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On p-dependent local spectral properties of certain linear differential operators in $L^p(\mathbb{R}^N)$

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Abstract. The aim is to investigate certain spectral properties, such as decomposability, the spectral mapping property and the Lyubich–Matsaev property, for linear differential operators with constant coefficients (and more general Fourier multiplier operators) acting in $L^p(\mathbb{R}^N)$. The criteria developed for such operators are quite general and p-dependent, i.e. they hold for a range of p in an interval about 2 (which is typically not $(1,\infty)$). The main idea is to construct appropriate functional calculi: this is achieved via a combination of methods from the theory of Fourier multipliers and local spectral theory.

Introduction. The global nature of linear differential operators with constant coefficients acting in $L^p(\mathbb{R}^N)$ can be rather complicated when $p \neq 2$. For instance, such operators are no longer spectral (the L^p-analogue of normal operators in the L^2-setting); see [2]. So, it is natural to seek weaker spectral decomposition properties for such operators. There is a large class of operators which are not required to decompose the underlying L^p-space in such a strong way and whose members still enjoy the spectral mapping property; these are the differential operators (or more general unbounded Fourier p-multiplier operators) which are decomposable in the sense of C. Foiaş. An effective tool, when available, for the investigation of local spectral properties of such operators in $L^p(\mathbb{R}^N)$ is the existence of a sufficiently rich functional calculus. This approach was used in [2] and [3] to determine large classes of decomposable p-multiplier operators (and to exhibit some which are not decomposable). The basic idea to generate appropriate functional calculi, for all $p > 1$, was to use classical results on p-multipliers (e.g. Mikhlin, Marcinkiewicz, Littman–McCarthy–Rivière).

The aim of this paper is to develop quite general criteria for decomposability (again via suitable functional calculi) which, unlike those in [2] and [3], are p-dependent. For example, if Q is a polynomial on $\mathbb{R}^N$ satisfying $|Q(x)| \to \infty$, for $|x| \to \infty$, and $\overline{Q(\mathbb{R}^N)} \neq \mathbb{C}$, then $Q\big(\frac{1}{i}\frac{\partial}{\partial x}\big)$ (with its natural

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domain) is decomposable in $L^p(\mathbb{R}^N)$ if and only if $(\lambda - Q)^{-1}$ is a p-multiplier function for some $\lambda \in \mathbb{C}\setminus\overline{Q(\mathbb{R}^N)}$. This always happens for all p in some maximal interval (typically not all of $(1,\infty)$) containing 2 in its interior. Such results are established via multiplier theorems due to W. Littman (and, independently, J. Peetre) and M. Schechter, combined with techniques from local spectral theory. For instance, combining our criteria with some results of A. Ruiz, it is shown that the differential operator

$$\left(1 - \left[\frac{\partial}{\partial x} - (-i)^{n-1}\frac{\partial^n}{\partial y^n}\right]^2\right)\left(1 - \frac{\partial^2}{\partial x^2}\right)$$

is decomposable in $L^p(\mathbb{R}^2)$ for all p satisfying $\frac{4}{3+4/n} < p < \frac{4}{1-4/n}$ and not decomposable otherwise (whenever $n \geq 5$). The criteria developed to treat operators of this kind (and other Fourier multiplier operators) are also extended to include systems of linear differential operators with constant coefficients (cf. Theorem 2.16). The final section presents a detailed investigation of the local spectral behaviour of the class of all second order linear differential operators in $L^p(\mathbb{R}^N)$ with real (constant) coefficients.

1. Preliminaries. For $1 \leq p < \infty$ we denote by $\mathcal{M}^p(\mathbb{R}^N)$ the semisimple unital Banach algebra of p-multiplier functions on $\mathbb{R}^N$, $N \in \mathbb{N} = \{1, 2, \ldots\}$. As in [2] and [3], let $\mathcal{U}^p(\mathbb{R}^N)$ be the set of all local p-multiplier functions, i.e. the set of all those $f \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ with the property that $\varphi f \in \mathcal{M}^p(\mathbb{R}^N)$ for all $\varphi \in C^\infty_c(\mathbb{R}^N)$. Then $\mathcal{U}^p(\mathbb{R}^N)$ contains all C^∞-functions on $\mathbb{R}^N$ and hence, in particular, all polynomials. Let $\widehat{\mathcal{D}}(\mathbb{R}^N)$ be the set of all those g in the Schwartz space $\mathcal{S}(\mathbb{R}^N)$ such that the Fourier transform $\widehat{g}$ has compact support. If $f \in \mathcal{U}^p(\mathbb{R}^N)$ and $g \in \widehat{\mathcal{D}}(\mathbb{R}^N)$, then we define

$$f(D)g := \mathcal{F}^{-1}(f\widehat{g}), \tag{1}$$

where the right-hand side is an element of $L^p(\mathbb{R}^N)$ and $\mathcal{F}$ denotes the Fourier transform. The operator $f(D)$ with domain $\widehat{\mathcal{D}}(\mathbb{R}^N)$ is closable in $L^p(\mathbb{R}^N)$. Its closure will be denoted by S^p_f. Although this definition appears different from the one given in [3; p. 153] it is easy to see (using the fact that $\widehat{\mathcal{D}}(\mathbb{R}^N)$ is dense in $L^p(\mathbb{R}^N)$) that the operator S^p_f from [3] coincides with S^p_f as defined here. Moreover, if $Q(x) = \sum_{|\alpha|\leq m} a_\alpha x^\alpha$, for $x \in \mathbb{R}^N$, is a polynomial, then $Q(D)$ coincides with $Q\big(\frac{1}{i}\frac{\partial}{\partial x_1}, \ldots, \frac{1}{i}\frac{\partial}{\partial x_N}\big)$ and it easily seen that, in this case, S^p_Q is just the minimal (or strong) extension of $Q(D)$ in the sense of [27; Ch. 4, §1]. A straightforward computation also shows that $D(S^p_Q)$ is translation invariant and that S^p_Q commutes with all translations. Hence, if $(S^p_Q)^{-1}$ exists as a bounded linear operator on $L^p(\mathbb{R}^N)$, then it must be translation invariant and, hence, is a multiplier operator. Moreover, in this case $(S^p_Q)^{-1} = S^p_{1/Q}$. Thus, for $f \in \mathcal{U}^p(\mathbb{R}^N)$, by a similar argument, the

operator S^p_f is invertible if and only if $1/f \in \mathcal{M}^p(\mathbb{R}^N)$ (cf. also [3; p. 153], [27; p. 66]). This shows that we always have $\overline{\operatorname{ess\,range}(f)} \subseteq \sigma(S^p_f)$, where the closure is taken in $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Since, in general, operators of the form S^p_f are no longer spectral for $p \neq 2$, it is natural to look for weaker spectral decomposition properties of such operators.

Recall that a closed linear operator T with domain $D(T)$ which is a subspace of a Banach space X is said to be *decomposable* (in the sense of C. Foiaş, [10], [28]) if, for every finite open covering $U_1, \ldots, U_m$ of $\overline{\mathbb{C}}$, there are closed subspaces $X_1, \ldots, X_m$ of X satisfying $X = X_1 + \ldots + X_m$ such that $T(D(T) \cap X_j) \subset X_j$ and $\sigma(T|X_j) \subset \overline{U_j}$ for $j = 1, \ldots, m$. (Here $T|X_j$ is the closed linear operator in X_j with domain $D(T|X_j) = D(T) \cap X_j$.) If $F \subset \overline{\mathbb{C}}$ is closed we write $X_T(F)$ for the set of all those $x \in X$ for which there exists some analytic function $f : \overline{\mathbb{C}} \setminus F \to X$ with values in $D(T)$ and satisfying $(z - T)f(z) \equiv x$ on $\mathbb{C} \setminus F$.

A closed linear operator T has the *Lyubich–Matsaev property* [21] if $X_T(F)$ is closed in X, for every closed set $F \subset \overline{\mathbb{C}}$, and if for every locally finite open covering $(U_j)_{j=1}^\infty$ of $\mathbb{C}$ consisting of bounded open sets, the Banach space X is the closed linear span of the subspaces $X_T(\overline{U_j})$, $j \in \mathbb{N}$.

Often, decomposability for multiplier operators S^p_f is obtained by showing that they have a sufficiently rich functional calculus. In particular, one tries to prove that the class of functions $\varphi : \mathbb{C} \to \mathbb{C}$ for which $\varphi \circ f \in \mathcal{M}^p(\mathbb{R}^N)$ is "rich" enough.

Recall that an algebra $\mathcal{A}$ of functions on a closed set $\Omega \subset \overline{\mathbb{C}}$ is said to be *quasiadmissible* [28; Definition IV.9.2] if it has the following properties:

(i) $\mathcal{A}$ is *normal*, i.e. for every open covering $U_1, \ldots, U_m$ of $\overline{\mathbb{C}}$ there exist functions $\varphi_1, \ldots, \varphi_m \in \mathcal{A}$ satisfying $\operatorname{supp}(\varphi_j) \subset U_j$ $(j = 1, \ldots, m)$ and $\varphi_1 + \ldots + \varphi_m \equiv 1$ on Ω.

(ii) For all $\varphi \in \mathcal{A}$ such that $\operatorname{supp}(\varphi)$ is compact in $\mathbb{C}$, the function $\varphi \cdot \mathrm{id}_{\mathbb{C}}$ is in $\mathcal{A}$, where $\mathrm{id}_{\mathbb{C}}$ denotes the identity function on $\mathbb{C}$.

(iii) For every $\varphi \in \mathcal{A}$ and every $\lambda \in \mathbb{C} \setminus \operatorname{supp}(\varphi)$ the function φ_λ belongs to $\mathcal{A}$, where

$$\varphi_\lambda(z) = \begin{cases} 0 & \text{for } z \in \overline{\mathbb{C}} \setminus \operatorname{supp}(\varphi), \\ \dfrac{\varphi(z)}{\lambda - z} & \text{for } z \in \operatorname{supp}(\varphi) \cap \mathbb{C}. \end{cases}$$

Let $\mathcal{A}$ be a quasiadmissible algebra on $\Omega = \overline{\Omega} \subset \overline{\mathbb{C}}$. We say that S^p_f (where $f \in \mathcal{U}^p(\mathbb{R}^N)$) has a *translation invariant $\mathcal{A}$-functional calculus* if $\overline{f(\mathbb{R}^N)} \subseteq \Omega$ and $\varphi \circ f \in \mathcal{M}^p(\mathbb{R}^N)$ holds for all $\varphi \in \mathcal{A}$. In this case a unital homomorphism $\Phi : \mathcal{A} \to \mathcal{L}(L^p(\mathbb{R}^N))$ is given by $\Phi(\varphi) = S^p_{\varphi\circ f}$ for $\varphi \in \mathcal{A}$. We then see by (i) and (ii) that $\Phi(\varphi)S^p_f \subset S^p_f\Phi(\varphi) = \Phi(\varphi\cdot\mathrm{id}_{\mathbb{C}})$, for all $\varphi \in \mathcal{A}$ having compact support contained in $\mathbb{C}$. Then S_f^p is known to be decomposable (cf. [28; Corollary IV.9.8 and the remark following its proof]). Moreover, the subspaces

$$\mathcal{E}_{\mathcal{A}}(F) = \bigcap\{\ker(S^p_{\varphi\circ f}) : \varphi \in \mathcal{A},\ \operatorname{supp}(\varphi)\cap F = \emptyset\}$$

are closed in $L^p(\mathbb{R}^N)$ for all closed sets $F \subset \overline{\mathbb{C}}$. If F is a compact subset of $\mathbb{C}$ then, using the properties of $\mathcal{A}$, it follows that $\mathcal{E}_{\mathcal{A}}(F) \subset D(S_f^p)$ and $S_f^p\mathcal{E}_{\mathcal{A}}(F) \subset \mathcal{E}_{\mathcal{A}}(F)$ with $\sigma(S_f^p|\mathcal{E}_{\mathcal{A}}(F)) \subset F$. In particular, for all compact subsets F in $\mathbb{C}$ we have $\mathcal{E}_{\mathcal{A}}(F) = L^p(\mathbb{R}^N)_{S_f^p}(F)$.

Examples and Definitions 1.1. Let $k \in \mathbb{N}_0 = \{0,1,2,\ldots\}$. The following algebras are quasiadmissible.

(a) Let $C^k(\overline{\mathbb{C}})$ denote the algebra of all functions $\varphi \in C^k(\mathbb{C})$ for which the function φ_∞ given by $\varphi_\infty(z) := \varphi(1/z)$, for $z \neq 0$, has a continuous extension to $\mathbb{C}$ (again denoted by φ_∞) with $\varphi_\infty \in C^k(\mathbb{C})$. Then $C^k(\overline{\mathbb{C}})$ is a Banach algebra when endowed with the norm $\|\varphi\|_{C^k(\overline{\mathbb{C}})} = \max\{\|\varphi\|_k, \|\varphi_\infty\|_k\}$ for $\varphi \in C^k(\overline{\mathbb{C}})$, where

$$\|\varphi\|_k = \sum_{\mu_1+\mu_2\le k} \frac{1}{\mu_1!\mu_2!} \sup_{z=u+iv}\left|\frac{\partial^{\mu_1+\mu_2}\varphi}{\partial u^{\mu_1}\partial v^{\mu_2}}(z)\right|.$$

(b) As in [3], we denote by $\mathcal{A}^k$ the Banach algebra of all functions $\varphi \in C^k(\mathbb{C})$ satisfying

$$\|\varphi\|_{\mathcal{A}^k} = \sum_{\mu_1+\mu_2\le k} \frac{1}{\mu_1!\mu_2!} \sup_{z=u+iv}(1+|z|)^{\mu_1+\mu_2}\left|\frac{\partial^{\mu_1+\mu_2}\varphi}{\partial u^{\mu_1}\partial v^{\mu_2}}(z)\right| < \infty.$$

Note that $\mathcal{A}^k$ is not quasiadmissible since, for $\varphi \in \mathcal{A}^k$, the value at ∞ is in general not well defined. However, the subalgebra $\mathcal{B}^k$ of all those $\varphi \in \mathcal{A}^k$ having a continuous extension to $\overline{\mathbb{C}}$ is quasiadmissible.

(c) Let $\mathcal{K}^k$ denote the Banach algebra of all functions $\varphi \in C(\overline{\mathbb{C}}) \cap C^{2k}(\mathbb{C}\setminus\{0\})$ satisfying

$$\|\varphi\|_{\mathcal{K}^k} = \sum_{\mu_j\le k} \frac{1}{\mu_1!\mu_2!} \sup_{z=u+iv\neq 0}\left|u^{\mu_1}v^{\mu_2}\frac{\partial^{\mu_1+\mu_2}\varphi}{\partial u^{\mu_1}\partial v^{\mu_2}}(z)\right| < \infty.$$

(d) Let $C^k_\infty(\mathbb{C})$ denote the set of all those functions $\varphi \in C^k(\mathbb{C})$ which are analytic in some neighbourhood U_φ of ∞ (which may depend on φ), i.e. the function $w \mapsto \varphi(1/w)$ is analytic in $\{w \in \mathbb{C} : 1/w \in \mathbb{C}\cap U_\varphi\}$ and has a removable singularity at 0. The algebra $C^k_\infty(\mathbb{C})$ is in a natural way an inductive limit of Fréchet spaces. ■

We collect some permanence properties for unbounded multiplier operators admitting a translation invariant $\mathcal{A}$-functional calculus for some quasiadmissible algebra $\mathcal{A}$.

Lemma 1.2. *Let f be a pointwise defined function in $\mathcal{U}^p(\mathbb{R}^N)$ and suppose that S_f^p has a translation invariant $\mathcal{A}$-functional calculus for a quasiadmissible algebra $\mathcal{A}$ of functions on some closed set $\Omega \subset \overline{\mathbb{C}}$ containing the closure (in $\overline{\mathbb{C}}$) of $f(\mathbb{R}^N)$.*

(a) *For every $n \geq N$ the function f_n defined by $f_n(x_1,\ldots,x_n) := f(x_1,\ldots,x_N)$, for $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$, is in $\mathcal{U}^p(\mathbb{R}^n)$ and $S^p_{f_n}$ has a translation invariant $\mathcal{A}$-functional calculus. In particular, $S^p_{f_n}$ is decomposable.*

(b) *If f is continuous and satisfies $|f(x)| \to \infty$ for $|x| \to \infty$, then S_f^p has the Lyubich–Matsaev property.*

(c) *If $A : \mathbb{R}^N \to \mathbb{R}^N$ is an affine mapping of $\mathbb{R}^N$ onto $\mathbb{R}^N$, then $A^*f \in \mathcal{U}^p(\mathbb{R}^N)$ where $A^*f = f\circ A$. Moreover, $S^p_{A^*f}$ also has a translation invariant $\mathcal{A}$-functional calculus. The operators S_f^p and $S^p_{A^*f}$ are similar and hence have the same spectral behaviour.*

(d) *If f is continuous, $\mathcal{A} \subseteq C(\overline{\mathbb{C}})$ and $m < N$, then for each $y \in \mathbb{R}^{N-m}$ the function $f_y : x \mapsto f(x,y)$ from $\mathbb{R}^m$ to $\mathbb{C}$ is in $\mathcal{U}^p(\mathbb{R}^m)$ and $S^p_{f_y}$ has a translation invariant $\mathcal{A}$-functional calculus.*

Proof. (a) To prove that $f_n \in \mathcal{U}^p(\mathbb{R}^n)$ fix an arbitrary $\psi \in C_c^\infty(\mathbb{R}^n)$. As $f \in \mathcal{U}^p(\mathbb{R}^N)$, we know that $\widetilde{\psi}f \in \mathcal{M}^p(\mathbb{R}^N)$ where we choose $\widetilde{\psi} \in C_c^\infty(\mathbb{R}^N)$ such that

$$P_N(\operatorname{supp}(\psi)) \subset \{x \in \mathbb{R}^N : \widetilde{\psi}(x) = 1\}$$

and $P_N : \mathbb{R}^n \to \mathbb{R}^N$ is the canonical projection onto the first N coordinates. By [9; Corollary B.2.2] the function $(\widetilde{\psi}f)_n$, where

$$(\widetilde{\psi}f)_n(x_1,\ldots,x_n) := \widetilde{\psi}(x_1,\ldots,x_N)f(x_1,\ldots,x_N),\quad \text{for } x = (x_1,\ldots,x_n) \in \mathbb{R}^n,$$

is a p-multiplier function on $\mathbb{R}^n$. Since $\mathcal{M}^p(\mathbb{R}^n)$ is an algebra we see that $\psi f_n = \psi(\widetilde{\psi}f)_n \in \mathcal{M}^p(\mathbb{R}^n)$. Hence, $f_n \in \mathcal{U}^p(\mathbb{R}^n)$. By [9; Corollary B.2.2] we also see that $\varphi\circ f_n = (\varphi\circ f)_n \in \mathcal{M}^p(\mathbb{R}^n)$ for all $\varphi \in \mathcal{A}$ and so (a) is proved.

(b) Fix an arbitrary locally finite open covering $(U_j)_{j=1}^\infty$ of $\mathbb{C}$ consisting of bounded open sets. Let $g \in L^p(\mathbb{R}^N)$ and $\varepsilon > 0$ be arbitrary. Since $\widehat{\mathcal{D}}(\mathbb{R}^N)$ is dense in $L^p(\mathbb{R}^N)$ there exists some $h \in \widehat{\mathcal{D}}(\mathbb{R}^N)$ with $\|g-h\|_{L^p(\mathbb{R}^N)} < \varepsilon$. Since f is continuous and satisfies $|f(x)| \to \infty$ for $|x| \to \infty$, the set $f(\operatorname{supp}(\widehat{h}))$ is a compact subset of $\mathbb{C}$. Hence, $f(\operatorname{supp}(\widehat{h})) \subset U_1\cup\ldots\cup U_m$ for some finite $m \in \mathbb{N}$. Since $\mathcal{A}$ is normal there are functions $\varphi_1,\ldots,\varphi_m \in \mathcal{A}$ with $\operatorname{supp}(\varphi_j) \subset U_j$, for $1 \le j \le m$, such that $\sum_{j=1}^m \varphi_j \equiv 1$ on $V\cap\Omega$ for some open neighbourhood V of $\operatorname{supp}(\widehat{h})$. It follows that $\sum_{j=1}^m \varphi_j(f(x))\widehat{h}(x) \equiv \widehat{h}(x)$ on $\mathbb{R}^N$ and, hence, that $h = \sum_{j=1}^m S^p_{\varphi_j\circ f}h$, where $S^p_{\varphi_j\circ f}h \in \mathcal{E}_{\mathcal{A}}(\operatorname{supp}(\varphi_j)) \subset$

$\mathcal{E}_A(\overline{U}_j) \subset L^p(\mathbb{R}^N)_{S^p_f}(\overline{U}_j)$ for $j = 1, \ldots, m$. This proves that the linear hull of the spaces $L^p(\mathbb{R}^N)_{S^p_f}(\overline{U}_j)$, for $j \in \mathbb{N}$, is dense in $L^p(\mathbb{R}^N)$.

(c) If $A : \mathbb{R}^N \to \mathbb{R}^N$ is a surjective affine mapping, then A is bijective and A^{-1} is affine. By [12; Theorem 1.13], A^* defines an isometric isomorphism from $\mathcal{M}^p(\mathbb{R}^N)$ onto $\mathcal{M}^p(\mathbb{R}^N)$. Hence, for all $\psi \in C_c^\infty(\mathbb{R}^N)$, we have $(A^* f)\psi = A^*(f \cdot ((A^{-1})^*\psi)) \in \mathcal{M}^p(\mathbb{R}^N)$, since $(A^{-1})^*\psi = \psi \circ A^{-1} \in C_c^\infty(\mathbb{R}^N)$. This shows that $A^* : \mathcal{U}^p(\mathbb{R}^N) \to \mathcal{U}^p(\mathbb{R}^N)$ is a bijective mapping (where $(A^*)^{-1} = (A^{-1})^*$ with A^{-1} affine again). A straightforward computation shows that by $C_A g := \mathcal{F}^{-1}A^*\widehat{g}$, for $g \in L^p(\mathbb{R}^N)$, where A^* is extended to act on tempered distributions in the canonical way, we define a continuous isomorphism $C_A : L^p(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ satisfying $C_A\widehat{\mathcal{D}}(\mathbb{R}^N) = \widehat{\mathcal{D}}(\mathbb{R}^N)$ and

$$C_A S^p_f g = S^p_{A^*f} C_A g, \quad S^p_f C_{A^{-1}} g = C_{A^{-1}} S^p_{A^*f} g \quad \text{for all } g \in \widehat{\mathcal{D}}(\mathbb{R}^N).$$

From this we conclude that $C_A S^p_f = S^p_{A^*f} C_A$ and $C_A D(S^p_f) = D(S^p_{A^*f})$. By [12; Theorem 1.13] we also see that $\varphi \circ (A^* f) = A^*(\varphi \circ f) \in \mathcal{M}^p(\mathbb{R}^N)$ for all $\varphi \in \mathcal{A}$. Hence $S^p_{A^*f}$ has a translation invariant $\mathcal{A}$-functional calculus.

(d) This follows in a straightforward way from the fact that for all $y \in \mathbb{R}^{N-m}$ the mapping $h \mapsto h(\cdot, y)$ is continuous and linear from $C(\mathbb{R}^N) \cap \mathcal{M}^p(\mathbb{R}^N)$ into $C(\mathbb{R}^m) \cap \mathcal{M}^p(\mathbb{R}^m)$ and has norm at most 1 with respect to the $\mathcal{M}^p$-norms ([15], [19]). ∎

2. Decomposability criteria and functional calculi. In the first part of this section we formulate two multiplier theorems (Theorems 2.2 and 2.4), one due to W. Littman (and, independently, J. Peetre) and the other due to M. Schechter. The point is that these two multiplier theorems are "p-dependent", unlike the classical multiplier theorems of Mikhlin, Marcinkiewicz, and Littman–McCarthy–Rivière which yield p-multipliers for all $p \in (1, \infty)$. These two multiplier theorems are then used to establish the existence of suitable functional calculi and/or decomposability (and the spectral mapping property) for certain operators S^p_f, with appropriate f and for a certain range of p (typically not $(1, \infty)$); see Theorems 2.6, 2.8 and 2.12. These results are applied to some non-trivial examples from the class of constant coefficient linear differential operators (i.e. f is a polynomial). In particular, we exhibit examples of such differential operators which are decomposable for some p, but not for all p in $(1, \infty)$; see Examples 2.11(c), (d). We also show, for any hypoelliptic (and more general) constant coefficient linear differential operator S, that there exists an interval J_S about 2 such that the operator S exhibits "good" local spectral behaviour in $L^p(\mathbb{R}^N)$ for all $p \in J_S$ (Corollary 2.9). Of course, for elliptic operators S it is known that $J_S = (1, \infty)$ can be chosen independent of S; see [2], [3].

In the last part of the section we formulate a result concerning the spectral mapping property and decomposability (with some restrictions on p) for certain systems of linear partial differential operators with constant coefficients acting in $L^p(\mathbb{R}^N)^m$.

We begin with the following fact which facilitates certain continuity investigations.

LEMMA 2.1. *Let $f : \mathbb{R}^N \setminus \{0\} \to \mathbb{C}$ be a continuous function and suppose that $\mathcal{A}$ is a Fréchet space of continuous functions (or an inductive limit of such spaces) on a set $\Omega \subset \mathbb{C}$ satisfying the following three conditions.*

(i) *$f(\mathbb{R}^N \setminus \{0\}) \subset \Omega$.*

(ii) *The restriction map $\varphi \mapsto \varphi|_{f(\mathbb{R}^N\setminus\{0\})}$ is a continuous linear map from $\mathcal{A}$ to the Banach space of all bounded $\mathbb{C}$-valued functions on $f(\mathbb{R}^N \setminus \{0\})$, endowed with the sup-norm.*

(iii) *$\varphi \circ f \in \mathcal{M}^p(\mathbb{R}^N)$ for all $\varphi \in \mathcal{A}$.*

Then the linear mapping $\Phi : \varphi \mapsto S^p_{\varphi\circ f}$ from $\mathcal{A}$ to $\mathcal{L}(L^p(\mathbb{R}^N))$ is continuous.

Proof. Because of (i) and (ii) the linear mapping $\varphi \mapsto \varphi\circ f$ is continuous from $\mathcal{A}$ to $L^\infty(\mathbb{R}^N)$. Since the mapping $S^p_g \mapsto g$ from the Banach space of all translation invariant operators on $L^p(\mathbb{R}^N)$ to $L^\infty(\mathbb{R}^N)$ is known to be continuous, the statement follows in a standard way by means of the closed graph theorem. ∎

As a consequence of this lemma, all functional calculi described in this paper will automatically be continuous algebra homomorphisms.

As in [3] we denote by $\mathcal{N}^k(\mathbb{R}^N)$ the Banach algebra of those functions $f \in C^k(\mathbb{R}^N \setminus \{0\})$ for which there exists $C > 0$ such that, for each $R > 0$,

$$\max_{R\le|x|\le 2R} \left|\frac{\partial^\mu f}{\partial x^\mu}(x)\right| \le \frac{C}{R^{|\mu|}},$$

for all $\mu \in \mathbb{N}_0^N$ with $|\mu| \le k$. Here, $\mathcal{N}^k(\mathbb{R}^N)$ is endowed with the norm $\|\cdot\|_{\mathcal{N}^k}$ given by

$$\|f\|_{\mathcal{N}^k} = \sum_{|\mu|\le k} \frac{1}{\mu!} \sup_{x\ne 0} |x|^{|\mu|} \left|\frac{\partial^\mu f}{\partial x^\mu}(x)\right|.$$

The following result is a direct consequence of a multiplier theorem due to W. Littman [20; Theorem A] and J. Peetre [22; Théorème 6.1].

THEOREM 2.2. *Let $k \in \mathbb{N}$. Then $\mathcal{N}^k(\mathbb{R}^N) \subset \mathcal{M}^p(\mathbb{R}^N)$ for all $p \in (1, \infty)$ satisfying*

$$\left|\frac{1}{2} - \frac{1}{p}\right| < \frac{k}{N}. \tag{2}$$

A similar argument as in the proof of Lemma 2.1 (or looking more closely at the proofs of [20], [22]) shows that the inclusion mapping $\mathcal{N}^k(\mathbb{R}^N) \hookrightarrow \mathcal{M}^p(\mathbb{R}^N)$ in the statement of the theorem is necessarily continuous. In particular, we have $\mathcal{N}^k(\mathbb{R}^N) \subset \mathcal{M}^p(\mathbb{R}^N)$, for every $p \in (1,\infty)$, provided that $k \in \mathbb{N}$ satisfies $k \geq N/2$. Thus, for even dimensions N, this consequence is a slight generalization of the Mikhlin multiplier theorem.

It is known [3; Lemma 1.3] that for all $g \in \mathcal{N}^k(\mathbb{R}^N)$, the multiplication operators $M_g : f \mapsto gf$ from $\mathcal{N}^k(\mathbb{R}^N)$ to $\mathcal{N}^k(\mathbb{R}^N)$ are generalized scalar of class C^k and have a functional calculus $\Phi : C^k(\mathbb{C}) \to \mathcal{L}(\mathcal{N}^k(\mathbb{R}^N))$ given by $\Phi(\varphi) = M_{\varphi\circ g}$, for $\varphi \in C^k(\mathbb{C})$. This fact together with Theorem 2.2 implies the following result.

COROLLARY 2.3. *For all $f \in \mathcal{N}^k(\mathbb{R}^N)$ and p satisfying* (2), *the multiplier operator $S_f^p \in \mathcal{L}(L^p(\mathbb{R}^N))$ is generalized scalar and, hence, is also decomposable. A $C^k(\mathbb{C})$-functional calculus Φ (necessarily continuous by Lemma* 2.1) *for S_f^p is given by $\Phi(\varphi) = S^p_{\varphi\circ f}$, for $\varphi \in C^k(\mathbb{C})$.*

Using a more refined version of Theorem 2.2 (based on Theorem C in [20]), M. Schechter [26], [27] derived the following multiplier theorem which is very useful in the theory of linear differential operators in $L^p(\mathbb{R}^N)$.

THEOREM 2.4 ([26], [27]). *Let $f \in C^k(\mathbb{R}^N)$, where $k, N \in \mathbb{N}$, and suppose that there exist real constants $b > 0$ and $a \leq 1$ such that, for some constant $C > 0$, we have*

$$\left|\frac{\partial^\mu f}{\partial x^\mu}(x)\right| \leq C|x|^{-|\mu|a-b}, \quad |x| > 1, \tag{3}$$

for all $\mu \in \mathbb{N}_0^N$ with $|\mu| \leq k$. Then $f \in \mathcal{M}^p(\mathbb{R}^N)$ for all $p \in (1,\infty)$ satisfying both (2) *and*

$$(1-a)\left|\frac{1}{2} - \frac{1}{p}\right| < \frac{b}{N}. \tag{4}$$

By means of this fact we obtain the following result.

PROPOSITION 2.5. *Let f be as in Theorem* 2.4. *Then, for all $p \in (1,\infty)$ satisfying* (2) *and* (4), *the multiplier operator $S_f^p \in \mathcal{L}(L^p(\mathbb{R}^N))$ is $C^k(\mathbb{C})$-scalar in the sense of* [7].

Proof. It suffices to show that the mapping $\varphi \mapsto S^p_{\varphi\circ f}$ defines a unital homomorphism from the algebra $C^k(\mathbb{C})$ to $\mathcal{L}(L^p(\mathbb{R}^N))$, for all p satisfying (2) and (4). This is obvious if we can show that $\varphi \circ f \in \mathcal{M}^p(\mathbb{R}^N)$ for all $\varphi \in C^k(\mathbb{C})$. Because of $(\varphi \circ f)(x) = \varphi(0) + (\varphi(f(x)) - \varphi(0))$, for all $x \in \mathbb{R}^N$, and because of Theorem 2.4, it suffices to show that the function $x \mapsto \varphi(f(x)) - \varphi(0)$ satisfies condition (3) with the same constants a and b as for f and all $\mu \in \mathbb{N}_0^N$ with $|\mu| \leq k$ (the constant C will be different, of course). For $\mu = 0$ this follows from (3) for f (with $\mu = 0$) via the mean value theorem. For $1 \leq |\mu| \leq k$ one uses (3) for f and some basic facts on higher derivatives of composite functions (cf. [3; Lemma 2.2] and [18; Satz 12.1]). ∎

Since we have a particular interest in linear differential operators we now give some criteria for this class of unbounded multiplier operators. Let k, N be positive integers and $f \in C^k(\mathbb{R}^N \setminus \{0\})$. We shall say that f has the *LP-property* (for W. Littman & J. Peetre) with respect to an integer $m \geq 0$ if the following conditions are satisfied:

(i) $1/|f(x)| = O(|x|^{-m})$ for $|x| \to \infty$.

(ii) For all $\mu \in \mathbb{N}_0^N$ with $|\mu| \leq k$ we have
$$\left|\frac{\partial^\mu f}{\partial x^\mu}(x)\right| = O(|x|^{m-|\mu|}) \quad \text{for } |x| \to \infty.$$

(iii) For all $\mu \in \mathbb{N}_0^N$ with $|\mu| \leq k$ we have
$$\left|\frac{\partial^\mu f}{\partial x^\mu}(x)\right| = O(|x|^{-|\mu|}) \quad \text{for } |x| \to 0.$$

The following result extends Theorem 2.1 of [3]. The proof is omitted as it is almost the same; at the place where the Mikhlin multiplier theorem is used we now apply Theorem 2.2.

THEOREM 2.6. *Let k, N be positive integers and $f \in C^k(\mathbb{R}^N \setminus \{0\})$ be a function having the LP-property with respect to an integer $m \geq 0$. Then, for all $p \in (1,\infty)$ satisfying* (2), *we have $f \in \mathcal{U}^p(\mathbb{R}^N)$; cf. Theorem* 2.2. *Moreover, for such p and every $\varphi \in \mathcal{A}^k$ the function $\varphi \circ f \in \mathcal{M}^p(\mathbb{R}^N)$. In particular, S_f^p has a translation invariant $\mathcal{B}^k$-functional calculus. The unital homomorphism $\Phi : \mathcal{A}^k \to \mathcal{L}(L^p(\mathbb{R}^N))$ given by $\Phi(\varphi) = S^p_{\varphi\circ f}$ for $\varphi \in \mathcal{A}^k$ is continuous and has the following properties*:

(a) *For all $\varphi \in \mathcal{A}^k$ having compact support, $\Phi(\varphi)S_f^p \subset S_f^p\Phi(\varphi) = \Phi(\varphi \cdot \mathrm{id}_{\mathbb{C}})$.*

(b) *There exists a sequence $(\varrho_n)_{n=1}^\infty$ in $\mathcal{A}^k$, with each ϱ_n having compact support, such that $\Phi(\varrho_n)g \to g$ in $L^p(\mathbb{R}^N)$ for all $g \in L^p(\mathbb{R}^N)$. The domain $D(S_f^p)$ of S_f^p is given by*

$$D(S_f^p) = \{g \in L^p(\mathbb{R}^N) : \lim_{n\to\infty} \Phi(\varrho_n \cdot \mathrm{id}_{\mathbb{C}})g \text{ exists in } L^p(\mathbb{R}^N)\}$$

and we have $S_f^p g = \lim_{n\to\infty}(\varrho_n \cdot \mathrm{id}_{\mathbb{C}})g$, for all $g \in D(S_f^p)$.

(c) *The operator S_f^p is decomposable and has the Lyubich–Matsaev property. Moreover, $\sigma(S_f^p)$ is the closure (in $\overline{\mathbb{C}}$) of $f(\mathbb{R}^N \setminus \{0\})$ and coincides with the support of Φ.*

(d) *For every* $\varphi \in \mathcal{A}^k$ *the operator* $\Phi(\varphi) = S^p_{\varphi\circ f} \in \mathcal{L}(L^p(\mathbb{R}^N))$ *is generalized scalar of class* $C^k(\mathbb{C})$. *In particular, for every* $\lambda \in \varrho(S^p_f)$ *the resolvent operator* $(\lambda I - S^p_f)^{-1}$ *is generalized scalar.*

EXAMPLE 2.7. The function $f(x) = |x|^2 + \sin(|x|^2) + 2$, for $x \in \mathbb{R}^N$, satisfies conditions (i)–(iii) of the LP-property with $m = 2$ and $k = 1$ (but *not* with $k = 2$). Thus the statements (a)–(d) of Theorem 2.6 hold for S^p_f for all p with $|1/p - 1/2| < 1/N$.

On the other hand, the bounded function $1/f$ satisfies the conditions of Theorem 2.4 and Proposition 2.5 with $b = k = 2$ and $a = 0$. Hence $1/f \in \mathcal{M}^p(\mathbb{R}^N)$ and $S^p_{1/f}$ is decomposable (even $C^2(\mathbb{C})$-generalized scalar) for all p with $|1/p - 1/2| < 2/N$. Since $S^p_{1/f} = (S^p_f)^{-1}$, we conclude (by [3; Lemma 2.4]) that S^p_f is decomposable for p with $|1/p - 1/2| < 2/N$. It follows that $\sigma(S^p_f) = \overline{f(\mathbb{R}^N)}$, where the closure is taken in $\overline{\mathbb{C}}$. Using the C^2-functional calculus for $S^p_{1/f}$ one can then derive a functional calculus for S^p_f. ■

Even in cases where $f(\mathbb{R}^N) = \mathbb{C}$ (and hence $\sigma(S^p_f) = \overline{\mathbb{C}}$) or where the range of f is not known, one can sometimes obtain a sufficiently rich functional calculus and decomposability for the corresponding multiplier operator.

THEOREM 2.8. *Let* $N, k \in \mathbb{N}$ *and let* $f \in C^k(\mathbb{R}^N)$ *be a function satisfying both*

$$\text{(i)} \qquad \frac{1}{f(x)} = O\Big(\frac{1}{|x|^b}\Big) \quad \text{as } |x| \to \infty, \quad \text{and}$$

$$\text{(ii)} \qquad \frac{\partial^\mu f/\partial x^\mu}{f(x)} = O\Big(\frac{1}{|x|^{a|\mu|}}\Big) \quad \text{as } |x| \to \infty, \text{ for } 1 \le |\mu| \le k,$$

for some real constants $a \in (-\infty, 1]$ *and* $b \in (0, \infty)$. *If* $p \in (1, \infty)$ *satisfies* (2) *and* (4) *then*:

(a) S^p_f *has a translation invariant* $C^k(\overline{\mathbb{C}})$*-functional calculus* $\Phi : C^k(\overline{\mathbb{C}}) \to \mathcal{L}(L^p(\mathbb{R}^N))$ *given by* $\Phi(\varphi) = S^p_{\varphi\circ f}$, *for* $\varphi \in C^k(\overline{\mathbb{C}})$, *with all the properties listed prior to Definition* 1.1. *In particular,* S^p_f *is decomposable and has the Lyubich–Matsaev property.*

(b) $\sigma(S^p_f) = \overline{f(\mathbb{R}^N)} = \operatorname{supp}(\Phi)$.

(c) *For all* $\lambda \in \mathbb{C} \setminus \overline{f(\mathbb{R}^N)} = \varrho(S^p_f) \cap \mathbb{C}$ *the resolvent operator* $(\lambda I - S^p_f)^{-1} = S^p_{1/(\lambda - f)}$ *is generalized scalar of class* $C^k(\mathbb{C})$.

Proof. (a) Notice that $f \in \mathcal{U}^p(\mathbb{R}^N)$; see Theorem 2.4. We prove that $\Phi(\varphi) := S^p_{\varphi\circ f} \in \mathcal{L}(L^p(\mathbb{R}^N))$, i.e. that $\varphi \circ f \in \mathcal{M}^p(\mathbb{R}^N)$ for all $\varphi \in C^k(\overline{\mathbb{C}})$. Because of $\varphi\circ f = \varphi \circ f - \varphi_\infty(0) + \varphi_\infty(0)$ it suffices to show that $\varphi \circ f - \varphi_\infty(0) \in \mathcal{M}^p(\mathbb{R}^N)$. By (i) there exists $r \ge 1$ such that $|x| \ge r$ implies $|f(x)| \ge 1$. By the mean value theorem and (i) we have, for $|x| \ge r$,

$$(5) \quad \varphi(f(x)) - \varphi_\infty(0) = \varphi_\infty\Big(\frac{1}{f(x)}\Big) - \varphi_\infty(0) \le C_1 \|\varphi_\infty\|_k \frac{1}{|f(x)|} \le C_2 \frac{1}{|x|^b},$$

where the constant C_2 depends only on φ, f and r. Let $|\mu| \le k$. By Lemma 1.2 in [3] (see also [14], [27; p. 67]) $\frac{\partial^\mu(1/f)}{\partial x^\mu}(x)$ is a finite sum of terms of the form

$$\frac{A}{f(x)^{s+1}} \prod_{j=1}^{s} \frac{\partial^{\alpha_j} f_{\eta(j)}(x)}{\partial x^{\alpha_j}}$$

with $\sum_{j=1}^s \alpha_j = \mu$, where A is a constant, $\eta(j) \in \{1, 2\}$ and $f_1 = \mathrm{Re}(f)$ and $f_2 = \mathrm{Im}(f)$. Hence, by (i) and (ii), we have, for $|x| \ge r$,

$$(6) \qquad \Big|\frac{\partial^\mu(1/f)}{\partial x^\mu}(x)\Big| \le C_\mu \frac{1}{|x|^{a|\mu| + b}}.$$

Using again Lemma 1.2 in [3] for the composite function $\varphi \circ (1/f)$, we see that for each $1 \le |\mu| \le k$, the function $(\partial^\mu \varphi_\infty \circ (1/f)/\partial x^\mu)(x)$ can be written in the form

$$\frac{\partial^\mu \varphi_\infty \circ (1/f)}{\partial x^\mu}(x) = \sum_{|\alpha| \le |\mu|} c_\alpha \frac{\partial^{\alpha_1 + \alpha_2} \varphi_\infty}{\partial u_1^{\alpha_1} \partial u_2^{\alpha_2}} \Big(\frac{1}{f(x)}\Big) \cdot \sum_{m=1}^{s(\alpha,\mu)} \prod_{j=1}^{|\alpha|} \frac{\partial^{\gamma(j,\alpha,m)} \psi_{\eta(j,\alpha,m)}}{\partial x^{\gamma(j,\alpha,m)}}(x),$$

where $c_\alpha \in \{0, 1\}$ and $\gamma(j, \alpha, m) \in \mathbb{N}_0^N \setminus \{0\}$ with $\sum_{j=1}^{|\alpha|} \gamma(j, \alpha, m) = \mu$. Also, $\eta(j, \alpha, m) \in \{1, 2\}$ and $\psi_1 = \mathrm{Re}(1/f)$, $\psi_2 = \mathrm{Im}(1/f)$. So, by (6), for $1 \le |\mu| \le k$ and $|x| \ge r$ we have

$$\Big|\frac{\partial^\mu \varphi_\infty \circ (1/f)}{\partial x^\mu}(x)\Big| \le \sum_{|\alpha| \le |\mu|} C_\alpha(\varphi) \cdot \sum_{m=1}^{s(\alpha,\mu)} \prod_{j=1}^{|\alpha|} C_{\gamma(j,\alpha,m)} \frac{1}{|x|^{a|\gamma(j,\alpha,m)| + b}}$$

$$\le \sum_{|\alpha| \le |\mu|} \widetilde{C}_\alpha(\varphi) \sum_{m=1}^{s(\alpha,\mu)} \frac{1}{|x|^{a|\mu| + |\alpha| b}} \le C \frac{1}{|x|^{a|\mu| + b}}.$$

From this and (5) we see that $\varphi \circ f - \varphi_\infty(0)$ satisfies (3) in Theorem 2.4. So, $\varphi \circ f - \varphi_\infty(0)$ and thus also $\varphi \circ f$ belongs to $\mathcal{M}^p(\mathbb{R}^N)$ for all $p \in (1, \infty)$ satisfying (2) and (4). It follows that Φ is a continuous homomorphism from $C^k(\overline{\mathbb{C}})$ to $\mathcal{L}(L^p(\mathbb{R}^N))$; cf. Lemma 2.1.

The decomposability is now clear; the Lyubich–Matsaev property follows from Lemma 1.2(b).

(b) By the decomposability we must have $\sigma(S_f^p) = \overline{f(\mathbb{R}^N)}$ (closure in $\overline{\mathbb{C}}$). Clearly, $\operatorname{supp}(\Phi) = \overline{f(\mathbb{R}^N)}$ follows.

(c) can be obtained in the same way as (d) in [3; Theorem 2.2]. ■

Recall that a constant coefficient linear partial differential operator $Q(D)$ in $\mathbb{R}^N$ is called *hypoelliptic* if and only if there exists a positive constant c such that

$$\left|\frac{\frac{\partial^\alpha Q}{\partial x^\alpha}(x)}{Q(x)}\right| = O(1+|x|^{-c|\alpha|}), \quad |x| \to \infty, \tag{7}$$

for all $\alpha \in \mathbb{N}_0^N$; see [13; Theorem 11.1.3]. If d is the degree of the polynomial Q, then there exists an α with $|\alpha| = d$ such that $\partial^\alpha Q/\partial x^\alpha$ is a non-zero constant. Hence, we see that

$$\frac{1}{|Q(x)|} = O(1+|x|)^{-cd}, \quad |x| \to \infty.$$

It follows that the conditions of Theorem 2.8 are satisfied with $b = cd$ and $a = \min\{1, c\}$. Hence we obtain the following result.

COROLLARY 2.9. *Let the differential operator $Q(D)$ be hypoelliptic and $c>0$ be as in* (7). *Then, for $p \in (1,\infty)$ satisfying $(1-\min\{1,c\})|1/2-1/p| < cd/N$ and for all $k \in \mathbb{N}$ satisfying*

$$k/N > |1/2-1/p| \quad \textit{if } c \ge 1 \quad (\textit{resp. } cd/(1-c) < k \textit{ if } c<1),$$

the statements (a)–(c) *of Theorem* 2.8 *are valid for $f = Q$.*

Let $Q : \mathbb{R}^N \to \mathbb{C}$ be a polynomial satisfying $|Q(x)| \to \infty$ as $|x| \to \infty$. As has been noted in [27] this implies, by a result of Hörmander [11; Theorem 3.2], that there exists some $b > 0$ such that

$$\frac{1}{|Q(x)|} = O(|x|^{-b}) \quad \text{as } |x| \to \infty. \tag{8}$$

As shown in [27; Ch. 4, Corollary 4.3] this ensures that Q satisfies condition (ii) in Theorem 2.8 for $a = (b+1-m) \le 1$ where m is the degree of Q. Hence we obtain the following consequence of Theorem 2.8.

COROLLARY 2.10. *Let $Q : \mathbb{R}^N \to \mathbb{C}$ be a polynomial satisfying $|Q(x)| \to \infty$ as $|x| \to \infty$. Then there exists some $s > 0$ such that for every $p \in (1,\infty)$ satisfying $|1/2-1/p| < s$ all the statements* (a)–(c) *in Theorem* 2.8 *are valid for the operator S_Q^p.*

Of course, there exist polynomials Q which are not hypoelliptic but still satisfy the condition $|Q(x)| \to \infty$ as $|x| \to \infty$.

EXAMPLE 2.11. (a) Consider $Q_1(t,x) = it + x^3$ for $t, x \in \mathbb{R}$. By [3; Theorem 2.2], for every $p > 1$ the differential operator $Q_1\left(\frac{1}{i}\frac{\partial}{\partial t}, \frac{1}{i}\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial t} - i\frac{\partial^3}{\partial x^3}$ with domain $D(S_{Q_1}^p)$ (i.e. the operator $S_{Q_1}^p$) is decomposable and has a nice functional calculus. Theorem 2.2 in [3] is, however, not applicable to the polynomial $Q_2(t,x) = it + x^3 + x$, for $t, x \in \mathbb{R}$, since its real part is not a product of real affine functionals on $\mathbb{R}^2$. Still, a straightforward computation shows that Q_2 is hypoelliptic and satisfies (7) with $c = 1/3$. Hence it follows from Corollary 2.9 that $S_{Q_2}^p$ is decomposable for all $p \in (1,\infty)$. Notice that $Q_2(\mathbb{R}^2) = \mathbb{C}$.

(b) Let $Q_3(t,x) = it + |x|^{2l}$, for $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$. Then a computation similar to that in [27; p. 70] shows that Q_3 is hypoelliptic with $c = 1/(2l)$. By Corollary 2.9 we conclude that the operator $S_{Q_3}^p$ is decomposable for all $p \in (1,\infty)$ satisfying

$$\frac{2l}{(2l-1)(N+1)} > \left|\frac{1}{2} - \frac{1}{p}\right|. \tag{9}$$

So, given any $l \in \mathbb{N}$, this operator is decomposable (and hence satisfies the spectral mapping property $\sigma(S_{Q_3}^p) = \overline{Q_3(\mathbb{R}^N)}$) provided that $1/(N+1) > |1/2-1/p|$. For $N = 1$ this is satisfied for all $p \in (1,\infty)$. As noticed in [27], if $l = 1$ (i.e. for the heat operator) condition (9) is satisfied for all $p \in (1,\infty)$ provided that $N \le 3$. It would be interesting to know what happens for $N \ge 4$. For this example, $Q_3(\mathbb{R}^{N+1}) = \{z \in \mathbb{C} : \operatorname{Re}(z) \ge 0\}$.

(c) The following example has been considered in [14] and [27; p. 295]. Let

$$\begin{aligned} Q_4(x_1,x_2,x_3) &= (x_1 - x_2^2 - x_3^2 - i)(x_1 + x_2^2 + x_3^2 + i) \\ &= Q_-(x_1,x_2,x_3)Q_+(x_1,x_2,x_3). \end{aligned}$$

Kenig and Tomas have proved in [16], [17] that for each of the two factors Q_+ and Q_- we have a proper inclusion $\overline{Q_\pm(\mathbb{R}^3)} \subset \sigma(S_{Q_\pm}^p)$ for all $p \in (1,\infty)\setminus\{2\}$. Hence, as noted in [2], the operators $S_{Q_+}^p$ and $S_{Q_-}^p$ cannot be decomposable for any $p \ne 2$. Iha and Schubert proved in [14] that $\overline{Q_4(\mathbb{R}^3)} \ne \sigma(S_{Q_4}^p)$ for all p satisfying $|1/2-1/p| > 3/8$, so that $S_{Q_4}^p$ cannot be decomposable for such p. On the other hand, as shown in [27; p. 295], Q_4 does satisfy the conditions of Theorem 2.8 with $b = 1$ and $a = -1/2$ (it suffices to consider $k = 1$). Hence we conclude that $S_{Q_4}^p$ is decomposable for all p satisfying $|1/2-1/p| < 2/9$. This shows that there exist differential operators which are decomposable on $L^p(\mathbb{R}^N)$ for certain p but not for all $p \in (1,\infty)$.

(d) In [23]–[25] it is shown, for $n \ge 4$, that the polynomial

$$Q_5(x,y) = ((x-y^n)^2+1)(1+x^2)$$

has the property that $0 \notin \sigma(S_{Q_5}^p)$ whenever

$$\frac{4}{3+4/n} < p < \frac{4}{1-4/n}, \tag{10}$$

and that $0 \in \sigma(S_{Q_5}^p)$ for all $p \in (1,\infty)$ not satisfying (10).

Assume now that n is an arbitrary positive integer. To investigate the behaviour of $Q_5(x,y)$ as $|(x,y)| \to \infty$ we first consider points $(x,y) \in \mathbb{R}^2$ satisfying $|x-y^n| \le |x| + \frac{1}{2}|y|$. Then $|y^n| - |x| \le |x| + \frac{1}{2}|y|$ and, when $|y| \ge 1$, we have $|y| \le 4|x|$. Accordingly,

$$\frac{1}{|Q_5(x,y)|} \le \frac{1}{1+x^2} \le \frac{1}{x^2/2 + x^2/2} \le \frac{1}{y^2/32 + x^2/2} \le 32|(x,y)|^{-2}.$$

On the other hand, for $|y| \le 1$, we have

$$\frac{1}{|Q_5(x,y)|} \le \frac{1}{1+x^2} \le \frac{1}{y^2+x^2} = |(x,y)|^{-2}.$$

Now consider points $(x,y) \in \mathbb{R}^2$ satisfying $|x-y^n| > |x| + \frac{1}{2}|y|$. Then

$$\frac{1}{|Q_5(x,y)|} \le \frac{1}{|x-y^n|^2+1} \le \frac{1}{(|x|+|y|/2)^2} \le 4|(x,y)|^{-2}.$$

Accordingly, Q_5 satisfies

$$\frac{1}{|Q_5(x,y)|} \le 32|(x,y)|^{-2}, \tag{11}$$

for all $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$. By (11) and Theorem 1 in [14] we have, for $n \ge 5$,

$$\sigma(S^p_{Q_5}) = \begin{cases} \overline{Q(\mathbb{R}^2)} = [1,\infty] & \text{if } p \text{ satisfies (10)},\\ \overline{\mathbb{C}} & \text{otherwise.}\end{cases}$$

In particular, for $n \ge 5$ and p not satisfying (10), the operator $S^p_{Q_5}$ cannot be decomposable.

Now, for all $n \in \mathbb{N}$ and $(x,y) \in \mathbb{R}^2$, we have

$$\left|\frac{\frac{\partial Q_5}{\partial x}(x,y)}{Q_5(x,y)}\right| = \left|\frac{2(x-y^n)}{(x-y^n)^2+1} + \frac{2}{1+x^2}\right| \le 3$$

since

$$\left|\frac{2t}{t^2+1}\right| \le 1, \quad t \in \mathbb{R}. \tag{12}$$

By (12) again we also have, for $(x,y) \in \mathbb{R}^2$,

$$\left|\frac{\frac{\partial Q_5}{\partial y}(x,y)}{Q_5(x,y)}\right| = \left|\frac{-2ny^{n-1}(x-y^n)}{(x-y^n)^2+1}\right| \le n|y|^{n-1} \le n|(x,y)|^{n-1}.$$

Accordingly, the assumptions of Theorem 2.8 are satisfied with $k=1$ and $a = 1-n$ and $b=2$. It follows from this theorem that $S^p_{Q_5}$ is decomposable and has a translation invariant $C^1(\overline{\mathbb{C}})$-functional calculus for all p satisfying

$$\frac{n+2}{2n} < p < \frac{2n}{n-2} \quad \text{if } n > 2,$$

and for all $p \in (1,\infty)$ if $n = 1,2$.

By means of the following result (which provides a different criterion for decomposability than that of Theorems 2.6 and 2.8) we will see that $S^p_{Q_5}$ is actually decomposable for all p satisfying (10). ■

THEOREM 2.12. *Let $p \in (1,\infty)$ and $N \in \mathbb{N}$ and suppose that k is the smallest integer satisfying $k/N > |1/2 - 1/p|$. Suppose that $f \in C^k(\mathbb{R}^N)$ satisfies the following conditions:*

(i) *$1/|f(x)| = O(1/|x|^s)$ for some $s > 0$ as $|x| \to \infty$.*
(ii) *For all $\mu \in \mathbb{N}_0^N$ with $1 \le |\mu| \le k$ we have*

$$\left|\frac{\partial^\mu f}{\partial x^\mu}(x)\right| = O(|x|^c) \quad \textit{for some } c \ge 0 \textit{ as } |x| \to \infty.$$

Then either $\sigma(S^p_f) = \overline{\mathbb{C}}$ (in which case S^p_f is not decomposable if $\overline{f(\mathbb{R}^N)} \ne \overline{\mathbb{C}}$), or $\sigma(S^p_f) = \overline{f(\mathbb{R}^N)} \ne \overline{\mathbb{C}}$ and S^q_f is decomposable for all $q \in (1,\infty)$ satisfying $|1/2 - 1/q| \le |1/2 - 1/p|$.

Proof. It follows from Theorem 2.2 that $f \in \mathcal{U}^q(\mathbb{R}^N)$ for all q satisfying $|1/2 - 1/q| \le |1/2 - 1/p|$. Now suppose that $\sigma(S^p_f) \ne \overline{\mathbb{C}}$. Hence there exists some $\lambda \in \mathbb{C}$ such that $(\lambda I - S^p_f)^{-1} \in \mathcal{L}(L^p(\mathbb{R}^N))$. As noticed earlier we must then have $1/(\lambda - f) \in \mathcal{M}^p(\mathbb{R}^N)$ and $(\lambda I - S^p_f)^{-1} = S^p_{1/(\lambda - f)}$. Since also $\lambda - f$ satisfies conditions (i) and (ii) we may assume, without loss of generality, that $\lambda = 0$ and $(S^p_f)^{-1} = S^p_{1/f}$ exists in $\mathcal{L}(L^p(\mathbb{R}^N))$. Since $\mathcal{M}^p(\mathbb{R}^N) \subset \mathcal{M}^q(\mathbb{R}^N)$ for all q such that $|1/2 - 1/q| \le |1/2 - 1/p|$ we also have $S^q_{1/f} \in \mathcal{L}(L^q(\mathbb{R}^N))$ and so $0 \notin \sigma(S^q_f)$ as $S^q_{1/f} = (S^q_f)^{-1}$.

Let m be an integer satisfying $m \ge 1 + (k(1+c)+1)/s$. We show that the function $1/f^m$ then satisfies condition (3) in Theorem 2.4 with $a = b = 1$ for $|\mu| \le k$. If $|\mu| = 0$, then by (i) and the fact that $ms \ge 1$, there exists some $C_1 > 0$ such that for $|x| \ge 1$ we have

$$\frac{1}{|f(x)^m|} \le C_1 \frac{1}{|x|^{ms}} \le C_1 \frac{1}{|x|}.$$

For $1 \le |\mu| \le k$ we write $f^{-m} = \varphi \circ f$ with $\varphi(z) = 1/z^m$ for $z \in \mathbb{C} \setminus \{0\}$. Using Lemma 2.1 in [3] and conditions (i) and (ii) we can find a constant $C_2 > 0$ such that, for $|x| \ge 1$,

$$\left|\frac{\partial^\mu (f^{-m})}{\partial x^\mu}(x)\right| \le C_2 \frac{1}{|x|^{(m+1)s}} |x|^{|\mu|c} \le C_2 \frac{1}{|x|^{(m+1)s - kc}} \le C_2 \frac{1}{|x|^{1+|\mu|}}.$$

Hence, by Proposition 2.5, the operator $(S^p_{1/f})^m = S^p_{1/f^m}$ is $C^k(\mathbb{C})$-scalar and therefore decomposable. A result of Apostol [5; III, Theorem 1.5] then implies that $S^q_{1/f} = (S^q_f)^{-1}$ is decomposable for all q satisfying $|1/2 - 1/q| \le |1/2 - 1/p|$. From Lemma 2.4 in [3] we conclude that also S^q_f is decomposable for all such q. Finally, by Corollary 3.4 in [2] it follows that the spectral mapping theorem holds, that is, $\sigma(S_f^q) = \overline{f(\mathbb{R}^N)}$. ∎

Notice that, with $f = Q$ a polynomial and $c = \operatorname{degree}(Q)$, assumption (ii) in the preceding theorem is always satisfied. This observation leads to the following result.

COROLLARY 2.13. *Let Q be a polynomial such that $\mathbb{C} \setminus \overline{Q(\mathbb{R}^N)} \neq \emptyset$ and $|Q(x)| \to \infty$ for $|x| \to \infty$. Then the set $\operatorname{Dec}(Q)$, consisting of all $p \in (1,\infty)$ for which S_Q^p is decomposable, coincides with $\{p \in (1,\infty) : \sigma(S_Q^p) \neq \overline{\mathbb{C}}\}$ and is an interval containing 2 in its interior.*

Proof. Since $\{p \in (1,\infty) : \sigma(S_Q^p) \neq \overline{\mathbb{C}}\}$ coincides with the set of all $p \in (1,\infty)$ such that, for some $\lambda \in \mathbb{C}$ (dependent on p), the function $(\lambda - Q)^{-1} \in \mathcal{M}^p(\mathbb{R}^N)$ [27; Ch. 4, Theorem 4.1], this set must be an interval. That this set coincides with $\operatorname{Dec}(Q)$ follows from Theorem 2.12; the fact that 2 is an interior point of $\operatorname{Dec}(Q)$ is a consequence of Corollary 2.10. ∎

In particular, if $n \geq 5$, then the operator Q_5 of Ruiz (cf. Example 2.11(d)) is decomposable for all p satisfying (10).

Actually, as seen in our next statement, in this and similar cases we even obtain a translation invariant C_∞^k-functional calculus; see Definition 1.1(d).

THEOREM 2.14. *Let Q be a polynomial such that $\overline{Q(\mathbb{R}^N)} \neq \overline{\mathbb{C}}$ and $|Q(x)| \to \infty$ for $|x| \to \infty$. Let $p \in \operatorname{Dec}(Q)$ and k be the smallest integer satisfying $k/N > |1/2 - 1/p|$. Then S_Q^p has a translation invariant $C_\infty^k(\mathbb{C})$-functional calculus $\Phi : \varphi \mapsto S_{\varphi\circ Q}^p$. In particular, S_Q^p has the Lyubich–Matsaev property.*

Proof. It suffices to show that the operator $S_{\varphi\circ Q}^p$ is continuous in $L^p(\mathbb{R}^N)$ for all $\varphi \in C_\infty^k(\mathbb{C})$. Hence, fix an arbitrary $\varphi \in C_\infty^k(\mathbb{C})$. Without loss of generality we may assume that $0 \notin \overline{Q(\mathbb{R}^N)}$. Since S_Q^p is decomposable we must have $0 \notin \sigma(S_Q^p)$ [2; Corollary 3.4].

By the proof of Theorem 2.12 there exists some $m \in \mathbb{N}$ such that $S_{Q^{-m}}^p$ has a translation invariant $C^k(\overline{\mathbb{C}})$-functional calculus Ψ given by $\Psi(\psi) := S_{\psi\circ(Q^{-m})}^p$ for $\psi \in C^k(\overline{\mathbb{C}})$. Hence, a translation invariant $C^k(\overline{\mathbb{C}})$-functional calculus Φ_m for $S_{Q^m}^p$ is given by $\Phi_m(\tau) = S_{\tau^*\circ(Q^{-m})}^p$ for $\tau \in C^k(\overline{\mathbb{C}})$, where we define $\tau^*(z) = \tau(\infty)\chi_{\{0\}}(z) + \tau(z^{-1})\chi_{\mathbb{C}\setminus\{0\}}(z)$, for $z \in \mathbb{C}$. Note that $\tau^* \in C^k(\mathbb{C})$. Since φ is analytic in a neighbourhood of ∞ there exists an $r \geq 1$ such that φ is analytic on $R_r := \{z \in \overline{\mathbb{C}} : |z| > r\}$.

Fix $\chi \in C^\infty(\overline{\mathbb{C}})$ with the properties $\operatorname{supp}(\chi) \subset R_{r^m}$ and $\operatorname{supp}(1-\chi) \cap \overline{R_{(2r)^m}} = \emptyset$. Since $|Q(x)| \to \infty$ for $|x| \to \infty$, we see that $1 - \chi^* \circ Q^{-m}$ is a C^∞-function with compact support in $\mathbb{R}^N$. Hence $(\varphi\circ Q)\cdot(1-\chi^*\circ Q^{-m})$ is a p-multiplier function and so it suffices to show that also $(\varphi\circ Q)\cdot(\chi^*\circ Q^{-m})$ is a p-multiplier function. To prove this let us first prove that $S_{\chi^*\circ Q^{-m}}^p = S_{\chi\circ Q^m}^p$ has its range in the closed subspace

$$X := \mathcal{E}_m(\operatorname{supp}(\chi)) = \bigcap \{\ker \Phi_m(\varrho) : \varrho \in C^k(\overline{\mathbb{C}}),\ \operatorname{supp}(\varrho) \cap \operatorname{supp}(\chi) = \emptyset\}$$

of $L^p(\mathbb{R}^N)$. Obviously, X is invariant for all translation invariant operators in $\mathcal{L}(L^p(\mathbb{R}^N))$ and for $(S_Q^p)^n = S_{Q^n}^p$, $n \in \mathbb{N}$. Moreover, $D(S_{Q^n}^p|X) = S_{Q^{-n}}^p(X)$. Hence, the inclusion "$\supseteq$" is obvious. To prove "$\subseteq$" we fix an arbitrary $f \in D(S_{Q^n}^p|X) = X \cap D(S_{Q^n}^p) = X \cap \operatorname{ran}(S_{Q^{-n}}^p)$. Thus $f = S_{Q^{-n}}^p g$ for some $g \in L^p(\mathbb{R}^N)$. For all $\varrho \in C^k(\overline{\mathbb{C}})$ with $\operatorname{supp}(\varrho) \cap \operatorname{supp}(\chi) = \emptyset$ we then have

$$0 = \Phi_m(\varrho)f = \Phi_m(\varrho)S_{Q^{-n}}^p g = S_{Q^{-n}}^p \Phi_m(\varrho)g,$$

and hence $\Phi_m(\varrho)g = 0$ since $S_{Q^{-n}}^p$ is injective. It follows that $g \in X$ and we indeed have $D(S_{Q^n}^p|X) = S_{Q^{-n}}^p(X)$ for all $n \in \mathbb{N}$. It now follows that $S_{Q^n}^p|X$ is closed for all $n \in \mathbb{N}$ and that $S_{Q^n}^p|X = (S_Q^p|X)^n$ and $0 \in \mathbb{C} \setminus \sigma(S_{Q^n}^p|X)$.

In the next step we prove that $\sigma(S_{Q^m}^p|X) \subseteq \operatorname{supp}(\chi)$. Hence, fix an arbitrary $\lambda \in \mathbb{C} \setminus \operatorname{supp}(\chi)$. Since the algebra $C^k(\overline{\mathbb{C}})$ is normal we can find $\psi \in C^k(\overline{\mathbb{C}})$ such that $\operatorname{supp}(1-\psi) \cap \operatorname{supp}(\chi) = \emptyset$ and $\lambda \notin \operatorname{supp}(\psi)$. Hence, the function $\psi_\lambda : z \mapsto (\lambda - z)^{-1}\psi(z)\chi_{\overline{\mathbb{C}}\setminus\{\lambda,\infty\}}(z)$ is in $C^\infty(\overline{\mathbb{C}})$. For $f \in D(S_{Q^m}^p|X)$ we then have

$$\begin{aligned}\Phi_m(\psi_\lambda)(\lambda I - S_{Q^m}^p)f &= \mathcal{F}^{-1}((\psi_\lambda^* \circ Q^{-m}) \cdot (\lambda - Q^m)\widehat{f}) \\ &= \mathcal{F}^{-1}((\psi^* \circ Q^{-m}) \cdot \widehat{f}) = \Phi_m(\psi)f = f\end{aligned}$$

since $f = \Phi_m(\psi)f + \Phi_m(1-\psi)f = \Phi_m(\psi)f$ by the definition of X.

Hence, we have proved that $\sigma(S_{Q^m}^p|X) \cap \mathbb{C} \subseteq \operatorname{supp}(\chi)$ and thus $\sigma(S_{Q^m}^p|X) \subset R_{r^m}$. By the spectral mapping theorem [8; VII, Theorem 9.10] we conclude that $\sigma(S_Q^p|X) \subset R_r$. Since φ is analytic in R_r we have a Laurent series expansion $\varphi(z) = \sum_{n=0}^\infty a_n z^{-n}$ for φ which converges uniformly on all compact subsets of R_r. Because of $\operatorname{ran}(S_{\chi\circ Q^m}^p) \subseteq X$ and the continuity of the analytic functional calculus for $S_Q^p|X$ we obtain the convergence of $\sum_{n=0}^\infty a_n (S_Q^p|X)^{-n} S_{\chi\circ Q^m}^p$ in $\mathcal{L}(L^p(\mathbb{R}^N))$. It is clear that this operator is translation invariant with symbol $(\varphi \circ Q) \cdot (\chi \circ Q^m)$. Thus $(\varphi \circ Q) \cdot (\chi \circ Q^m) = (\varphi \circ Q) \cdot (\chi^* \circ Q^{-m})$ is a p-multiplier function. ∎

Finally, we wish to give a decomposability criterion (dependent on p) for systems of linear partial differential operators with constant coefficients operating in $L^p(\mathbb{R}^N)^m$. First we recall some definitions and notations from [4]. In particular, if Q is a matrix-valued function on $\mathbb{R}^N$ with polynomial entries recall that $\Sigma(Q) = \overline{\bigcup_{x\in\mathbb{R}^N} \sigma(Q(x))}$; the closure is taken in $\mathbb{C}$.

For a matrix-valued C^∞-function $a = [a_{ij}]_{i,j=1}^m$ defined on $\mathbb{R}^N$ let S_a^p denote the closure of the linear operator S in $L^p(\mathbb{R}^N)^m$ with domain

$$D(S) = D^{(p)} := \{f \in L^p(\mathbb{R}^N)^m : \operatorname{supp}(\widehat{f}) \text{ is compact}\}$$

and given by $Sf := \mathcal{F}^{-1}(\chi a \widehat{f})$, for $f \in D^{(p)}$, where χ is any C^∞-function with compact support, satisfying $\chi \equiv 1$ in a neighbourhood of $\operatorname{supp}(\widehat{f})$. Since χa is an $(m \times m)$-matrix whose entries are p-multiplier functions we see that $Sf \in L^p(\mathbb{R}^N)^m$. Moreover, this definition is independent of the choice of $\chi \in C_c^\infty(\mathbb{R}^N)$ with the property that $\chi \equiv 1$ in a neighbourhood of $\operatorname{supp}(\widehat{f})$. If $a = Q$ is actually a matrix polynomial, then it can be shown (using standard regularization techniques) that S_Q^p coincides with the closed linear operator $Q_p(D)$ with domain $\{f \in L^p(\mathbb{R}^N)^m : Q_p(D)f \in L^p(\mathbb{R}^N)^m\}$ and given by $f \mapsto Q_p(D)f$, where $Q_p(D)f$ is defined in the sense of distributions. From this second definition it follows that we always have $(S_Q^p)^k \subseteq S_{Q^k}^p$, for $k \in \mathbb{N}$. Since both operators obviously coincide on $D^{(p)}$ and since $(S_Q^p)^k$ is closed whenever S_Q^p has non-empty resolvent set [8; VII, Theorem 9.7], we obtain the following result.

LEMMA 2.15. *Let Q be an $(m \times m)$-matrix polynomial. If $p \in [1, \infty)$ has the property that the operator S_Q^p has non-empty resolvent set, then $(S_Q^p)^k = S_{Q^k}^p$ for all $k \in \mathbb{N}$.*

In the following theorem this fact is needed for the proof of part (c), which may be viewed as a matrix version of Theorem 3.1 in [14]. Moreover, $\|\cdot\|$ denotes any matrix norm on the set $M_m(\mathbb{C})$ of all $(m \times m)$-matrices over $\mathbb{C}$. Given $\lambda \in \mathbb{C}$ and a matrix polynomial Q in N real variables, we say that $(\lambda - Q)^{-1}$ *vanishes at infinity* if there exists $\delta > 0$ with the property that $(\lambda - Q(x))^{-1}$ exists for all $|x| > \delta$ and

$$\lim_{|x| \to \infty} \|(\lambda - Q(x))^{-1}\| = 0. \tag{13}$$

THEOREM 2.16. *Let $Q = [Q_{jk}]_{j,k=1}^m$ be a non-constant matrix polynomial in N real variables such that $(\lambda - Q)^{-1}$ vanishes at infinity for some $\lambda \in \mathbb{C}$. Then we have the following properties:*

(a) *There exists some $r > 0$ such that $\|(\lambda - Q(x))^{-1}\| = O(|x|^{-r})$, for $|x| \to \infty$.*

(b) *If r is as in* (a), *then $\det(\lambda - Q(x)) = O(|x|^{mr})$, for $|x| \to \infty$.*

(c) *Let $p \in [1, \infty)$. If $\sigma(S_Q^p) \neq \overline{\mathbb{C}}$, then $\Sigma(Q) = \sigma(S_Q^p) \cap \mathbb{C}$ and the operator S_Q^p is decomposable. If $\sigma(S_Q^p) = \overline{\mathbb{C}}$ but $\Sigma(Q) \neq \sigma(S_Q^p) \cap \mathbb{C}$, then S_Q^p is not decomposable.*

(d) *There exists some $s > 0$ such that, for all $p \in (1, \infty)$ satisfying $|1/2 - 1/p| < s$, the operator S_Q^p is decomposable in $L^p(\mathbb{R}^N)^m$.*

Proof. (b) Write $a_{ij}(\lambda, x)$ for the (i,j)-entry of the matrix $(\lambda - Q(x))^{-1}$ and define $q_Q(\lambda, x) = \det(\lambda - Q(x))$, for $x \in \mathbb{R}^N$. Because of the identity

$$q_Q(\lambda, x)^{-1} = \det(\lambda - Q(x))^{-1} = \sum_{\pi \in S_m} \operatorname{sgn}(\pi) \prod_{j=1}^m a_{j,\pi(j)}(\lambda, x) \tag{14}$$

we see that the growth condition in (a) indeed implies that of (b).

(a) Note that $a_{ij}(\lambda, x) = A_{ij}(\lambda, x)/q_Q(\lambda, x)$ (for $i, j = 1, \ldots, m$), where $A_{ij}(\lambda, x)$ is the minor of $\lambda - Q(x)$ corresponding to (i, j). By (13) we have $a_{ij}(\lambda, x) \to 0$ for $|x| \to \infty$ and hence, by (14), also $q_Q(\lambda, x)^{-1} \to 0$ as $|x| \to \infty$. In particular, $q_Q(\lambda, \cdot)^{-1}(\{0\})$ is a compact subset of $\mathbb{R}^N$. We conclude from the arguments in the proof of Theorem 3.1 in [11], by replacing ξ with x, grad $P(\xi)$ with $A_{ij}(\lambda, x)$ and $P(\xi)$ with $q_Q(\lambda, x)$ in those arguments, that there exists some $C_1 > 0$ and a constant $b_1 > 0$ such that

$$|A_{ij}(\lambda, x)|^2 \le C_1(1 + |q_Q(\lambda, x)|^2)^{1-b_1} \tag{15}$$

for $i, j = 1, \ldots, m$. Moreover, by Theorem 3.2 in [11], for some other constants $C_2 > 0$ and $b_2 > 0$, we also have

$$|x|^4 \le C_2(1 + |q_Q(\lambda, x)|^2)^{b_2}. \tag{16}$$

From (15) and (16) we conclude that

$$|a_{ij}(\lambda, x)| = \frac{|A_{ij}(\lambda, x)|}{|q_Q(\lambda, x)|} = O(|x|^{-2b_1/b_2}), \quad |x| \to \infty,$$

from which (a) follows with $r = 2b_1/b_2$.

(c) Assume that $\sigma(S_Q^p) \neq \overline{\mathbb{C}}$ and fix an arbitrary point $\mu \in \mathbb{C} \setminus \Sigma(Q)$. Such a point μ exists since $\Sigma(Q) \subseteq \sigma(S_Q^p) \setminus \{\infty\}$ [4; Lemma 2.2]. Then, by Lemma 1.2 in [4] and the discussion preceding that lemma, we see that $(\mu - Q)^{-1}$ is a bounded matrix-valued function on $\mathbb{R}^N$ with rational functions in its entries. Hence, for $|x| \to \infty$,

$$\begin{aligned} &\|(\mu - Q(x))^{-1}\| \\ &\quad = \|(\lambda - Q(x))^{-1} + (\lambda - \mu)(\lambda - Q(x))^{-1}(\mu - Q(x))^{-1}\| = O(|x|^{-r}), \end{aligned} \tag{17}$$

where $r > 0$ is as in (a).

Let K denote the maximum of the degrees of $q_Q(\mu, x)$ and each of the minors of the matrix polynomial $\mu - Q$. Let n be the smallest integer exceeding $N/2$ and let $k \in \mathbb{N}$ satisfy $kr \ge (K+1)n$. In the next step we show that the entries of the matrix function $(\mu - Q)^{-k}$ are actually Fourier transforms of elements from $L^1(\mathbb{R}^N)$, and hence are p-multiplier functions. First, if $\alpha \in \mathbb{N}_0^N$, then by (b) and by the arguments in the proof of Lemma 4 in [14] we see that

$$\left|\frac{\partial^\alpha q_Q(\mu, x)^{-1}}{\partial x^\alpha}\right| = O(|x|^{|\alpha|(K-mr-1)-rm}), \quad |x| \to \infty.$$

Let $A_{ij}(\mu,x)$ be the minor of $\mu - Q(x)$ corresponding to the indices $i,j \in \{1,\ldots,m\}$. Then

$$\left|\frac{\partial^\alpha A_{ij}(\mu,x)}{\partial x^\alpha}\right| = O(|x|^{K-|\alpha|}), \quad |x| \to \infty.$$

Using the Leibniz rule we see, for $\alpha \in \mathbb{N}_0^N$ with $|\alpha| > 0$, that

$$\left|\frac{\partial^\alpha}{\partial x^\alpha}(A_{ij}(\mu,x)q_Q(\mu,x)^{-1})\right| = O(|x|^{K|\alpha|-rm}), \quad |x| \to \infty. \tag{18}$$

Now, each entry of $(\mu - Q(x))^{-k}$ is a sum of m^{k-1} terms of the form

$$\prod_{t=1}^{k} \frac{A_{j_t,j_{t+1}}(\mu,x)}{q_Q(\mu,x)}, \quad \text{where } \left|\frac{A_{j_t,j_{t+1}}(\mu,x)}{q_Q(\mu,x)}\right| = O(|x|^{-r}) \quad \text{for} \quad |x| \to \infty,$$

by (17). Applying the product rule for differentiation and the estimates (18) we see, for all $\alpha \in \mathbb{N}_0^N$, that

$$\frac{\partial^\alpha}{\partial x^\alpha}\Big(\prod_{t=1}^{k} \frac{A_{j_t,j_{t+1}}(\mu,x)}{q_Q(\mu,x)}\Big)$$

is a sum of at most $k^{|\alpha|}$ terms, each of the form

(19)
$$C_{t(\cdot)}\Big(\prod_{u=v+1}^{k} \frac{A_{j_{t(u)},j_{t(u)+1}}(\mu,x)}{q_Q(\mu,x)}\Big)\Big(\prod_{w=1}^{v} \frac{\partial^{\alpha_w}}{\partial x^{\alpha_w}}\Big(\frac{A_{j_{t(w)},j_{t(w)+1}}(\mu,x)}{q_Q(\mu,x)}\Big)\Big),$$

where $\alpha_1,\ldots,\alpha_v \in \mathbb{N}_0^N$ satisfy $\sum_{w=1}^{v} \alpha_w = \alpha$ and $t(\cdot) : \{1,\ldots,k\} \to \{1,\ldots,k\}$ is a bijection, and $C_{t(\cdot)}$ is a constant. Because of (17) and (18), as $|x| \to \infty$ this term has order $O(|x|^{(\sum_{w=1}^{v} K|\alpha_w|)-(k-v)r-mrv})$. It follows, for $\alpha \in \mathbb{N}_0^N$ with $|\alpha| > 0$, that

$$\left\|\frac{\partial^\alpha}{\partial x^\alpha}(\mu - Q(x))^{-k}\right\| = O(|x|^{K|\alpha|-kr}), \quad |x| \to \infty. \tag{20}$$

Let $b_{ij}(x)$ be an arbitrary entry of the matrix $(\mu - Q(x))^{-k}$. As in formula (10) of [14] we see that

$$\int_{\mathbb{R}^N} |\mathcal{F}^{-1}(b_{ij})(\xi)|\,d\xi \le C\Big[\sum_{|\alpha|\le n} d_\alpha \int_{\mathbb{R}^N} \left|\frac{\partial^\alpha b_{ij}(x)}{\partial x^\alpha}\right|^2 dx\Big]^{1/2}, \tag{21}$$

where the d_α are the multinomial coefficients. By the definition of n and (20) we see that the right-hand side of (21) must be finite, and hence that $\mathcal{F}^{-1}(b_{ij}) \in L^1(\mathbb{R}^N)$. It follows that the matrix-valued function $(\mu - Q(x))^{-k}$ has entries from $\mathcal{F}(L^1(\mathbb{R}^N)) \subset \mathcal{M}^p(\mathbb{R}^N)$. Thus $S^p_{(\mu-Q)^{-k}}$ is a bounded linear operator in $L^p(\mathbb{R}^N)^m$. A straightforward computation shows that $S^p_{(\mu-Q)^{-k}} = (S^p_{(\mu-Q)^k})^{-1}$. On the other hand, Lemma 2.15 implies that $S^p_{(\mu-Q)^k} = (S^p_{\mu-Q})^k$ since, by hypothesis, S^p_Q and hence also $\mu I - S^p_Q = S^p_{\mu-Q}$ has non-empty resolvent set. By the spectral mapping theorem [8; VII, Theorem 9.10], 0 belongs to the resolvent set of $S^p_{\mu-Q}$ and therefore, $\mu \in \mathbb{C}\setminus\sigma(S^p_Q)$. Since $\Sigma(Q) \subset \sigma(S^p_Q)$ we obtain $\Sigma(Q) = \mathbb{C} \cap \sigma(S^p_Q)$.

The convolution algebra $L^1(\mathbb{R}^N)$ is known to be a regular, semisimple Banach algebra and so the same is true of the algebra $\mathcal{B} = L^1(\mathbb{R}^N) \oplus \mathbb{C}$ that we obtain by adjoining a unit. The mapping $\Lambda : M_m(\mathcal{B}) \to \mathcal{L}(L^p(\mathbb{R}^N)^m)$ defined by

$$\Lambda([g_{ij} + \alpha_{ij}]_{i,j=1}^m) := [S^p_{\widehat{g}_{ij}+\alpha_{ij}}]_{i,j=1}^m, \quad g_{ij} \in L^1(\mathbb{R}^N),\ \alpha_{ij} \in \mathbb{C},$$

is a unital homomorphism containing $(S^p_{(\mu-Q)^k})^{-1}$ in its range. Hence, by [4; Proposition 1.4] the operator $S^p_{(\mu-Q)^{-k}} = (S^p_{(\mu-Q)^k})^{-1}$ is decomposable. But it was noted above that $S^p_{(\mu-Q)^k} = (S^p_{\mu-Q})^k = (\mu I - S^p_Q)^k$ and so $(\mu I - S^p_Q)^{-k} = (S^p_{(\mu-Q)^k})^{-1}$. Hence, Theorem 1.5 in [5; Section III] shows that $(\mu I - S^p_Q)^{-1}$ is decomposable. By [3; Lemma 2.4] the operator $\mu I - S^p_Q$ is also decomposable, from which the decomposability of S^p_Q is immediate.

Suppose now that $\sigma(S^p_Q) = \overline{\mathbb{C}}$ but $\Sigma(Q) \ne \sigma(S^p_Q) \cap \mathbb{C}$. Then S^p_Q cannot be decomposable by Proposition 2.3 of [4].

(d) We proceed along the lines of the proof of Proposition 4.1 in [4]. By part (b), the set $N(\lambda) := \{x \in \mathbb{R}^N : q_Q(\lambda,x) = 0\}$ is compact and hence contained in a ball $U_R(0) = \{x \in \mathbb{R}^N : |x| < R\}$ for some $R > 0$. For a closed set $F \subset \mathbb{R}^N$ the closed linear subspace $\mathcal{E}^p(F) := \{f \in L^p(\mathbb{R}^N)^m : \operatorname{supp}(\widehat{f}) \subset F\}$ of $L^p(\mathbb{R}^N)^m$ is invariant for S^p_Q (in the sense that $S^p_Q(D(S^p_Q) \cap \mathcal{E}^p(F)) \subseteq \mathcal{E}^p(F)$). We note that if F happens to be compact, then $\mathcal{E}^p(F) \subset D(S^p_Q)$ for all $p \in [1,\infty)$. Fix $\varphi \in C^\infty(\mathbb{R}^N)$ such that $\operatorname{supp}(\varphi) \subset U_{R+1}(0)$ and $\operatorname{supp}(1-\varphi) \cap \overline{U_R(0)} = \emptyset$. Then the range $\operatorname{ran}(S^p_\varphi) \subset \mathcal{E}^p(\operatorname{supp}(\varphi)) \subset \mathcal{E}^p(\overline{U_{R+1}(0)})$ and $\operatorname{ran}(S^p_{1-\varphi}) \subset \mathcal{E}^p(\operatorname{supp}(1-\varphi)) \subset \mathcal{E}^p(\mathbb{R}^N \setminus U_R(0))$. From this we see that $L^p(\mathbb{R}^N)^m = \mathcal{E}^p(\overline{U_{R+1}(0)}) + \mathcal{E}^p(\mathbb{R}^N \setminus U_R(0))$. As in the proof of part (a) of Proposition 4.1 in [4] one shows that the restriction $S^p_Q|\mathcal{E}^p(\overline{U_{R+1}(0)})$ is decomposable (for all $p \in [1,\infty)$).

We now show, for $p \in (1,\infty)$ satisfying $|1/2 - 1/p| < s := r/(N(K+1))$ (with K defined as in the proof of (c)), that the operator $S^p_Q|\mathcal{E}^p(\mathbb{R}^N \setminus U_R(0))$ is also decomposable. Fix a function $\varrho \in C^\infty(\mathbb{R}^N)$ such that $\operatorname{supp}(\varrho) \cap N(\lambda) = \emptyset$ and $\operatorname{supp}(1-\varrho) \subset U_R(0)$. Then the function $\varrho(\lambda - Q)^{-1}$ (defined to be identically 0 on $N(\lambda)$) is a C^∞-matrix function. An inspection of the proof of (18) shows that the growth behaviour of (18) is also true for the entries $A_{ij}(\lambda,x)q_Q(\lambda,x)^{-1}$ of $(\lambda - Q(x))^{-1}$. Since the derivatives of ϱ have compact support we conclude from this and part (a) that the entries e_{ij} of $\varrho(\lambda - Q)^{-1}$, for $i,j = 1,\ldots,m$, satisfy the conditions of Theorem 2.4 and Proposition 2.5

with $a=-K$ and $b=r$ and for every $k \geq r/(K+1)$. In particular, the functions e_{ij} are p-multiplier functions for each $i,j=1,\ldots,m$ and we deduce that $T := S^p_{\varrho(\lambda-Q)^{-1}}$ is a bounded linear operator on $L^p(\mathbb{R}^N)^m$. Obviously, $\mathcal{E}^p(\mathbb{R}^N \setminus U_R(0))$ is invariant for T and a straightforward computation (as in the proof of part (b) of Proposition 4.1 in [4]) shows that

$$(22)\qquad \begin{aligned} T|\mathcal{E}^p(\mathbb{R}^N \setminus U_R(0)) &= (S^p_{\lambda-Q}|\mathcal{E}^p(\mathbb{R}^N \setminus U_R(0)))^{-1} \\ &= (\lambda I - S^p_Q|\mathcal{E}^p(\mathbb{R}^N \setminus U_R(0)))^{-1}. \end{aligned}$$

By Proposition 2.5 each of the operators $S^p_{e_{ij}} \in \mathcal{L}(L^p(\mathbb{R}^N))$, for $i,j=1,\ldots,m$, is generalized scalar and has a $C^\infty(\mathbb{C})$-functional calculus Ψ_{ij} given by $\Psi_{ij}(\psi) := S^p_{\psi\circ e_{ij}}$, for $\psi \in C^\infty(\mathbb{C})$. Obviously, with $\mathcal{E}^p_0(\mathbb{R}^N \setminus U_R(0)) := \{f \in L^p(\mathbb{R}^N) : \operatorname{supp}(\widehat{f}) \subset \mathbb{R}^N \setminus U_R(0)\}$, the continuous homomorphism $\theta_{ij} : C^\infty(\mathbb{C}) \to \mathcal{L}(\mathcal{E}^p_0(\mathbb{R}^N \setminus U_R(0)))$ defined by

$$\theta_{ij}(\psi) := S^p_{\psi\circ e_{ij}}|\mathcal{E}^p_0(\mathbb{R}^N \setminus U_R(0)), \qquad \psi \in C^\infty(\mathbb{C}),$$

is a $C^\infty(\mathbb{C})$-functional calculus for $S^p_{e_{ij}}|\mathcal{E}^p_0(\mathbb{R}^N \setminus U_R(0))$. We note that the ranges of these homomorphisms mutually commute, i.e.

$$\begin{aligned} \theta_{i_1j_1}(\psi_1)\theta_{i_2j_2}(\psi_2) &= \theta_{i_2j_2}(\psi_2)\theta_{i_1j_1}(\psi_1) \\ &= S^p_{(\psi_1\circ e_{i_1j_1})\cdot(\psi_2\circ e_{i_2j_2})}|\mathcal{E}^p_0(\mathbb{R}^N \setminus U_R(0)). \end{aligned}$$

Hence, the continuous m^2-linear mapping $(\psi_{ij})^m_{i,j=1} \mapsto \prod^m_{i,j=1}\theta_{ij}(\psi_{ij})$ extends to a continuous unital homomorphism $\theta : C^\infty(\mathbb{C}^{m^2}) \to \mathcal{L}(\mathcal{E}^p_0(\mathbb{R}^N \setminus U_R(0)))$, since $C^\infty(\mathbb{C}^{m^2})$ coincides with the completed projective tensor product $\widehat{\bigotimes}^m_{i,j=1}C^\infty(\mathbb{C})$. Note that $\theta(z_{ij}) = S^p_{e_{ij}}$, for $i,j=1,\ldots,m$, where z_{ij} denotes the (i,j)-coordinate function $z_{ij} : \mathbb{C}^{m^2} \to \mathbb{C}$. Let

$$M := \max_{1\leq i,j\leq m}\sup_{x\in\mathbb{R}^N}|e_{ij}(x)| + 1.$$

With $B_M := \{z \in \mathbb{C} : |z| \leq M\}$ and $\mathcal{A} := \{\psi|_{B^{m^2}_M} : \psi \in C^\infty(\mathbb{C}^{m^2})\}$ one easily verifies that $\mathcal{A}$ is a normal spectrally closed subalgebra of $C(B^{m^2}_M)$ in the sense of [1; §3]. Moreover, θ induces a (well-defined) unital homomorphism $\Phi_1 : \mathcal{A} \to \mathcal{L}(\mathcal{E}^p_0(\mathbb{R}^N \setminus U_R(0)))$ by the formula $\Phi_1(\psi|_{B^{m^2}_M}) := \theta(\psi)$, for $\psi \in C^\infty(\mathbb{C}^{m^2})$. By means of Φ_1 we can construct a homomorphism

$$\Phi_m : M_m(\mathcal{A}) = \mathcal{A} \otimes M_m(\mathbb{C}) \to \mathcal{L}(\mathcal{E}^p_0(\mathbb{R}^N \setminus U_R(0))^m) = \mathcal{L}(\mathcal{E}^p(\mathbb{R}^N \setminus U_R(0)))$$

by $\Phi_m([\psi_{ij}]^m_{i,j=1}) := [\Phi_1(\psi_{ij})]^m_{i,j=1}$, for $[\psi_{ij}]^m_{i,j=1} \in M_m(\mathcal{A})$. Then by [1; Corollary 3.14] every operator in the range of Φ_m is decomposable. In particular, this applies to

$$\Phi_m([z_{ij}]^m_{i,j=1}) = [S^p_{e_{ij}}|\mathcal{E}^p_0(\mathbb{R}^N \setminus U_R(0))]^m_{i,j=1} = T|\mathcal{E}^p(\mathbb{R}^N \setminus U_R(0)).$$

Hence, by (22) and [3; Lemma 2.4] the operator $S^p_Q|\mathcal{E}^p(\mathbb{R}^N \setminus U_R(0))$ is decomposable.

The decomposability of S^p_Q on all of $L^p(\mathbb{R}^N)^m$ now follows as in part (c) of the proof of [4; Proposition 4.1]. ∎

3. Second order linear partial differential operators with real coefficients. In this section we investigate the local spectral behaviour of linear second order partial differential operators on $L^p(\mathbb{R}^N)$ with real coefficients. In this connection let us remark that a complete investigation of the invertibility problem for second order linear differential operators (with complex coefficients) which have $(N-1)$-dimensional level sets is contained in the work [6] of Chang and Tomas.

So, let us now consider the operator

$$\sum^N_{j,k=1} a_{jk}\frac{\partial^2}{\partial x_j\partial x_k} + \sum^N_{j=1} b_j\frac{\partial}{\partial x_j} + c$$

with real constants a_{jk}, b_j and c, for $j,k=1,\ldots,N$. The corresponding minimal differential operator in $L^p(\mathbb{R}^N)$ is S^p_Q, where Q is the polynomial

$$Q(\xi) = -\sum^N_{j,k=1} a_{jk}\xi_j\xi_k + i\sum^N_{j=1} b_j\xi_j + c, \qquad \xi \in \mathbb{R}^N.$$

This may also be written as

$$Q(\xi) = \xi^{\mathrm{t}}A\xi + ib^{\mathrm{t}}\xi + c$$

where $\xi^{\mathrm{t}} = [\xi_1,\ldots,\xi_N]$, $A = \big[-\frac{1}{2}(a_{jk}+a_{kj})\big]^N_{j,k=1}$ and $b^{\mathrm{t}} = [b_1,\ldots,b_N]$. Since an affine change of variables does not change the spectral behaviour of such an operator (cf. Lemma 1.2(c)), we may assume that Q is of the form

$$(23)\qquad Q(x,y,z) = |x|^2 - |y|^2 + c + i(\alpha^{\mathrm{t}}x + \beta^{\mathrm{t}}y + \gamma^{\mathrm{t}}z),$$

where now $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $z \in \mathbb{R}^k$ are the variables (with $n,m,k \in \mathbb{N}_0$ satisfying $n+m+k=N$) and $\alpha \in \mathbb{R}^n$, $\beta \in \mathbb{R}^m$, $\gamma \in \mathbb{R}^k$, $c \in \mathbb{R}$ are constants.

Accordingly, we now have various cases to consider in (23).

Case I: If $m=0$, $n \neq 0$ and $\gamma = 0$ (resp. $n=0$, $m\neq 0$ and $\gamma=0$), then Q is elliptic with respect to the variables in x (resp. in y) and does not depend on z. In this situation it follows from [2] that for all $p \in (1,\infty)$ the operator S^p_Q is decomposable, admits a functional calculus as in Theorem 2.6 and has the Lyubich–Matsaev property.

Case II: Suppose that $n \neq 0, m = 0$ and $\gamma \neq 0$ (or $m \neq 0$, $n = 0$ and $\gamma \neq 0$), in which case $k \neq 0$. We shall only consider the first of the two situations, since the other is of the same type. Without loss of generality we may assume that $\gamma_1 \neq 0$. By an additional linear change of variables (i.e. $x_j \mapsto x_j$, $\alpha^t x+\gamma^t z \mapsto z_1$ and $z_l \mapsto z_l$ for $2 \le l \le k$) we may suppose that Q has the form $Q(x,z) = |x|^2 + c + iz_1$, where $\gamma_1 \ne 0$. By Example 2.11(b), S_Q^p is decomposable (considered on $L^p(\mathbb{R}^{n+1})$) and has a translation invariant $C^2(\overline{\mathbb{C}})$-functional calculus for all $p \in (1,\infty)$ satisfying $2/(n+1) > |1/2 - 1/p|$. By Lemma 1.2, the operator S_Q^p considered on $L^p(\mathbb{R}^{n+k})$ must also be decomposable and have a translation invariant $C^2(\overline{\mathbb{C}})$-functional calculus. Moreover, S_Q^p also has the Lyubich–Matsaev property. If $n \le 3$ and $k \in \mathbb{N}$, then this is the case for all $p \in (1,\infty)$. Notice that, independent of n and k, we have $\sigma(S_Q^p) = \overline{\mathbb{C}} = \overline{Q(\mathbb{R}^N)}$ for all $p \in (1,\infty)$.

CASE III: Both $n \ne 0$ and $m \ne 0$. Here we have to distinguish between various subcases of a qualitatively different nature.

SUBCASE (a): $m = n = 1$ and $k \in \mathbb{N}_0$ arbitrary. Then (13) becomes

$$Q(x,y,z) - c = (x-y)(x+y) + i(\alpha x + \beta y + \gamma^t z) \tag{24}$$

with $\alpha,\beta \in \mathbb{R}$ and $\gamma \in \mathbb{R}^k$. Thus the real part of (24) is a product of real linear functionals and the imaginary part is real linear. It follows from [3; Theorem 2.2] that, for all $p \in (1,\infty)$, we have $S_{Q-c}^p = S_Q^p - cI$ and this operator is decomposable, admits a translation invariant $\mathcal{K}^2$-functional calculus and has the Lyubich–Matsaev property. The same is then true of S_Q^p. ■

We will require the following result due to Kenig and Tomas [16], [17].

THEOREM 3.1. *For some $q \in (0,\infty)$ let $\varphi \in L^q(\mathbb{R}) \cap L^\infty(\mathbb{R})$ be a non-constant function.*

(a) *If $n \ge 1$, then the function $m_a : (t,x) \mapsto \varphi(t - |x|^2)$, for $(t,x) \in \mathbb{R}\times\mathbb{R}^n$, is a p-multiplier only when $p = 2$.*

(b) *If $n,m \in \mathbb{N}$ and $\max\{n,m\} \ge 2$, then the function $m_b : (x,y) \mapsto \varphi(|x|^2 - |y|^2)$, for $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, is not in $\mathcal{M}^p(\mathbb{R}^n\times\mathbb{R}^m)$ unless $p = 2$.*

Part (a) has been proved in [17], Theorem A, for $n = 1$. As noted there, the general case can be obtained by applying a theorem of de Leeuw [15], [19]. Part (b) is proved in [17], Theorem B (independently of (a)). It may be of interest to note that (b) can also be obtained as a direct consequence of (a).

To see this, assume that $m_b \in \mathcal{M}^p(\mathbb{R}^{n+m})$ for some $p \ne 2$. With the linear change of variables $x_j \mapsto \xi_j$ (for $1 \le j \le n-1$), $x_n + y_m \mapsto u$, $x_n - y_m \mapsto v$ and $y_l \mapsto \eta_l$ (for $1 \le l \le m-1$) we see that the function $(\xi,u,v,\eta) \mapsto \varphi(|\xi|^2 + uv - |\eta|^2)$, for $(\xi,u,v,\eta) \in \mathbb{R}^{n-1}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{m-1}$, is also in $\mathcal{M}^p(\mathbb{R}^{n+m})$ [12; Theorem 1.13]. Without loss of generality we may assume that $m \ge 2$. Hence, by the theorem of de Leeuw [15], [19], for almost all $(\xi,u) \in \mathbb{R}^{n-1}\times\mathbb{R}$ the function $m_{\xi,u} : (v,\eta) \mapsto \varphi(|\xi|^2 + uv - |\eta|^2)$ is in $\mathcal{M}^p(\mathbb{R}\times\mathbb{R}^{m-1})$. Choose and fix any point $(\xi,u) \in \mathbb{R}^{n-1}\times\mathbb{R}$ (with $u \ne 0$) having this property. After one more affine change of variables, namely $|\xi|^2 + uv \mapsto \varrho$ and $\eta \mapsto \eta$, we deduce by [12; Theorem 1.13] that the function $(\varrho,\eta) \mapsto \varphi(\varrho - |\eta|^2)$, for $\varrho \in \mathbb{R}$ and $\eta \in \mathbb{R}^{m-1}$, is in $\mathcal{M}^p(\mathbb{R}\times\mathbb{R}^{m-1})$. However, because of (a) this is impossible.

SUBCASE (b): Suppose that $\min\{n,m\} \ge 1$, $\max\{n,m\} \ge 2$, $k \ge 0$, $\alpha = 0$, $\beta = 0$, $\gamma = 0$.

COROLLARY 3.2. *In this situation S_Q^p is never decomposable for any $p \in (1,\infty)\setminus\{2\}$. In particular, S_Q^p cannot have a translation invariant $\mathcal{A}$-functional calculus for any quasiadmissible algebra $\mathcal{A}$.*

Proof. By Corollary 3.4 in [2] it suffices to show that $\mathbb{R}\cup\{\infty\} = \overline{Q(\mathbb{R}^N)}$ does not contain $\sigma(S_Q^p)$. By Theorem 3.1(b) (with $\varphi(t) = 1/(i-t)$, for $t \in \mathbb{R}$) we see that $\varphi\circ Q$, considered as a function on $\mathbb{R}^n\times\mathbb{R}^m$, is not in $\mathcal{M}^p(\mathbb{R}^{n+m})$ for $p \ne 2$. Hence, by the theorem of de Leeuw [15], [19] we conclude that $\varphi\circ Q$ considered as function on $\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^k$ cannot be in $\mathcal{M}^p(\mathbb{R}^N)$ for $p \ne 2$. So, as mentioned in Section 1, the point $i \in \mathbb{C}$ must be in $\sigma(S_Q^p)$, considering S_Q^p as an operator on $L^p(\mathbb{R}^N)$. ■

SUBCASE (c): Suppose that $\min\{n,m\} \ge 1$, $\max\{n,m\} \ge 2$, $k \ge 1$ and $\gamma \ne 0$ in (23).

THEOREM 3.3. *In the situation of Subcase* (c), *for $p \in (1,\infty)\setminus\{2\}$, the operator S_Q^p cannot have a translation invariant $\mathcal{A}$-functional calculus for any quasiadmissible algebra $\mathcal{A}$ of functions on $\overline{\mathbb{C}}$ whose restrictions to $\mathbb{C}$ belong to $L^\infty_{\mathrm{loc}}(\mathbb{C})$.*

Proof. Under the assumption that the theorem is false, by means of a linear change of variable, we may assume that Q is of the form $Q(x,y,u) = |x|^2 - |y|^2 + c + iu_1$ and that, for some $p \ne 2$, the operator S_Q^p has a translation invariant $\mathcal{A}$-functional calculus for some quasiadmissible algebra $\mathcal{A}$ of functions for which $\{\varphi|_{\mathbb{C}} : \varphi \in \mathcal{A}\} \subset L^\infty_{\mathrm{loc}}(\mathbb{C})$. By the properties of $\mathcal{A}$ there exists a non-zero function $\varphi \in \mathcal{A}$ with compact support. In particular, the set V of all $u_1 \in \mathbb{R}$ such that $s \mapsto \varphi(s + c + iu_1)$, for $s \in \mathbb{R}$, is a non-constant function belonging to $L^\infty(\mathbb{R})$, is not a null set. By the already quoted theorem of de Leeuw [15], [19] there exists some $u \in \mathbb{R}^k$ such that $u_1 \in V$ and the function $(x,y) \mapsto \varphi(|x|^2 - |y|^2 + c + iu_1)$ is in $\mathcal{M}^p(\mathbb{R}^{n+m})$. But by Theorem 3.1(b) the function $\varphi\circ(|x|^2 - |y|^2 + c + iu_1)$ cannot be a p-multiplier function on $\mathbb{R}^{n+m}$ and we have a contradiction. ■

SUBCASE (d): Suppose that $\min\{n,m\} \ge 2$, $\gamma = 0$ and $|\alpha|^2 + |\beta|^2 \ne 0$. Without loss of generality assume that $\alpha \ne 0$. By a rotation in the x-variables and another one in the y-variables and by using Lemma 1.2(c), we may suppose that

$$Q(x,y,z) = |x|^2 - |y|^2 + c + i(\alpha_n x_n + \beta_m y_m)$$

and $\alpha_n \neq 0$. With the additional linear change of variables $x_j \mapsto \xi_j$ (for $1 \le j \le n-1$), $x_n + (\beta_m/\alpha_n)y_m \mapsto t$, $y \mapsto y$ and $z \mapsto z$ this is transformed to

$$Q(\xi,t,y,z) = |\xi|^2 + \left(t - \frac{\beta_m y_m}{\alpha_n}\right)^2 - |y|^2 + c + i\alpha_n t. \tag{25}$$

THEOREM 3.4. *In the situation of Subcase* (d) *we have the following properties.*

(a) $\sigma(S_Q^p) = \overline{\mathbb{C}} = \overline{Q(\mathbb{R}^N)}$ *for all* $p \in (1,\infty)$.

(b) *If* $p \in (1,\infty)\setminus\{2\}$, *then* S_Q^p *does not have a translation invariant* $\mathcal{A}$*-functional calculus for any quasiadmissible algebra of functions on* $\overline{\mathbb{C}}$ *whose restrictions to* $\mathbb{C}$ *are in* $L^\infty_{\mathrm{loc}}(\mathbb{C})$.

Proof. Assuming that, for some $p \neq 2$, the operator S_Q^p does have a translation invariant $\mathcal{A}$-functional calculus, where $\mathcal{A}$ is a quasiadmissible algebra of locally essentially bounded functions on $\mathbb{C}$, we choose φ and V as in the proof of Theorem 3.3. By the theorem of de Leeuw [15], [19] we can find (and now fix) some $t \neq 0$ such that $\alpha_n t \in V$ (so the function $\psi : s \mapsto \varphi(s + c + i\alpha_n t)$ is not a null function on $\mathbb{R}$, is compactly supported and is in $L^\infty(\mathbb{R})$) has the property that the function from $\mathbb{R}^{n-1} \times \mathbb{R}^m$ to $\mathbb{C}$ given by

$$(\xi, y) \mapsto \psi\left(|\xi|^2 + \left(t - \frac{\beta_m y_m}{\alpha_n}\right)^2 - |y|^2\right)$$

is a p-multiplier on $\mathbb{R}^{n-1} \times \mathbb{R}^m$.

(A) Suppose first that $\beta_m^2/\alpha_n^2 < 1$. Then we make the following linear change of variable: $\xi \mapsto \xi$, $y_j \mapsto u_j$ (for $1 \le j \le m-1$) and

$$\sqrt{\left(1 - \frac{\beta_m^2}{\alpha_n^2}\right)} \cdot \left(y_m + \frac{\beta_m t}{\alpha_n}\left(1 - \frac{\beta_m^2}{\alpha_n^2}\right)^{-1}\right) \mapsto u_m.$$

It follows that $(\xi, u) \mapsto \psi(|\xi|^2 - |u|^2 + d)$ is in $\mathcal{M}^p(\mathbb{R}^{n-1} \times \mathbb{R}^m)$ where $d = t^2(1 - \beta_m^2/\alpha_n^2)^{-1}$. However, by Theorem 3.1 this is not possible (since ψ is a not identically vanishing $L^\infty(\mathbb{R})$ function with compact support, $(n-1) \ge 1$ and $m \ge 2$).

(B) If $\beta_m^2/\alpha_n^2 > 1$ we make the linear change of variables $\xi_j \mapsto u_j$ (for $1 \le j \le n-1$), $y_l \mapsto v_l$ (for $1 \le l \le m-1$) and

$$\sqrt{\left(\frac{\beta_m^2}{\alpha_n^2} - 1\right)} \cdot \left(y_m - \frac{\beta_m}{\alpha_n} t\left(\frac{\beta_m^2}{\alpha_n^2} - 1\right)^{-1}\right) \mapsto u_m.$$

It follows that the function $(u,v) \mapsto \psi(|u|^2 - |v|^2 + d)$ is in $\mathcal{M}^p(\mathbb{R}^n \times \mathbb{R}^{m-1})$, where now $d = -t^2((\beta_m^2/\alpha_n^2) - 1)^{-1}$. Since $n \ge 2$, $(m-1) \ge 1$ and ψ is a non-constant $L^\infty(\mathbb{R})$ function with compact support, this is again a contradiction to Theorem 3.1.

(C) Suppose now that $\beta_m^2/\alpha_n^2 = 1$. In this case (25) has the following form:

$$Q(\xi,t,y,z) = |\xi|^2 + t^2 - \frac{2ty_m\beta_m}{\alpha_n} - \sum_{j=1}^{m-1} y_j^2 + c + i\alpha_n t.$$

Again we assume, for some $p \in (1,\infty)\setminus\{2\}$, that S_Q^p admits a translation invariant $\mathcal{A}$-functional calculus with $\mathcal{A}$ as above and choose φ and V as in the proof of Theorem 3.3. By the theorem of de Leeuw we find some t with $\alpha_n t \in V$ and $z \in \mathbb{R}^k$, $\xi \in \mathbb{R}^{n-1}$ (so that $\psi : s \mapsto \varphi(s + c + i\alpha_n t)$ is non-zero) such that

$$\begin{aligned}(y_1,\ldots,y_m) \mapsto \varphi\Big(|\xi|^2 - \frac{2t\beta_m y_m}{\alpha_n} + t^2 - \sum_{j=1}^{m-1} y_j^2 + c + i\alpha_n t\Big) \\ = \psi\Big(-\sum_{j=1}^{m-1} y_j^2 - \frac{2t\beta_m y_m}{\alpha_n} + d\Big)\end{aligned}$$

is in $\mathcal{M}^p(\mathbb{R}^{m-1} \times \mathbb{R})$, where $d = t^2 + \sum_{j=1}^{n-1} \xi_j^2$. It follows that

$$(y_1,\ldots,y_{m-1},u) \mapsto \psi(u - y_1^2 - \ldots - y_{m-1}^2)$$

is in $\mathcal{M}^p(\mathbb{R}^{m-1} \times \mathbb{R})$, where $y_j \mapsto y_j$ for $1 \le j \le m-1$ and $-2t\beta_m y_m/\alpha_n \mapsto u$. This contradicts Theorem 3.1(a). Notice that this part of the proof also works for $n = 1$ and $m \ge 2$. ■

Hence, we now have to consider the last subcase.

SUBCASE (e): Suppose that $n = 1$ and $m = 2$ (resp. $n = 2$ and $m = 1$) with $|\alpha|^2 + |\beta|^2 \neq 0$ and $\gamma = 0$. Since the problem is symmetric in n and m we need only consider the situation when $n = 1$, $m = 2$, $|\alpha|^2 + |\beta|^2 \neq 0$ and $\gamma = 0$. As in the previous case we may assume that Q has the form

$$Q(x,y_1,y_2,z) = x^2 - y_1^2 - y_2^2 + c + i(\alpha x + \beta y_2) \quad \text{for } x, y_1, y_2 \in \mathbb{R},\ z \in \mathbb{R}^k, \tag{26}$$

where $\alpha^2 + \beta^2 \neq 0$. In this situation we have the following result.

THEOREM 3.5. *Suppose that Q is given in the form* (26).

(a) *If* $\alpha^2 = \beta^2$ *and* $p \in (1,\infty)\setminus\{2\}$, *then* S_Q^p *does not have a translation invariant* $\mathcal{A}$*-functional calculus for any quasiadmissible algebra of functions on* $\overline{\mathbb{C}}$ *whose restrictions to* $\mathbb{C}$ *are in* $L^\infty_{\mathrm{loc}}(\mathbb{C})$.

(b) *If* $\beta^2 < \alpha^2$ *then, for all* p *satisfying* $|1/2 - 1/p| < 1/3$, *the operator* S_Q^p *admits a translation invariant* $C^1(\overline{\mathbb{C}})$*-functional calculus, is decomposable and has the Lyubich–Matsaev property if* $k = 0$.

(c) *For all choices of* α, β *with* $\beta^2 \geq \alpha^2$ *we have* $\sigma(S_Q^p) = \overline{Q(\mathbb{R}^N)} = \overline{\mathbb{C}}$ *for all* $p \in (1, \infty)$.

Proof. (a) follows from part (C) of the proof of Theorem 3.4.

(b) Fix some $d > 0$ with $\beta^2/\alpha^2 < d$ and choose an $\varepsilon > 0$ with $d < (1-\varepsilon)^2$. By Lemma 1.2(a) we may assume without loss of generality that $k = 0$. We shall show that

$$\left|\frac{1}{Q(x, y_1, y_2)}\right| = O\left(\frac{1}{|(x, y_1, y_2)|}\right) \quad \text{for } |(x, y_1, y_2)| \to \infty. \tag{27}$$

For points (x, y_1, y_2) satisfying

$$x^2 \leq \frac{d}{(1-\varepsilon)^2}(y_1^2 + y_2^2)$$

we have (because of $1 - d/(1-\varepsilon)^2 > 0$)

$$\begin{aligned}|Q(x, y_1, y_2) - c|^2 &= (x^2 - y_1^2 - y_2^2)^2 + \alpha^2(x + \beta\alpha^{-1}y_2)^2 \\ &\geq (x^2 - y_1^2 - y_2^2)^2 \geq (1 - d(1-\varepsilon)^{-2})^2(y_1^2 + y_2^2)^2 \\ &\geq \tfrac{1}{2}(1 - d(1-\varepsilon)^{-2})^2(y_1^2 + y_2^2)^2 \\ &\quad + \tfrac{1}{2}(1 - d(1-\varepsilon)^{-2})^2(1-\varepsilon)^4 d^{-2} x^4 \\ &\geq C_1(d, \varepsilon)|(x, y_1, y_2)|^4.\end{aligned}$$

For points (x, y_1, y_2) satisfying

$$x^2 > \frac{d}{(1-\varepsilon)^2}(y_1^2 + y_2^2)$$

we obtain

$$\begin{aligned}|Q(x, y_1, y_2) - c|^2 &\geq \alpha^2(x + \beta\alpha^{-1}y_2)^2 \geq \alpha^2(|x| - |\beta\alpha^{-1}||y_2|)^2 \\ &\geq \alpha^2\big(|x| - \sqrt{d(y_1^2 + y_2^2)}\big) \geq \alpha^2\varepsilon^2 x^2 \\ &\geq \tfrac{1}{2}\alpha^2 x^2 + \alpha^2\varepsilon^2(2(1-\varepsilon)^2)^{-1} d^2 (y_1^2 + y_2^2) \\ &\geq C_2(d, \varepsilon)|(x, y_1, y_2)|^2.\end{aligned}$$

Hence (27) is proved. Further direct computation (or the argument in the proof of [27; Ch. 4, Corollary 4.3]) now shows that Q satisfies the conditions of Theorem 2.8 with $b = 1$, $a = 0$ and $k = 1$. It follows from this theorem that S_Q^p (considered on $L^p(\mathbb{R}^3)$) has a translation invariant $C^1(\overline{\mathbb{C}})$-functional calculus, is decomposable and has the Lyubich–Matsaev property.

(c) By direct computation we see that $Q(\mathbb{R}^N) = \mathbb{C}$. This implies the statement. ∎

REMARK. Suppose that $\alpha^2 < \beta^2$. Although the spectrum of S_Q^p is known (cf. Theorem 3.5(c)), we have been unable to decide whether or not S_Q^p admits a reasonable translation invariant $\mathcal{A}$-functional calculus or even whether or not S_Q^p is decomposable.

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Two-parameter maximal functions associated with homogeneous surfaces in $\mathbb{R}^n$

by

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Abstract. Given a hypersurface $x_n = \Gamma(x_1, \ldots, x_{n-1})$ in $\mathbb{R}^n$, where Γ is homogeneous of degree $d > 0$, we define the two-parameter maximal operator

$$Mf(x) = \sup_{a,b>0} \int\limits_{s\in\mathbb{R}^{n-1},\, |s|<1} |f(x - (as, b\Gamma(s)))|\, ds.$$

We prove that if $d \neq 1$ and the hypersurface has non-vanishing Gaussian curvature away from the origin, then M is bounded on L^p if and only if $p > n/(n-1)$. If $d = 1$, i.e. if the surface is a cone, the same conclusion holds in dimension $n \geq 3$ if the surface has $n-1$ non-vanishing principal curvatures away from the origin and it intersects the hyperplane $x_n = 0$ only at the origin.

Maximal operators defined by averages on curves or surfaces have been extensively considered. Restricting our attention to translation invariant operators in $\mathbb{R}^n$, the usual way to construct such operators is to take the surface measure on some bounded part of the manifold and then act on it by a one-parameter family of dilations. If μ is the basic measure and μ_δ is the same measure dilated by $\delta > 0$ and appropriately normalised, the operator is

$$Mf(x) = \sup_{\delta>0} |f| * \mu_\delta(x).$$

Two different situations can arise. If the manifold is homogeneous under the given dilations, one obtains basically the same operator by restricting the supremum to $\delta = 2^j$. Under appropriate assumptions on the manifold, one then proves that M is bounded on L^p for $p > 1$ (see [SW]).

If the manifold is not homogeneous under the given dilations, then the various μ_δ are supported on different manifolds and the problem becomes much more subtle. Most of the results available concern the case where

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