}_{\inf}(\Delta^p_n f) = \mathfrak{S}_{\inf}(f)$.*

(ii) *$(\Delta^p_n f)_{n\in\mathbb{N}} \subset C^{1,\mathrm{u}}_B(X)$ and $(\Delta^p_n f)_{n\in\mathbb{N}} \subset C^{1,\alpha}_B(X)$ provided that the modulus of smoothness of the norm $\|\cdot\|$ is of power type $1+\alpha$; actually, $(\Delta^{1+\alpha}_n f)_{n\in\mathbb{N}} \subset C^{1,\alpha}(X)$.*

(iii) *$\Delta^p_n f \xrightarrow{\tau_{\rm p}} f$ pointwise and $\Delta^p_n f \xrightarrow{\tau_{\rm K}} f$ as $n\to\infty$ if $f: X\to\mathbb{R}$ is continuous.*

If moreover the norm $\|\cdot\|$ is uniformly convex then:

(iv) *$\Delta^p_n f \xrightarrow{\tau_{\rm b}} f$ as $n\to\infty$ whenever f is uniformly continuous on bounded sets.*

(v) *$\Delta^p_n f \xrightarrow{\tau_{\rm u}} f$ as $n\to\infty$ provided that f is uniformly continuous on X (not necessarily bounded below) and the modulus of convexity of the norm $\|\cdot\|$ is of power type p $(p\ge 2)$.*

REMARK. It is well known that the existence of a uniformly smooth norm on a Banach space X implies the superreflexivity of X (and conversely, the articles [E] of P. Enflo and [Pi] of G. Pisier tell us that any superreflexive Banach space admits an equivalent uniformly smooth norm). Similarly, we want to point out that the conclusions of Theorem 1 cannot be expected to hold outside the superreflexive setting.

First, the $\tau_{\rm b}$-density of the set of Δ-convex functions defined on X in the set of functions on X that are uniformly continuous on bounded sets is equivalent to the superreflexivity of X (see in [C]). On the other hand, the existence of $C^{1,\alpha}$ bump functions (for some $0<\alpha\le1$) on X implies the existence of an equivalent norm on X with modulus of smoothness of power type $1+\alpha$ (see Theorem V.3.1 of [DGZ]).

REMARK. The optimal application of Theorem 1 is for a Hilbertian norm $\|\cdot\|$. In this case, taking $p=2$ in Theorem 1 we obtain approximation results similar to those given by the Lasry–Lions approximation method (see [LL]). Nevertheless, the sequences of approximants are not the same even in this setting (see remark after Proposition 8).

We proceed to state some corollaries to Theorem 1. They are related to certain results following from the existence of *smooth partitions of unity* on superreflexive Banach spaces (see Theorem VIII.3.2 of [DGZ]). Their proof is easily obtained by appealing to Theorem 1 and Pisier's renorming theorem (the original proof can be found in [Pi]; we refer to [L] for a simpler and more geometrical proof).

The first corollary improves Corollary 1 of [St$_2$] for superreflexive Banach spaces.

COROLLARY 2. *Let X be a superreflexive Banach space. Then there exists some $0 < \alpha \le 1$ such that any non-empty closed set F of X is the set of zeros of a Δ-convex $C^{1,\alpha}$-differentiable function on X. Moreover, F is the limit in the Hausdorff distance of a sequence of sets*

$$\mathfrak{S}_n = \{x \in X : f_n(x) < \sigma_n \in \mathbb{R}\} \quad (n \in \mathbb{N}),$$

where the functions $(f_n)_n$ are Δ-convex and in $C_B^{1,\alpha}(X)$.

Proof. Pisier's renorming theorem ensures the existence of an equivalent norm $\|\cdot\|$ on X with modulus of smoothness of power type q $(1 < q \le 2)$. Given a closed set F in X, consider the proper function d defined by $d(x) := \operatorname{dist}(x, F) = \inf_{y \in F} \|x - y\|$ for $x \in X$ (d is proper because F is not empty). By Theorem 1(i)–(ii), $\Delta_1^q d$ is Δ-convex, $C^{1,q-1}$-differentiable and satisfies $\mathfrak{S}_{\inf}(\Delta_1^q(d)) = \mathfrak{S}_{\inf}(d) = F$.

Moreover, using the Asplund averaging technique (see Proposition IV.5.2 of [DGZ]), we can assume that the modulus of convexity of $\|\cdot\|$ is in addition of power type p (for some $p \ge 2$). Since d is Lipschitz continuous on X, from Theorem 1(iv) it follows for every n that

$$F \subseteq \{x \in X : \Delta_n^p d(x) < 1/n\} := \mathfrak{S}_n,$$

where $\Delta_n^p d$ is a Δ-convex $C_B^{1,q-1}$-differentiable function and $(\mathfrak{S}_n)_n$ converges to F in the Hausdorff distance. ∎

The next corollary gives a slightly stronger version of some other approximation results obtained by using partition of unity techniques (for instance, see Theorem 1 of [NS]).

COROLLARY 3. *For any superreflexive Banach space X there is $0 < \alpha \le 1$ so that every function f on X uniformly continuous on bounded sets (resp. uniformly continuous) is the uniform limit on any fixed bounded subset B of X (resp. on X) of a sequence of Δ-convex $C^{1,\alpha}$-differentiable (resp. $C_B^{1,\alpha}$-differentiable) functions having the same infimum and set of minimizers on B as f.*

Proof. Appealing again to Pisier's renorming theorem, we can suppose that there is an equivalent norm $\|\cdot\|$ on X with modulus of smoothness of power type q $(1 < q \le 2)$. Fix some bounded set B of X and define $\widetilde{f} := \max\{f, \inf_B f\}$. Since f is uniformly continuous on B, we have $\inf_B f > -\infty$. Therefore, $\widetilde{f}$ is uniformly continuous on bounded sets and bounded below. Note that trivially $\widetilde{f}(x) = f(x)$ for all $x \in B$ and so the infimum and set of minimizers on B of f and $\widetilde{f}$ are the same. Hence, Theorem 1(ii), (vi) tells us that the sequence $(\Delta_n^q \widetilde{f})_n$ satisfies the conditions of the claim for $\alpha = q - 1$.

If f is uniformly continuous on X, the proof follows the same lines, using the existence of an equivalent norm with non-trivial moduli of convexity and smoothness and Theorem 1(v). ∎

The last corollary is an extension of Remark (viii) of [LL]. It deals with extending and regularizing functions defined on subsets of superreflexive Banach spaces.

COROLLARY 4. *Let X be a superreflexive Banach space. The following holds true for some $0 < \alpha \le 1$ depending only on X:*

Let S be a subset of X and $f : S \to \mathbb{R}$ be a function that is uniformly continuous on bounded subsets of S. Then for every $r > 0$ and $\varepsilon > 0$ there exists a Δ-convex function $F_{r,\varepsilon} : X \to \mathbb{R}$ satisfying

(i) $\inf_S f = \inf_X F_{r,\varepsilon}$ *and* $\mathfrak{S}_{\inf}(f) = \mathfrak{S}_{\inf}(F_{r,\varepsilon})$,
(ii) $F_{r,\varepsilon} \in C^{1,\alpha}(X)$, *and*
(iii) $f(x) - \varepsilon \le F_{r,\varepsilon}(x) \le f(x)$ *for every* $x \in S \cap B_X(r)$.

Proof. By the same argument as above, let $\|\cdot\|$ be an equivalent norm on X with modulus of smoothness $1+\alpha$ (for some $0 < \alpha \le 1$). Consider the following simple extension of f:

$$F(x) := \begin{cases} f(x) & \text{for } x \in S, \\ +\infty & \text{otherwise.} \end{cases}$$

Notice that $\mathfrak{S}_{\inf}(F) = \mathfrak{S}_{\inf}(f) \subset S$. It is not hard to see using Proposition 8(i) and the proof of Proposition 6(v) that the sequence $(\Delta_n^{1+\alpha} F)_n$, which satisfies (i) and (ii) of Theorem 1, also converges to f uniformly on bounded subsets of S. ∎

The proof of Theorem 1 will follow a general scheme involving two main steps. First, we explain an *extended inf-convolution* formula that gives us a standard way to approximate functions on X. Then we develop some convexity techniques in order to get smooth Δ-convex functions from the functions given by the extended inf-convolution formula.

2. The extended inf-convolution. In this section we explain the convergence results we need in the proof of Theorem 1. First, we introduce the definition of extended inf-convolution. This definition generalizes the classical one of inf-convolution (see [St$_1$] for a general survey of the subject) and will be an important tool in our work.

DEFINITION. For any map $K : X \times X \to \mathbb{R} \cup \{+\infty\}$ and any function $f : X \to \mathbb{R} \cup \{+\infty\}$ we define the *extended inf-convolution* of f and K as the function

$$(f \,\square\, K)(x) := \inf_{y \in X}\{f(y) + K(x,y)\}, \quad x \in X.$$

K will be called the *kernel* of the extended inf-convolution.

EXAMPLE. For $g : X \to \mathbb{R} \cup \{+\infty\}$ set $K_g(x,y) := g(x-y)$. Then the extended inf-convolution $f \,\square\, K_g$ is nothing but the classical inf-convolution $f \,\square\, g$.

Before the statement of the main result of this section, we need to define some natural properties of kernels.

DEFINITION. A kernel K is *pointwise separating* if for every $x_0 \in X$ and every $\delta > 0$ there exists $C_{x_0,\delta} > 0$ such that $K(x_0, y) \geq C_{x_0,\delta}$ whenever $\|x_0 - y\| \geq \delta$.

A kernel K is called *uniformly separating on bounded sets* if for all $r > 0$ and $\delta > 0$ there exists $C_{r,\delta} > 0$ so that $K(x,y) \geq C_{r,\delta}$ provided $\|x\| \leq r$ and $\|x - y\| \geq \delta$.

A kernel K is *uniformly separating* if for every $\delta > 0$ there is some $\beta_\delta > 0$ such that $K(x,y) \geq \beta_\delta \|x - y\|$ whenever $\|x - y\| \geq \delta$.

DEFINITION. Given a function $f : X \to \mathbb{R} \cup \{+\infty\}$ and a kernel K, we define the following sequences of functions:

$$I_{K,n}f := f \,\square\, nK \quad \text{and} \quad S_{K,n}f := -(-f \,\square\, nK) \quad (n \in \mathbb{N}).$$

REMARK. For any Hilbert norm $\|\cdot\|$ consider the kernel $K_L(x,y) = \|x - y\|^2$. Then, with our notation, the sequence $(S_{K_L,m}(I_{K_L,n}f))_{m>n}$ consists of the *Lasry–Lions approximants* of f related to the norm $\|\cdot\|$.

REMARK. Note that the Lasry–Lions approximation commutes with translations in the same way as the classical inf-convolution does. This is a consequence of the following property of the kernel:

$$K_L(x-a, y) = K_L(x, y+a) \quad (\text{for all } x,\ y \text{ and } a).$$

However, the problem of regularizing (not necessarily convex) functions in a non-Hilbert space leads naturally to more general kernels which do not yield translation-invariant approximants.

The next facts are easy to check.

FACTS 5. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$.*

1. *For $x \in X$,*

$$I_{K,n}f = \inf_{y \in X}\{f(y) + nK(x,y)\},$$

$$S_{K,n}f(x) = -I_{K,n}(-f)(x) = \sup_{y \in X}\{f(y) - nK(x,y)\}.$$

2. *Let C be a constant. Then $I_{K,n}(f + C) = I_{K,n}f + C$ for any n.*

3. *Suppose that the kernel K is positive (i.e., $K(x,y) \geq 0$ for all x, y). Then:*

(i) *$(I_{K,n}f)_{n\in\mathbb{N}}$ is an increasing sequence of functions bounded below by $\inf f$.*

(ii) *If $f \leq g$, then $I_{K,n}f \leq I_{K,n}g$ for any n.*

(iii) *$I_{K,m}(I_{K,n}f) \leq I_{K,m}(I_{K,m}f)$ for any $m > n$.*

We now proceed to a technical proposition which is the main result of this section.

PROPOSITION 6. *Let $K : X \times X \to \mathbb{R}$ be a kernel satisfying:*

(a) *K is positive and $K(x,x) = 0$ for all $x \in X$,*

(b) *K is symmetric (i.e., $K(x,y) = K(y,x)$ for all $x, y \in X$),*

(c) *$K(x,y) \to +\infty$ as $y \to \infty$ uniformly on bounded sets,*

(d) *K is uniformly continuous (resp. Lipschitz continuous) on bounded sets, and*

(e) *K is pointwise separating.*

Then for every proper l.s.c. function $f : X \to \mathbb{R} \cup \{+\infty\}$ bounded below the following statements hold:

(i) *$I_{K,n}f \leq S_{K,n}(I_{K,n}f) \leq f$.*

(ii) *$\inf I_{K,n}f = \inf f$ and $\mathfrak{S}_{\inf}(I_{K,n}f) = \mathfrak{S}_{\inf}(f)$.*

(iii) *$I_{K,n}f$ is uniformly continuous (resp. Lipschitz continuous) on bounded sets.*

(iv) *$I_{K,n}(I_{K,n}f) \xrightarrow{\tau_p} f$ and $I_{K,n}(I_{K,n}f) \xrightarrow{\tau_K} f$ as $n \to \infty$ if f is continuous.*

If in addition K is uniformly separating on bounded sets then

(v) *$I_{K,n}(I_{K,n}f) \xrightarrow{\tau_b} f$ as $n \to \infty$ when f is uniformly continuous on bounded sets.*

Finally, if K is uniformly separating then

(vi) *$I_{K,n}(I_{K,n}f) \xrightarrow{\tau_u} f$ as $n \to \infty$ provided f is uniformly continuous on X (not necessarily bounded below).*

REMARK. The sequence of functions $I_{K,n}(I_{K,n}f)$ plays an important auxiliary rôle in this work; namely, it provides a lower bound for the sequence $(\Delta_{K,n}f)_{n\in\mathbb{N}}$ in Proposition 8(i).

Proof (of Proposition 6). The proof of (i)–(iii) can be found in several works on infimal convolution (e.g. [BPP]). Nonetheless, we provide a short proof for completeness.

(i) Since $K(x,x) = 0$ we get $I_{K,n}f \le f$ (take $y = x$ in the infimal definition of $I_{K,n}f$ at any point $x \in X$). Therefore,

$$S_{K,n}(I_{K,n})f = -I_{K,n}(-I_{K,n}f) \ge I_{K,n}f.$$

To see the other inequality, notice that from Fact 5-1 we obtain

$$S_{K,n}(I_{K,n}f)(x) = \sup_{y\in X} \inf_{z\in X} \{f(z) + n(K(y,z) - K(x,y))\} \tag{2}$$

for $x \in X$. If we take $z = x$ in (2) we conclude from the symmetry of K that $S_{K,n}(I_{K,n}f)(x) \le f(x)$.

(ii) From (i) and Fact 5-1(i) we deduce that $\inf I_{K,n}f = \inf f$ and $\mathfrak{S}_{\inf}(f) \subseteq \mathfrak{S}_{\inf}(I_{K,n}f)$. Consider any minimum point $x_0 \in X$ of $I_{K,n}f$. Then there exists a sequence $(y_k)_{k\in\mathbb{N}} \subset X$ so that

$$\inf f = I_{K,n}f(x_0) \le f(y_k) + nK(x_0,y_k) \xrightarrow[k\to\infty]{} \inf f. \tag{3}$$

Since K is positive it follows from (3) that

$$\lim_{k\to\infty} f(y_k) = \inf f \quad \text{and} \quad \lim_{k\to\infty} K(x_0,y_k) = 0. \tag{4}$$

But K is pointwise separating, so the second part of (4) implies that $y_k \to x_0$. Using the lower semicontinuity of f and the first part of (4) we conclude that

$$\inf f \le f(x_0) \le \lim_{k\to\infty} f(y_k) = \inf f,$$

and this proves assertion (ii).

Before proceeding with the rest of the proof, we set up the following useful definition:

$$\Omega_n(x) := \{y \in X : f(y) + nK(x,y) \le I_{K,n}f(x) + 1\} \quad (x \in X,\ n \in \mathbb{N}). \tag{5}$$

With these notations, we remark that for $n \in \mathbb{N}$ and $x \in X$,

$$I_{K,n}f(x) = \inf_{y\in\Omega_n(x)} \{f(y) + nK(x,y)\} \ge \inf_{\Omega_n(x)} f \tag{6}$$

(the last inequality coming from the positivity of K).

It is clear from (6) that the behaviour of $I_{K,n}f$ is directly linked with the size of the sets $\{\Omega_n(x)\}_{x\in X}$. We shall see that the growth condition (c) ensures that the sets $\Omega_n(x)$ are not arbitrarily large when x runs through bounded subsets of X. More precisely, we have the following.

CLAIM 6.1. *For any $r > 0$, the set $\Omega_r := \bigcup_{n\in\mathbb{N}} \bigcup_{\|x\|\le r} \Omega_n(x)$ is bounded.*

The proof of this claim is based on the next simple fact.

FACT 6.2. *For any $r > 0$,*

$$M_r := \sup\{I_{K,n}f(x)/n : x \in B_X(r),\ n \in \mathbb{N}\} < +\infty.$$

Proof. Since f is proper, choose y_0 with $f(y_0) \le \inf f + 1 < +\infty$. Then by definition of $I_{K,n}f$ it follows that for any $x \in X$,

$$\frac{I_{K,n}f(x)}{n} \le \frac{f(y_0)}{n} + K(x,y_0) \le \inf f + 1 + \sup\{K(x,y_0) : x \in B_X(r)\},$$

and this expression is bounded above on bounded sets because K is uniformly continuous (or Lipschitz continuous) on bounded sets. ■

Proof of Claim 6.1. For $r_0 > 0$, let $M_{r_0} > 0$ be the upper bound defined in Fact 6.2. For any $x \in B_X(r_0)$ and $n \in \mathbb{N}$ if $y \in \Omega_n(x)$ it follows from the definition (5) of $\Omega_n(x)$ that

$$K(x,y) \le \frac{1}{n}(I_{K,n}f(x) + 1 - f(y)) \le M_{r_0} + 1 - \inf f. \tag{7}$$

But (c) implies that the set of y satisfying (7) is uniformly bounded for $x \in B_X(r_0)$. ■

We can now continue with the proof of Proposition 6.

(iii) Suppose the kernel K is Lipschitz continuous on bounded sets (the proof for the uniformly continuous case is practically the same). For $r_0 > 0$ take $x, x' \in B_X(r_0)$ and let L_{K,r_0} be the Lipschitz constant of K on $B_X(r_0) \times \Omega_{r_0}$ (Ω_{r_0} being bounded by Claim 6.1). Using the equality of (6) we can construct a sequence $(y_k)_{k\in\mathbb{N}} \subset \Omega_{r_0}$ such that $f(y_k) + nK(x',y_k) \le I_{K,n}f(x') + 1/k$ for every $k \in \mathbb{N}$. Therefore

$$\begin{aligned} I_{K,n}f(x') - I_{K,n}f(x) &\le f(y_k) + nK(x',y_k) - f(y_k) - nK(x,y_k) + \frac{1}{k} \\ &\le nL_{K,r_0}\|x'-x\| + \frac{1}{k} \xrightarrow[n\to\infty]{} nL_{K,r_0}\|x'-x\|. \end{aligned}$$

This concludes the proof of (iii).

We first prove (iv), (v) and (vi) for $(I_{K,n}f)_n$ instead of $(I_{K,n}(I_{K,n}f))_n$.

(iv′) Fix $x_0 \in X$. If $\lim_{n\to\infty} I_{K,n}f(x_0) = \sup_n I_{K,n}f(x_0) = +\infty$ then by (i) one has $f(x_0) = +\infty$ and the result holds. Now, suppose that $I_{x_0} := \lim_n I_{K,n}f(x_0) < +\infty$. By the infimal definition of $I_{K,n}f$ at x_0, we can choose a sequence $(y_n)_{n\in\mathbb{N}} \subset X$ such that

$$I_{K,n}f(x_0) \le f(y_n) + nK(x_0,y_n) \le I_{K,n}f(x_0) + \frac{1}{n} \xrightarrow[n\to\infty]{} I_{x_0}. \tag{8}$$

It follows that for $n \in \mathbb{N}$,

$$K(x_0,y_n) \le \frac{I_{K,n}f(x_0) - f(y_n)}{n} + \frac{1}{n^2} \le \frac{I_{x_0} - \inf f}{n} + \frac{1}{n^2} \xrightarrow[n\to\infty]{} 0. \tag{9}$$

But K is pointwise separating, so (9) shows that $(y_n)_n$ norm converges to x_0. Using the lower semicontinuity of f, the positivity of K in (8) and (i), we get

$$f(x_0) \le \liminf_{n\to\infty} f(y_n) \le I_{x_0} \le f(x_0).$$

If f is continuous, then since by Fact 5-3(i) and (iii), $(I_{K,n}f)_n$ is an increasing sequence of continuous functions, Dini's theorem tell us that the pointwise convergence of $(I_{K,n}f)_n$ to f is actually uniform on compact sets.

(v′) Let f be a function uniformly continuous on bounded sets and O_{r_0} be the oscillation of f on $B_X(r_0)\cup\Omega_{r_0}$, for some fixed $r_0>0$. Then, for any $n\in\mathbb{N}$, $x\in B_X(r_0)$ and $y\in\Omega_n(x)$, by the first inequality of (7) and (i) we have

$$K(x,y) \le \frac{I_{K,n}f(x)+1-f(y)}{n} \le \frac{f(x)-f(y)+1}{n} \le \frac{O_{r_0}+1}{n} \underset{n}{\to} 0. \tag{10}$$

Suppose that K is uniformly separating on bounded sets. Then a direct consequence of (10) is that $\lim_n \operatorname{diam}(\Omega_n(x)) = 0$ uniformly on $B_X(r_0)$. Therefore, it follows from (i), (6) and the uniform continuity of f on $B_X(r_0)$ that

$$f(x) \ge \lim_{n\to\infty} I_{K,n}f(x) \ge \lim_{n\to\infty} \inf_{\Omega_n(x)} f \underset{n\to\infty}{\longrightarrow} f(x) \tag{11}$$

uniformly in $x\in B_X(r_0)$.

(vi′) Suppose that f is uniformly continuous on X. Then there exists $\alpha>0$ such that

$$f(x)-f(y) \le \max\{1,\alpha\|x-y\|\} \quad \text{for all } x,y\in X \tag{12}$$

(a simple proof is left to the reader). Now, in the same way as in (10) before, using this time (12), we deduce that for $n\in\mathbb{N}$, $x\in X$ and any $y\in\Omega_n(x)$,

$$K(x,y) \le \frac{1}{n}(f(x)-f(y)+1) \le \max\left\{\frac{1}{n},\frac{\alpha}{n}\|x-y\|\right\}+\frac{1}{n}. \tag{13}$$

For $1>\delta>0$, since K is uniformly separating there is some $\beta_\delta>0$ so that from (13) we deduce that for $x\in X$ and $y\in\Omega_n(x)$,

$$\|x-y\| \le \max\left\{\frac{1}{n\beta_\delta},\frac{\alpha}{n\beta_\delta}\|x-y\|\right\}+\frac{1}{n\beta_\delta} \quad \text{whenever } \|x-y\|>\delta. \tag{14}$$

Hence, if we take n so large that $\max\{2/(n\beta_\delta), 2\alpha/(n\beta_\delta)\} \le \delta < 1$, then (14) shows that $\operatorname{diam}(\Omega_n(x)) \le 2\delta$ for every $x\in X$.

That is, we have shown that $\operatorname{diam}(\Omega_n(x)) \to 0$ uniformly in $x\in X$. Therefore, as f is uniformly continuous on X we can repeat the same reasoning of (11) to conclude that $(I_{K,n}f)_n$ converges to f uniformly on X.

(iv) and (v) are straightforward corollaries of (iv′) and (v′) if we remark the following.

Suppose that for $\varepsilon>0$ there exists $n_0\in\mathbb{N}$ so that $f-\varepsilon/2 \le I_{K,n_0}f$ on some set S (S being a singleton, or a compact set or a bounded subset of X). By Fact 5-3(i) and (iii), we can then apply (iv′) (or (v′)) to the function $I_{K,n_0}f$ bounded below and uniformly continuous, to obtain $m>n_0$ such that $I_{K,n_0}f-\varepsilon/2 \le I_{K,m}(I_{K,n_0}f)$ on the same S. Thus, by Fact 5-3(iii) and (i) it follows that

$$f-\varepsilon \le I_{K,n_0}f-\varepsilon/2 \le I_{K,m}(I_{K,n_0}f) \le I_{K,m}(I_{K,m}f) \le f \quad \text{on } S.$$

(vi) is also easily deduced from (vi′) through the following argument.

If $f-\varepsilon \le I_{K,n}f \le f$, for some $\varepsilon>0$ and $n\in\mathbb{N}$, then applying Facts 5-2 and 5-3(ii) we get

$$f-2\varepsilon \le I_{K,n}f-\varepsilon = I_{K,n}(f-\varepsilon) \le I_{K,n}(I_{K,n}f) \le I_{K,n}f \le f. \ \blacksquare$$

REMARK. With the above techniques it is not difficult to check that the sequence $(I_{K,n}(I_{K,n}f))_n$ converges to f for the *epigraphical distance* (see [AW] for the definition). We refer to the proof of Lemma 3(v) in [St$_2$] for details.

3. Convexity techniques and smoothness results. In this section we show a procedure to obtain smooth functions from the operators $I_{K,n}(\cdot)$ and $S_{K,n}(\cdot)$. We will need to impose some additional conditions of convexity and smoothness on the kernel K to achieve the smooth regularization. The interesting feature of these convexity arguments is the preservation of the approximating properties obtained in the previous section.

The main tool is explained in the next theorem. It deals with the smoothness properties inherited by the convex envelope of a "somehow" smooth function.

THEOREM 7. *Let $c: X\to\mathbb{R}$ be a differentiable function, and $d: X\to\mathbb{R}$ be a convex function. Set $h := c-d$ and assume that* $\operatorname{co} h$ *makes sense. Then:*

(i) *If $c\in C^{1,\mathrm{u}}(X)$ (resp. $c\in C^{1,\alpha}(X)$ for some $0<\alpha\le 1$) then $\operatorname{co} h\in C^{1,\mathrm{u}}(X)$ (resp. $\operatorname{co} h\in C^{1,\alpha}(X)$).*

(ii) *If $c\in C^{1,\mathrm{u}}_B(X)$ (resp. $c\in C^{1,\alpha}_B(X)$ for some $0<\alpha\le 1$) and h is uniformly continuous on bounded sets and strongly coercive (i.e., $\lim_{x\to\infty} h(x)/\|x\| = +\infty$) then $\operatorname{co} h\in C^{1,\mathrm{u}}_B(X)$ (resp. $\operatorname{co} h\in C^{1,\alpha}_B(X)$).*

REMARK. A proof of the finite-dimensional version of Theorem 7(ii) with $d\equiv 0$ can be found in [GR]. Our more general proof does not require local compactness and relies upon ideas of [Fa]. The fact that the convex envelope of a smooth function c "perturbed" by a non-smooth concave function $-d$ is still smooth will be crucial later (namely, when we check the smoothness of the sequence $(\Delta_{K,n}f)_{n\in\mathbb{N}}$ in Proposition 8).

Notice that the uniform continuity hypothesis on the derivative of c cannot be weakened in the infinite-dimensional setting. There are $\mathcal{C}^\infty$-differentiable functions bounded below on ℓ_2 whose convex envelope is not even Gateaux differentiable (see Example II.5.6(a) of [DGZ]).

Proof of Theorem 7. Define $\nu := \operatorname{co} h = \operatorname{co}(c-d)$.

(i) Suppose that $c \in \mathcal{C}^{1,\alpha}(X)$ (the proof for the other case is similar). Since ν is convex, a necessary and sufficient condition for $\nu \in \mathcal{C}^{1,\alpha}(X)$ is that for every $x, y \in X$ one has

$$\nu(x+y)+\nu(x-y)-2\nu(x) \le L\|y\|^{1+\alpha} \quad \text{for some } L>0 \tag{15}$$

(see Lemma V.3.5 of [DGZ]). We now check this condition.

For $\varepsilon > 0$ and $x \in X$, by (1), we can choose $x_1,\ldots,x_n \in X$ and $\lambda_1,\ldots,\lambda_n > 0$ so that

$$\sum_{i=1}^n \lambda_i = 1, \quad \sum_{i=1}^n \lambda_i x_i = x \quad \text{and} \quad \sum_{i=1}^n \lambda_i h(x_i) \le \nu(x) + \frac{\varepsilon}{2}. \tag{16}$$

Note that from the first two parts of (16) we also have

$$x \pm y = \Big(\sum_{i=1}^n \lambda_i x_i\Big) \pm \Big(\sum_{i=1}^n \lambda_i y\Big) = \sum_{i=1}^n \lambda_i (x_i \pm y). \tag{17}$$

Thus, (1) gives

$$\nu(x\pm y) \le \sum_{i=1}^n \lambda_i h(x_i \pm y). \tag{18}$$

Let $L>0$ be the α-Hölder continuity constant of the derivative of c. Putting together the last part of (16), (18) and using the convexity of d, we get

$$\begin{aligned}
\nu(x+y)&+\nu(x-y)-2\nu(x)\\
&\le \sum_{i=1}^n \lambda_i h(x_i+y) + \sum_{i=1}^n \lambda_i h(x_i-y) - 2\sum_{i=1}^n \lambda_i h(x_i) + \varepsilon\\
&= \sum_{i=1}^n \lambda_i (c(x_i+y)+c(x_i-y)-2c(x_i))\\
&\quad + \sum_{i=1}^n 2\lambda_i \Big(d(x_i) - \frac{d(x_i+y)+d(x_i-y)}{2}\Big) + \varepsilon\\
&\le \sum_{i=1}^n \lambda_i (c(x_i+y)-c(x_i)+c(x_i-y)-c(x_i)) + \varepsilon\\
&\le \sum_{i=1}^n \lambda_i 2^\alpha L\|y\|^{1+\alpha} + \varepsilon = 2^\alpha L\|y\|^{1+\alpha} + \varepsilon,
\end{aligned}$$

because $c \in \mathcal{C}^{1,\alpha}(X)$ (and therefore satisfies (15) for $L' = 2^\alpha L$). As ε is arbitrary, the condition (15) holds for ν.

(ii) can be proved just as above, with the uniformity for the derivative of c replaced by the following "localization" property of the convex envelope of a strongly coercive function.

Claim 7.1. *Let $h : X \to \mathbb{R}$ be a function which is uniformly continuous on bounded sets and strongly coercive. Then for every $r > 0$ there exists $\varrho_r > 0$ so that for all $\|x\| \le r$ the value of $\operatorname{co} h(x)$ is given by*

$$\inf\Big\{\sum_{i=1}^n \lambda_i h(x_i) : (x_i)_{i=1}^n \subset B_X(\varrho_r),\ \lambda_i > 0,\ \sum_{i=1}^n \lambda_i = 1,\ x = \sum_{i=1}^n \lambda_i x_i\Big\}.$$

Proof. First, note that under the hypothesis of the claim, h is bounded below, so that $\operatorname{co} h$ makes sense. Fix $r_0 > 0$ and let m_{r_0} be the infimum of h on X and M_{r_0} be the supremum of h on $B_X(r_0+1)$ ($M_{r_0} < +\infty$, because of the uniform continuity of f on $B_X(r_0+1)$). Consider the following family of hyperplanes:

$$\mathcal{H}_{r_0} := \{H_{x,v} : x \in B_X(r_0),\ v \in B_X,\ v^* \in B_{X^*},\ v^*(v) = 1\} \tag{19}$$

where

$$H_{x,v}(z) = m_{r_0} + (h(x+v) - m_{r_0})v^*(z-x).$$

Notice that for $\|x\| \le r_0$ and $v \in B_X$ we have

$$H_{x,v}(x) = m_{r_0} \le \operatorname{co} h(x) \quad \text{and} \quad H_{x,v}(x+v) = h(x+v) \le M_{r_0}. \tag{20}$$

Since h is strongly coercive, we see from (19) that

$$\sup_{H \in \mathcal{H}_{r_0}} H(z) \le m_{r_0} + (M_{r_0} - m_{r_0})(\|z\| + r_0) < h(z) \tag{21}$$

provided $\|z\| > \varrho_{r_0}$, for some $\varrho_{r_0} > 0$. Let us show that this ϱ_{r_0} satisfies the conclusion of the claim.

The strategy is to replace any convex combination that appears in (1) by another smaller convex combination with "uniformly bounded vertices". This idea is formally explained in the next fact.

Fact 7.2. *Given x with $\|x\| \le r_0$, consider any finite convex combination $(x_1,\ldots,x_n \in X,\ \lambda_1,\ldots,\lambda_n > 0$ and $\sum_{i=1}^n \lambda_i = 1)$ so that $\sum_{i=1}^n \lambda_i x_i = x$. If $\|x_{i_0}\| > \varrho_{r_0}$ for some $1 \le i_0 \le n$, then there exist $x'_{i_0} \in B_X(\varrho_{r_0})$ and $\lambda'_1,\ldots,\lambda'_n > 0$ so that $\sum_{i=1}^n \lambda'_i = 1$,*

$$\sum_{\substack{i=1\\ i\ne i_0}}^n \lambda'_i x_i + \lambda'_{i_0} x'_{i_0} = x \quad \text{and} \quad \sum_{\substack{i=1\\ i\ne i_0}}^n \lambda'_i h(x_i) + \lambda'_{i_0} h(x'_{i_0}) \le \sum_{i=1}^n \lambda_i h(x_i).$$

Proof. For simplicity, take $i_0 = 1$. Since $\|x_1\| > \varrho_{r_0}$, it follows from (21) that $H_{x,v_{x_1}}(x_1) < h(x_1)$, where $v_{x_1} := (x_1 - x)/\|x_1 - x\|$. But by the first part of (20) we also have $H_{x,v_{x_1}}(x) = m_{r_0} \le \sum_{i=1}^n \lambda_i h(x_i)$. Hence, the segment $I_{x,x_1} := [(x, \sum_{i=1}^n \lambda_i h(x_i)), (x_1, h(x_1))] \subset X \times \mathbb{R}$ lies in the upper half-space defined by $H_{x,v_{x_1}}$. Therefore, the equality in the second part of (20) implies that

$$I_{x,x_1} \cap \{(x + v_{x_1}, t) : t \in \mathbb{R}\} = \{(x + v_{x_1}, s)\} \quad \text{where } h(x + v_{x_1}) \le s. \tag{22}$$

If we define

$$x' := \sum_{i=2}^n \frac{\lambda_i}{1 - \lambda_1} x_i$$

(so that $x = \lambda_1 x_1 + (1 - \lambda_1)x'$), then using barycentric coordinates on the segment

$$\left[\left(x', \sum_{i=2}^n \frac{\lambda_i}{1 - \lambda_1} h(x_i)\right), (x_1, h(x_1))\right]$$

we can find $\mu \ge 0$ such that

$$\begin{aligned}\left(x, \sum_{i=1}^n \lambda_i h(x_i)\right) &= \mu\left(x', \sum_{i=2}^n \frac{\lambda_i}{1 - \lambda_1} h(x_i)\right) + (1 - \mu)(x + v_{x_1}, s) \\ &= \sum_{i=2}^n \frac{\mu\lambda_i}{1 - \lambda_1}(x_i, h(x_i)) + (1 - \mu)(x + v_{x_1}, s).\end{aligned} \tag{23}$$

But (22) and (23) give

$$x = \sum_{i=2}^n \frac{\mu\lambda_i}{1 - \lambda_1} x_i + (1 - \mu)(x + v_{x_1})$$

and

$$\begin{aligned}&\sum_{i=2}^n \frac{\mu\lambda_i}{1 - \lambda_1} h(x_i) + (1 - \mu)h(x + v_{x_1}) \\ &\qquad \le \sum_{i=2}^n \frac{\mu\lambda_i}{1 - \lambda_1} h(x_i) + (1 - \mu)s = \sum_{i=1}^n \lambda_i h(x_i).\end{aligned}$$

This concludes the proof of Fact 7.2. ■

Thus the proof of Claim 7.1 is complete and, therefore, Theorem 7 is proved. ■

REMARK. Claim 7.1 is false for functions h failing the strong coerciveness condition $\liminf_{x\to\infty} h(x)/\|x\| = +\infty$. For instance, consider $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = \sqrt{|x|}$.

Theorem 7 can be applied as a useful tool to regularize functions on infinite-dimensional Banach spaces. Our next proposition, which provides the smoothness assertions we need for proving Theorem 1, is a good example of this feature. We keep the notation used in Section 2.

PROPOSITION 8. *Let $K : X \times X \to \mathbb{R}$ be a kernel satisfying*:

(a) *K is positive and $K(x,x) = 0$ for all $x \in X$,*
(b) *K is symmetric,*
(c) *$K(x,y) \to +\infty$ as $y \to \infty$, uniformly on bounded sets,*
(d) *K is uniformly continuous on bounded sets and*
(e) *$K(x,y) = c_K(x) - d_K(x,y)$ where $d_K(\cdot,y)$ is a lower semicontinuous convex function for all $y \in X$.*

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function and consider the sequence of Δ-convex functions $\Delta_{K,n} f$ defined as follows:

$$\Delta_{K,n} f := \mathrm{co}(I_{K,n} f + nc_K) - nc_K \quad (n \in \mathbb{N}).$$

Then:

(i) *$I_{K,n}(I_{K,n} f) \le \Delta_{K,n} f \le f$.*
(ii) *If $c_K \in C^{1,\mathrm{u}}(X)$ (resp. $c_K \in C^{1,\alpha}(X)$ for some $0 < \alpha \le 1$), then $(\Delta_{K,n} f)_{n\in\mathbb{N}} \subset C^{1,\mathrm{u}}(X)$ (resp. $(\Delta_{K,n} f)_{n\in\mathbb{N}} \subset C^{1,\alpha}(X)$).*
(iii) *If $c_K \in C_B^{1,\mathrm{u}}(X)$ (resp. $c_K \in C_B^{1,\alpha}(X)$ for some $0 < \alpha \le 1$) and c_K is strongly coercive then $(\Delta_{K,n} f)_{n\in\mathbb{N}} \subset C_B^{1,\mathrm{u}}(X)$ (resp. $(\Delta_{K,n} f)_{n\in\mathbb{N}} \subset C_B^{1,\alpha}(X)$) provided f is bounded below.*

REMARK. For every pair of functions f, g on X, we define

$$f \blacktriangle g := \mathrm{co}(f + g) - g.$$

For a Hilbert norm $\|\cdot\|$, consider the kernel $K_L(x-y) := \|x-y\|^2$. The Lasry–Lions approximants of a function f in the norm $\|\cdot\|$ satisfy the following relation for $m > n$ (see Proposition 2(i) of [St$_2$]):

$$S_{K_L,m}(I_{K_L,n} f) = I_{K_L,1/n-1/m}(f \blacktriangle nc).$$

Compare this with the expression given by Proposition 8:

$$\Delta_{K_L,n} f = (I_{K_L,n} f) \blacktriangle nc.$$

Proof (of Proposition 8). For any function $g : X \to \mathbb{R} \cup \{+\infty\}$, define

$$D_n g(x) := \sup\{g(y) + nd_K(x,y) : y \in X\}.$$

Since $d_K(\cdot,y)$ is a l.s.c. convex function, so is $D_n g$. Note that by Fact 5-1 we also have the decomposition

$$S_{K,n} g = \sup_{y\in X}\{g(y) - n(c_K(x) - d_K(x,y))\} = D_n g(x) - nc_K(x). \tag{24}$$

On the other hand, (a) and (b) ensure that (i) of Proposition 6 holds true. Hence, for all $n \in \mathbb{N}$ from (24) we get

$$I_{K,n}(I_{K,n}f) \le S_{K,n}(I_{K,n}(I_{K,n}f)) = D_n(I_{K,n}I_{K,n}f) - nc_K \le I_{K,n}f. \tag{25}$$

Now, we make the next simple but crucial observation.

FACT 8.1. *Let c, d and e be three functions such that $d - c \le e$ and suppose that d is l.s.c. and convex. Then $d - c \le \mathrm{co}(e + c) - c \le e$.*

Proof. It suffices to note that the convexity of d implies the equivalent inequality $d \le \mathrm{co}(e + c) \le e + c$. ∎

Applying Fact 8.1 to the inequality (25) we obtain

$$\begin{aligned} I_{K,n}I_{K,n}f &\le D_n(I_{K,n}I_{K,n}f) - nc_K \\ &\le \mathrm{co}(I_{K,n}f + nc_K) - nc_K = \Delta_{K,n}f \le I_{K,n}f. \end{aligned}$$

At this point, another important remark turns up. By definition of $I_{K,n}f$ at any point $x \in X$ one has

$$\begin{aligned} (I_{K,n}f + nc_K)(x) &= 2nc_K(x) + \inf_{y \in X}\{f(y) - d_K(x,y)\} \\ &= 2nc_K(x) - D_n(-f)(x), \end{aligned} \tag{26}$$

where $D_n(-f)$ is a convex function.

Therefore, if $c_K \in \mathcal{C}^{1,\mathrm{u}}(X)$ (or $c_K \in \mathcal{C}^{1,\alpha}(X)$) then Theorem 7(i) can be applied to $\mathrm{co}(I_{K,n}f(x) + nc_K(x))$ because of (26). This shows the statement (ii) of Proposition 8.

The proof for the case of $c_K \in \mathcal{C}^{1,\mathrm{u}}_B(X)$ (or $c_K \in \mathcal{C}^{1,\alpha}_B(X)$) can be done in a similar way from Theorem 7(ii). Notice that $I_{K,n}f + nc_K$ is uniformly continuous on bounded sets since K satisfies (a)–(d) and therefore Proposition 6(iii) holds. On the other hand, the strong coerciveness of c_K implies for f bounded below that

$$\frac{I_{K,n}f(x) + nc_K(x)}{\|x\|} \ge \frac{\inf f}{\|x\|} + n\frac{c_K(x)}{\|x\|} \xrightarrow[x \to \infty]{} +\infty. \quad ∎$$

4. The proof of the main result. With the tools of Sections 2 and 3, we are now ready to prove our main result.

Proof of Theorem 1. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be bounded below and l.s.c. For $p > 1$, we define $K_p : X \times X \to \mathbb{R}$ as

$$K_p(x,y) := 2^{p-1}\|x\|^p + 2^{p-1}\|y\|^p - \|x + y\|^p. \tag{27}$$

Let us first check the following two basic properties of K_p.

(1) Clearly, $K_p(x,x) = 0$ (for all $x \in X$). Also, K_p is positive since

$$\|x + y\|^p \le (\|x\| + \|y\|)^p \le 2^{p-1}(\|x\|^p + \|y\|^p).$$

(2) K_p is obviously symmetric.

Set $c_{K_p}(x) := 2^{p-1}\|x\|^p$ and $d_{K_p}(x,y) := \|x + y\|^p - 2^{p-1}\|y\|^p$. Then $K_p(\cdot,y) = c_{K_p}(\cdot) - d_{K_p}(\cdot,y)$.

With this notation and the definition of g_n^p given in Theorem 1, one has

$$I_{K_p,n}f + nc_{K_p} = g_n^p \quad \text{for every } n \in \mathbb{N}. \tag{28}$$

Hence, using again the notation of Theorem 1 and Proposition 8(i), it follows that

$$I_{K_p,n}(I_{K_p,n}f) \le \Delta_{K_p,n}f = \mathrm{co}\, g_n^p - 2^{p-1}n\|\cdot\|^p = \Delta_n^p f \le f. \tag{29}$$

Therefore, by (29) the statements (i), (iii), (iv) and (v) of Theorem 1 hold true if we check that K_p satisfies the assumptions of Proposition 6.

We proceed to show the following growth property of K_p that trivially implies the condition (c) of Proposition 6.

CLAIM 1.1. *For any $p > 1$ there exist $\gamma_p > 0$ and $\eta_p > 1$ so that $K_p(x,y) \ge \gamma_p\|y\|^p$ whenever $\|y\| \ge \eta_p\|x\|$.*

Proof. Take $\eta > 1$ and $x, y \in X$ such that $\eta\|x\| \le \|y\|$. Then

$$\begin{aligned} K_p(x,y) &\ge \|y\|^p\biggl(2^{p-1}\biggl|\frac{\|x\|}{\|y\|}\biggr|^p + 2^{p-1} - \biggl\|\frac{y}{\|y\|} + \frac{x}{\|y\|}\biggr\|^p\biggr) \\ &\ge \|y\|^p\biggl(2^{p-1} - \biggl|1 + \frac{\|x\|}{\|y\|}\biggr|^p\biggr) \ge \|y\|^p\biggl(2^{p-1} - \Bigl(1 + \frac{1}{\eta}\Bigr)^p\biggr). \end{aligned}$$

Now choose $\eta_p > 1$ so that $\gamma_p := (2^{p-1} - (1 + 1/\eta_p)^p) > 0$. ∎

It is clear that K_p is Lipschitz continuous on bounded sets. The next claim takes care of the *separating* properties of K_p.

CLAIM 1.2. *Suppose that the norm $\|\cdot\|$ is l.u.c. at $x_0 \in X$ (resp. UC). Then for every $\varepsilon > 0$ there exists $C_{\varepsilon,x_0} > 0$ such that $K_p(x_0,y) \ge C_{\varepsilon,x_0}\|x_0 - y\|^p$ whenever $\|x_0 - y\| \ge \varepsilon$ (resp. for all $r > 0$ and $\varepsilon > 0$ there exists $C_{\varepsilon,r} > 0$ such that $K_p(x,y) \ge C_{\varepsilon,r}\|x - y\|^p$ provided $\|x - y\| \ge \varepsilon$ and $\|x\| \le r$).*

Proof. We only prove the claim under the uniform convexity assumption. The proof for the l.u.c. case is completely similar.

Suppose that the claim is false. Then by definition of K_p (see (27)), there are sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in X so that $(x_n)_n$ is bounded, $\|x_n - y_n\| \ge \varepsilon_0 > 0$ for all $n \in \mathbb{N}$ and

$$K_p(x_n,y_n) = 2^{p-1}\|x_n\|^p + 2^{p-1}\|y_n\|^p - \|x_n + y_n\|^p \le \frac{1}{n}\|x_n - y_n\|^p. \tag{30}$$

Moreover, without loss of generality we can suppose that $\|y_n\| \ge \|x_n\| > 0$ for all n. We then consider $0 < \beta_n = \|x_n\|/\|y_n\| \le 1$. From (30) it follows that

$$0 \le 2^{p-1}(\beta_n^p + 1) - (\beta_n + 1)^p \le \frac{1}{n}(\beta_n + 1)^p \xrightarrow[n \to \infty]{} 0.$$

Hence,

$$2^{p-1}\frac{\beta_n^p+1}{(\beta_n+1)^p}\xrightarrow[n\to\infty]{}1. \tag{31}$$

It follows that $\lim_{n\to\infty}\beta_n=\lim_{n\to\infty}\|x_n\|/\|y_n\|=1$. Thus, since $(x_n)_n$ is bounded, so is $(y_n)_n$ and therefore $\lim_{n\to\infty}(\|x_n\|-\|y_n\|)=0$. But using (30) again, we see that the bounded sequences $(x_n)_n$ and $(y_n)_n$ satisfy

$$\lim_{n\to\infty}\left(\|x_n\|-\left\|\frac{x_n+y_n}{2}\right\|\right)=\lim_{n\to\infty}(\|x_n\|-\|y_n\|)=0.$$

Nonetheless, by hypothesis $\|x_n-y_n\|\ge\varepsilon_0>0$ for all n, contrary to the uniform convexity of the norm $\|\cdot\|$. ∎

Another important fact is that K_p is uniformly separating when the modulus of convexity of $\|\cdot\|$ is of power type p. This is a consequence of the results of [H]. Indeed, for all $x,y\in X$ we have the stronger inequality

$$K_p(x,y)=2^{p-1}\|x\|^p+2^{p-1}\|y\|^p-\|x+y\|^p\ge C_{\|\cdot\|}\|x-y\|^p$$

for some $0<C_{\|\cdot\|}\le1$ (for instance, see [C], Lemma 3.1).

Hence, using Proposition 6 together with (29) we deduce (i), (iii), (iv) and (v) of Theorem 1. It remains to prove (ii), for which we use Proposition 8.

More precisely, we observe that in the decomposition (28), $d_K(\cdot,y)$ is a convex function for every $y\in X$. Moreover, for $p>1$ it is easy to verify that c_{K_p} is strongly coercive; that is,

$$\frac{c_{K_p}(x)}{\|x\|}=2^{p-1}\|x\|^{p-1}\xrightarrow[x\to\infty]{}+\infty.$$

Therefore, by Proposition 8(iii) the regularity of $\Delta_n^pf=\Delta_{K_p,n}f$ can be deduced from the regularity of $c_{K_p}=2^{p-1}\|\cdot\|^p$.

Recall now that for any norm $\|\cdot\|$ on X, being US (resp. with modulus of smoothness of power type $1+\alpha$) is equivalent to $\|\cdot\|\in\mathcal{C}^{1,\mathrm{u}}(X)$ (resp. $\|\cdot\|\in\mathcal{C}^{1,\alpha}(X)$). Therefore, $c_{K_p}=2^{p-1}\|\cdot\|^p\in\mathcal{C}_B^{1,\mathrm{u}}(X)$ (or $c_{K_p}\in\mathcal{C}_B^{1,\alpha}(X)$) whenever $\|\cdot\|$ is US (or with modulus of smoothness of power type $1+\alpha$).

In the last case of (ii), for a norm $\|\cdot\|$ with modulus of smoothness of power type $1+\alpha$ (or equivalently $\|\cdot\|\in\mathcal{C}^{1,\alpha}(X)$), we can achieve a smoother behaviour of the sequence (Δ_n^pf) by choosing the proper value of p: $(\Delta_n^{1+\alpha}f)_n\subset\mathcal{C}^{1,\alpha}(X)$. This is a corollary of Proposition 8(ii) and the next lemma.

LEMMA 1.3. *If* $\|\cdot\|\in\mathcal{C}^{1,\alpha}(X)$ *then* $\|\cdot\|^{1+\alpha}\in\mathcal{C}^{1,\alpha}(X)$.

Proof. This fact relies strongly on the convexity and homogeneity of a norm. Since it is clear that $\|\cdot\|^{1+\alpha}\in\mathcal{C}_B^{1,\alpha}(X)$, let $C>0$ be the α-Hölder continuity constant of the derivative of the norm $\|\cdot\|$ in B_X. We shall show that the condition (15) holds true for $\|\cdot\|^{1+\alpha}$. Take any $x,y\in X$ and denote by ω the maximum of $\|x\|$ and $\|y\|$. The lemma is proved by the following computation:

$$\begin{aligned}&\|x+y\|^{1+\alpha}+\|x-y\|^{1+\alpha}-2\|x\|^{1+\alpha}\\&\quad=\omega^{1+\alpha}\left(\left\|\frac{x}{\omega}+\frac{y}{\omega}\right\|^{1+\alpha}-\left\|\frac{x}{\omega}\right\|^{1+\alpha}+\left\|\frac{x}{\omega}-\frac{y}{\omega}\right\|^{1+\alpha}-\left\|\frac{x}{\omega}\right\|^{1+\alpha}\right)\\&\quad\le\omega^{1+\alpha}2^\alpha C\left\|\frac{y}{\omega}\right\|^{1+\alpha}=2^\alpha C\|y\|^{1+\alpha}.\ ∎\end{aligned}$$

By the above, this concludes the proof of Theorem 1. ∎

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References

[AA] H. Attouch and D. Azé, *Approximation and regularization of arbitrary functions in Hilbert spaces by the Lasry–Lions method*, Ann. Inst. H. Poincaré Anal. Non Linéaire 10 (1993), 289–312.

[AW] H. Attouch and R. Wets, *Quantitative stability of variational systems: I. The epigraphical distance*, Trans. Amer. Math. Soc. 328 (1991), 695–729.

[BPP] M. Bougeard, J. P. Penot and A. Pommelet, *Towards minimal assumptions for the infimal convolution regularization*, J. Approx. Theory 64 (1991), 245–270.

[C] M. Cepedello-Boiso, *Approximation of Lipschitz functions by Δ-convex functions in Banach spaces*, Israel J. Math., to appear.

[DFH] R. Deville, V. Fonf and P. Hájek, *Analytic and C^k approximations of norms in separable Banach spaces*, Studia Math. 120 (1996), 61–74.

[DGZ] R. Deville, G. Godefroy and V. Zizler, *Smoothness and Renormings in Banach Spaces*, Pitman Monographs Surveys Pure Appl. Math. 64, Longman, Boston, 1993.

[E] P. Enflo, *Banach spaces which can be given an equivalent uniformly convex norm*, Proc. Internat. Sympos. Partial Differential Equations and the Geometry of Normed Linear Spaces II (Jerusalem, 1972), Israel J. Math. 13 (1973), 281–288.

[Fa] M. Fabián, *Lipschitz smooth points of convex functions and isomorphic characterizations of Hilbert spaces*, Proc. London Math. Soc. 51 (1985), 113–126.

[Fr] J. Frontisi, *Smooth partitions of unity in Banach spaces*, Rocky Mountain J. Math. 25 (1995), 1295–1304.

[GR] A. Griewank and P. J. Rabier, *On the smoothness of convex envelopes*, Trans. Amer. Math. Soc. 322 (1990), 691–709.

[H] J. Hoffman-Jørgensen, *On the modulus of smoothness and the G_*-conditions in B-spaces*, preprint series, Aarhus Universitet, Matematisk Inst., 1974.

[L] G. Lancien, *On uniformly convex and uniformly Kadec–Klee renormings*, Serdica Math. J. 21 (1995), 1–18.

[LL] J. M. Lasry and P. L. Lions, *A remark on regularization in Hilbert spaces*, Israel J. Math. 55 (1986), 257–266.

[NS] A. S. Nemirovskiĭ and S. M. Semenov, *The polynomial approximation of functions on Hilbert space*, Mat. Sb. (N.S.) 92 (1973), 257–281, 344 (in Russian).

[Ph] R. R. Phelps, *Convex Functions, Monotone Operators and Differentiability*, Lecture Notes in Math. 1364, Springer, Berlin, 1993.

[Pi] G. Pisier, *Martingales with values in uniformly convex spaces*, Israel J. Math. 20 (1975), 236–350.

[St1] T. Strömberg, *The operation of infimal convolution*, Dissertationes Math. (Rozprawy Mat.) 352 (1996).

[St2] —, *On regularization in Banach spaces*, Ark. Mat. 34 (1996), 383–406.

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The symmetric tensor product of a direct sum of locally convex spaces

by

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Abstract. An explicit representation of the n-fold symmetric tensor product (equipped with a natural topology τ such as the projective, injective or inductive one) of the finite direct sum of locally convex spaces is presented. The formula for $\bigotimes^n_{\tau,s}(F_1 \oplus F_2)$ gives a direct proof of a recent result of Díaz and Dineen (and generalizes it to other topologies τ) that the n-fold projective symmetric and the n-fold projective "full" tensor product of a locally convex space E are isomorphic if E is isomorphic to its square E^2.

1. Symmetric tensor products

1.1. If $E_1, \ldots, E_n, E$ and F are vector spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ we denote by $L(E_1, \ldots, E_n; F)$ the space of n-linear maps $E_1 \times \ldots \times E_n \to F$. We write briefly $L(^nE; F) := L(E, \ldots, E; F)$ and $L_s(^nE; F)$ for the space of n-linear symmetric maps $E \times \ldots \times E \to F$; the space of n-homogeneous polynomials $E \to F$ is denoted by $P^n(E; F)$ (they are the restrictions to the diagonal of $E \times \ldots \times E$ of elements in $L(^nE; F)$). The polarization formula gives a natural isomorphism $P^n(E; F) = L_s(^nE; F)$. If the underlying spaces are locally convex we denote by $\mathcal{L}(E_1, \ldots, E_n; F)$ or $\mathcal{L}(^nE; F)$ if $E = E_1 = \ldots = E_n$, $\mathcal{L}_s(^nE; F)$ and $\mathcal{P}^n(E; F)$ the spaces of continuous n-linear, continuous n-linear symmetric mappings and continuous n-homogeneous polynomials respectively. Moreover, we use $L(^nE) := L(^nE; \mathbb{K})$ and, similarly, $L_s(^nE), P^n(E), \mathcal{L}(^nE)$, $\mathcal{L}_s(^nE)$ and $\mathcal{P}^n(E)$ in the case of $F = \mathbb{K}$. We shall write $E \cong F$ if the two locally convex spaces E and F are topologically isomorphic.

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