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Weyl sequences: Asymptotic distributions of the partition lengths

by

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1. Introduction: Statement of the problem and formulation of the main results

1.1. *Weyl sequences.* Let θ be an irrational number in $[0,1)$ and $x_k = k\theta \pmod 1$ for $k=1,2,\ldots$ The collection of points $W_n(\theta)=\{x_1,\ldots,x_n\}$ is sometimes called the *Weyl sequence* of order n.

In the present work we derive asymptotic distributions of different characteristics associated with the interval lengths of the partitions of $[0,1)$ generated by $W_n(\theta)$. The main result establishes the two-dimensional asymptotic distribution of

$$(n\min\{x_1,\ldots,x_n\},\, n(1-\max\{x_1,\ldots,x_n\}))$$

as $n\to\infty$. It then yields a number of results concerning the asymptotic distributions of one-dimensional characteristics.

Assume that $y_{0,n}=0$, $y_{n+1,n}=1$ and let $y_{k,n}$ $(k=1,\ldots,n)$ be the members of $W_n(\theta)$ arranged in increasing order. Define

$$\delta_n(\theta)=y_{1,n}=\min_{k=1,\ldots,n} x_k, \qquad \Delta_n(\theta)=1-y_{n,n}=1-\max_{k=1,\ldots,n} x_k \tag{1}$$

and consider the partition of $[0,1)$ generated by $W_n(\theta)$:

$$\mathcal{P}_n(\theta)=\bigcup_{k=0}^{n} I_{k,n}, \quad \text{where} \quad I_{k,n}=[y_{k,n},y_{k+1,n}).$$

It is a well known property of the Weyl sequence (see e.g. [3], [4]) that for any $n\ge 1$ the partition $\mathcal{P}_n(\theta)$ of $[0,1)$ contains the intervals $I_{k,n}$ whose lengths $|I_{k,n}|$ can only get two or three different values, namely, $\delta_n(\theta)$, $\Delta_n(\theta)$ and perhaps $\delta_n(\theta)+\Delta_n(\theta)$.

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Set

$$\alpha_n(\theta) = \min_{k=1,\ldots,n} |I_{k,n}| = \min\{\delta_n(\theta), \Delta_n(\theta)\}, \quad A_n(\theta) = \max_{k=1,\ldots,n} |I_{k,n}|,$$

$$\beta_n(\theta) = \max\{\delta_n(\theta), \Delta_n(\theta)\}, \quad \gamma_n(\theta) = \delta_n(\theta) + \Delta_n(\theta), \quad \xi_n(\theta) = \alpha_n(\theta)/\beta_n(\theta).$$

All these quantities, namely $\delta_n(\theta)$, $\Delta_n(\theta)$, $\alpha_n(\theta)$, $A_n(\theta)$, $\beta_n(\theta)$, $\gamma_n(\theta)$ and $\xi_n(\theta)$, give a rather complete description of the partition $\mathcal{P}_n(\theta)$. We are interested in their asymptotic behaviour as $n \to \infty$. The main result of the paper is formulated in Theorem 1 below and presents the joint asymptotic distribution for $(n\delta_n(\theta), n\Delta_n(\theta))$. In Corollaries 1–4 and Theorem 2 we derive the one-dimensional asymptotic distributions for all characteristics introduced above.

As demonstrated in Section 2, there is a close relationship between the Weyl and Farey sequences, and the quantities introduced above also characterize certain properties of the Farey sequences. (For example, $\alpha_n(\theta)$, whose asymptotic distribution has been derived in [2], characterizes the error in approximation of θ by the Farey sequence of order n (see (11)).) The present paper thus also studies some distributional properties of the Farey sequences.

In what follows "meas" stands for the Lebesgue measure on $[0,1)$, $\{\cdot\}$ and $\lfloor\cdot\rfloor$ denote the fractional and integer part operations respectively, $\varphi(\cdot)$ is the Euler totient function and $\mathrm{dilog}(\cdot)$ is the dilogarithm function:

$$\mathrm{dilog}(t) = \int_1^t \frac{\log s}{1-s}\, ds.$$

Also, we shall say that a sequence of functions $\psi_n(\theta)$, $\theta \in [0,1)$, *converges in distribution* as $n \to \infty$ to a probability measure with a density $q(\cdot)$ if for any $t > 0$,

$$\lim_{n\to\infty} \mathrm{meas}\{\theta \in [0,1) : \psi_n(\theta) \le t\} = \int_0^t q(s)\, ds.$$

The rest of the paper is organized as follows: the main results are formulated in Subsection 1.2, a relationship between the Weyl and Farey sequences is discussed in Section 2, all proofs are given in Section 3.

1.2. *Formulation of the main results.* For $0 \le s, t < \infty$ define

$$\Phi_n(s,t) = \mathrm{meas}\{\theta \in [0,1) : n\delta_n(\theta) \le s,\ n\Delta_n(\theta) \le t\}.$$

One can interpret $\Phi_n(\cdot,\cdot)$ as the two-dimensional cumulative distribution function (c.d.f.) of the random variables $n\delta_n(\theta)$ and $n\Delta_n(\theta)$, assuming that θ is uniformly distributed on $[0,1)$.

THEOREM 1. *The sequence of functions $\Phi_n(\cdot,\cdot)$ pointwise converges, as $n \to \infty$, to the c.d.f. $\Phi(\cdot,\cdot)$ with density*

$$(2) \qquad \phi(s,t) = \frac{d^2\Phi(s,t)}{dsdt} = \frac{6}{\pi^2 st}\begin{cases} s+t-1 & \textit{for } 0 \le s,t \le 1,\ s+t \ge 1,\\ s(1-s)/(t-s) & \textit{for } 0 \le s \le 1 \le t,\\ t(1-t)/(s-t) & \textit{for } 0 \le t \le 1 \le s,\\ 0 & \textit{otherwise.}\end{cases}$$

This means that for all measurable sets A in $\mathbb{R}^2$,

$$\lim_{n\to\infty} \operatorname{meas}\{\theta \in [0,1) : (n\delta_n(\theta), n\Delta_n(\theta)) \in A\} = \int_A \phi(s,t)\,ds\,dt.$$

Corollary 1. *The sequences of functions $n\delta_n(\theta)$ and $n\Delta_n(\theta)$ converge in distribution, as $n \to \infty$, to the probability measure with density*

$$(3) \qquad \phi_\delta(t) = \frac{6}{\pi^2}\begin{cases} 0 & \textit{for } t < 0,\\ 1 & \textit{for } 0 \le t < 1,\\ \dfrac{t-1}{t}\log\dfrac{t-1}{t} + \dfrac{1}{t} & \textit{for } t \ge 1.\end{cases}$$

The proof of Corollary 1 consists in computation of $\int_0^\infty \phi(s,t)\,ds$ where $\phi(\cdot,\cdot)$ is defined in (2).

Corollary 2. *The sequence of functions $n\alpha_n(\theta)$ converges in distribution, as $n \to \infty$, to the probability measure with density*

$$(4) \qquad \phi_\alpha(t) = \frac{12}{\pi^2}\begin{cases} 1 & \textit{for } 0 \le t < 1/2,\\ \dfrac{1-t}{t}\left(1 - \log\dfrac{1-t}{t}\right) & \textit{for } 1/2 \le t < 1,\\ 0 & \textit{otherwise.}\end{cases}$$

(Note again that Corollary 2 has been proved in [2], by different arguments.)

Corollary 3. *The sequence of functions $n\beta_n(\theta)$ converges in distribution, as $n \to \infty$, to the probability measure with density*

$$(5) \qquad \phi_\beta(t) = \frac{12}{\pi^2}\begin{cases} 0 & \textit{for } t < 1/2,\\ \dfrac{1-t}{t}\log\dfrac{1-t}{t} - \dfrac{1}{t} + 2 & \textit{for } 1/2 \le t < 1,\\ \dfrac{t-1}{t}\log\dfrac{t-1}{t} + \dfrac{1}{t} & \textit{for } t \ge 1.\end{cases}$$

Corollary 4. *The sequences of functions $n\gamma_n(\theta)$ and $nA_n(\theta)$ converge in distribution, as $n \to \infty$, to the probability measure with density*

$$(6) \qquad \phi_\gamma(t) = \frac{12}{\pi^2}\begin{cases} 0 & \textit{for } t < 1,\\ \dfrac{t-2}{2t}\log|t-2| - \dfrac{t-1}{t}\log(t-1) + \dfrac{1}{2}\log t & \textit{for } t \ge 1.\end{cases}$$

To make the difference between the asymptotic behaviour of δ_n, β_n and γ_n transparent, we provide Figure 1 which depicts the densities ϕ_δ, ϕ_β and ϕ_γ.

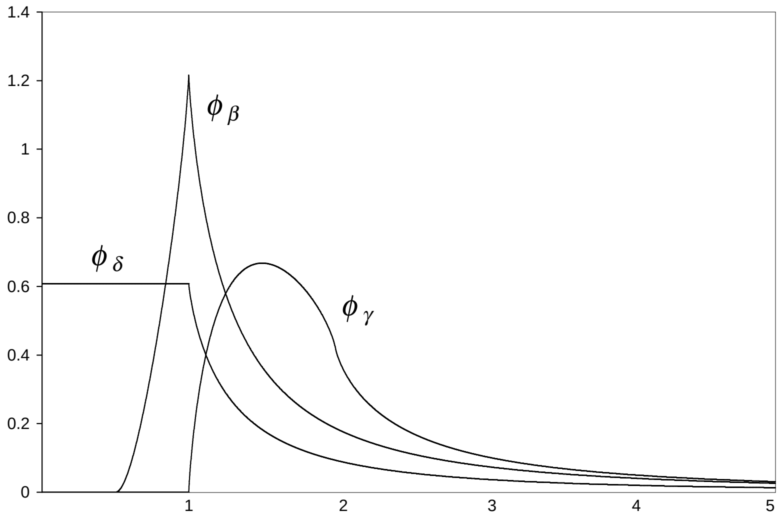

Fig. 1. Asymptotic densities for $n\delta_n(\theta)$, $n\beta_n(\theta)$ and $n\gamma_n(\theta)$

THEOREM 2. *The sequence of functions $\xi_n(\theta)$ converges in distribution, as $n \to \infty$, to the probability measure with density*

$$\phi_\xi(t) = -\frac{12}{\pi^2}\left(\frac{\log t}{1-t} + \frac{\log(1+t)}{t}\right), \quad t \in [0,1). \tag{7}$$

Theorem 2 can certainly be deduced from Theorem 1. This however would require evaluation of an unpleasant integral; in Section 3 we instead give a straightforward proof.

2. Relationships with the Farey sequences and continued fractions

2.1. *Relationship with the Farey sequences.* The *Farey sequence* of order n, denoted by $\mathcal{F}_n$, is the collection of all rationals p/q with $p \le q$, $\gcd(p,q) = 1$ and $1 \le q \le n$. The numbers in $\mathcal{F}_n$ are arranged in increasing order, and 0 and 1 are included in $\mathcal{F}_n$ as $0/1$ and $1/1$ respectively. There are $|\mathcal{F}_n| = N(n) + 1$ points in $\mathcal{F}_n$ where

$$N(n) = \sum_{q=1}^{n} \varphi(q) = \frac{3}{\pi^2} n^2 + O(n \log n), \quad n \to \infty. \tag{8}$$

The following well known statement establishes an important relationship between the Weyl and Farey sequences.

LEMMA 1 (e.g. [3]). *Let θ be an irrational number in $[0,1)$ and $W_n(\theta)$ be the Weyl sequence of order n. Let $\{q\theta\}$ and $\{q'\theta\}$ correspond respectively to* *the smallest and largest members of* $W_n(\theta)$:

$$y_1 = \delta_n(\theta) = \{q\theta\}, \qquad y_n = 1 - \Delta_n(\theta) = \{q'\theta\}.$$

Define $p = \lfloor q\theta \rfloor$ *and* $p' = 1 + \lfloor q'\theta \rfloor$. *Then* p/q *and* p'/q' *are the consecutive fractions in the Farey sequence* $\mathcal{F}_n$ *such that* $p/q < \theta < p'/q'$.

Let us rewrite the quantities (1) in terms of the Farey fractions p/q and p'/q' introduced in Lemma 1:

$$\delta_n(\theta) = \{q\theta\} = q\theta - \lfloor q\theta \rfloor = q\theta - p, \tag{9}$$

$$\Delta_n(\theta) = 1 - \{q'\theta\} = 1 + \lfloor q'\theta \rfloor - q'\theta = p' - q'\theta. \tag{10}$$

This in particular implies

$$\alpha_n(\theta) = \min_{p/q \in \mathcal{F}_n} |q\theta - p|. \tag{11}$$

2.2. *An asymptotic property of the Farey sequences.* In the sequel we shall use an asymptotic property of the Farey sequences formulated as Lemma 2.

If p/q and p'/q' are two consecutive Farey fractions in $\mathcal{F}_n$ then we call (q, q') a *neighbouring pair of denominators*. It is easy to verify that for a fixed n the set of all neighbouring pairs of denominators is

$$Q_n = \{(q, q') : q, q' \in \{1, \ldots, n\},\ \gcd(q, q') = 1,\ q + q' > n\},$$

and these pairs, properly normalised, share the asymptotic two-dimensional uniformity. Specifically, the following result holds.

Lemma 2 (see [1]). *Let* ν_n *be the two-variate probability measure assigning equal masses* $1/N(n)$ *to the pairs* $(q/n, q'/n)$, *where* (q, q') *take all possible values in* Q_n. *Then the sequence of probability measures* $\{\nu_n\}$ *weakly converges*, *as* $n \to \infty$, *to the uniform probability measure on the triangle* $T = \{(x, y) : 0 \le x, y \le 1, x + y \ge 1\}$, *that is*, *for any continuous function* f *on* $\mathbb{R}^2$,

$$\frac{1}{N(n)} \sum_{(q,q') \in Q_n} f(q/n, q'/n) \to 2 \iint_T f(x, y)\, dx\, dy, \qquad n \to \infty.$$

2.3. *Association with continued fractions.* Let us now indicate an interesting analogy between the quantity $\xi_n(\theta)$ and the residuals in the continued fraction expansions.

Let θ be an irrational number in $[0, 1)$. We denote by $\theta = [a_1, a_2, \ldots]$ its continued fraction expansion and by $p_n/q_n = [a_1, a_2, \ldots, a_n]$ its nth convergent.

Let also

$$r_0 = \theta, \qquad r_n = \{1/r_{n-1}\} \quad \text{for } n = 1, 2, \ldots$$

be the associated dynamical system.

As is well known, the asymptotic density of $\{r_n\}$ is

$$p(t) = \frac{1}{\log 2} \cdot \frac{1}{1+t}, \qquad 0 \le t < 1.$$

For every $n \ge 0$, $r_n = r_n(\theta)$ allows the following continued fraction expansion: $r_n(\theta) = [a_{n+1}, a_{n+2}, \ldots]$. It is not difficult to check (see e.g. [4]), that

$$r_n(\theta) = \frac{|q_n\theta - p_n|}{|q_{n-1}\theta - p_{n-1}|}, \qquad n > 1.$$

The role of $r_n(\theta)$ for $\mathcal{F}_n$ is played by

$$\xi_n(\theta) = \frac{\min(|q\theta - p|, |q'\theta - p'|)}{\max(|q\theta - p|, |q'\theta - p'|)} = \frac{\alpha_n(\theta)}{\beta_n(\theta)},$$

where p/q, p'/q' are the members of $\mathcal{F}_n$ neighbouring to θ. Figure 2 compares the asymptotic densities for $r_n(\theta)$ and $\xi_n(\theta)$.

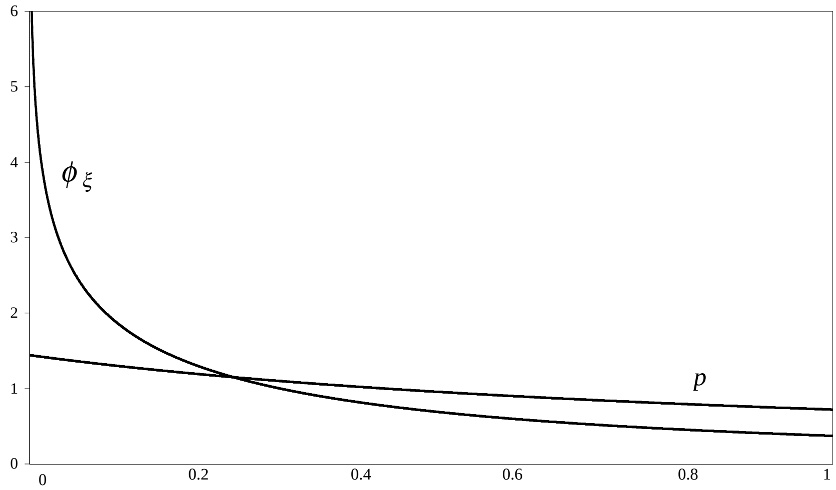

Fig. 2. Asymptotic densities for $r_n(\theta)$ and $\xi_n(\theta)$

3. Proofs

3.1. *Proof of Theorem 1.* Consider the two-variate function

$$\widetilde{\Phi}_n(s,t) = \text{meas}\{\theta \in [0,1) : n\delta_n(\theta) > s,\ n\Delta_n(\theta) > t\},$$

where $0 \le s, t < \infty$. The c.d.f. $\Phi(s,t)$ is related to $\widetilde{\Phi}(s,t)$ through the inclusion-exclusion formula

$$\Phi(s,t) = 1 - \widetilde{\Phi}(s,0) - \widetilde{\Phi}(0,t) + \widetilde{\Phi}(s,t). \tag{12}$$

Let p/q and p'/q' be consecutive fractions in $\mathcal{F}_n$. Define points θ_1, θ_2 in $[p/q, p'/q']$ such that

$$n\delta_n(\theta_1) = s, \qquad n\Delta_n(\theta_2) = t.$$

It is easily seen that

$$\operatorname{meas}\{\theta\in[p/q,p'/q'] : n\delta_n(\theta)>s,\ n\Delta_n(\theta)>t\} = \begin{cases}\theta_2-\theta_1 & \text{for } \theta_2-\theta_1>0,\\ 0 & \text{for } \theta_2-\theta_1\le 0.\end{cases}$$

We now try to find a simple expression for the difference $\theta_2-\theta_1$. First, formulas (9) and (10) yield

$$\theta_1=\frac{s/n+p}{q},\qquad \theta_2=\frac{p'-t/n}{q'},$$

and therefore

$$\theta_2-\theta_1=\frac{p'-t/n}{q'}-\frac{s/n+p}{q}=\frac{1}{qq'}\left(1-\frac{tq}{n}-\frac{sq'}{n}\right).$$

We thus get

$$\widetilde{\Phi}_n(s,t)=\sum_{(q,q')\in Q(n,s,t)}\frac{1}{qq'}\left(1-\frac{tq}{n}-\frac{sq'}{n}\right),$$

where

$$Q(n,s,t)=\{(q,q')\in Q_n : 1-tq/n-sq'/n>0\}.$$

Using formula (8) we have

$$\widetilde{\Phi}_n(s,t)=\frac{3}{\pi^2N(n)}\sum_{(q,q')\in Q(n,s,t)}\frac{n^2}{qq'}\left(1-\frac{tq}{n}-\frac{sq'}{n}\right)+O(n^{-1}\log n),\qquad n\to\infty.$$

Applying Lemma 2 we get

$$\widetilde{\Phi}_n(s,t)\to\widetilde{\Phi}(s,t)=\frac{6}{\pi^2}\iint_{Q(s,t)}\left(\frac{1-tx-sy}{xy}\right)dx\,dy, \tag{13}$$

where

$$Q(s,t)=\{x,y : 0\le x,y\le 1,\ x+y\ge 1,\ 1-tx-sy>0\}.$$

The formula for the integral on the right-hand side of (13) can be rewritten differently in 5 different regions:

1. For $s+t\le 1$:

$$\widetilde{\Phi}(s,t)=\frac{6}{\pi^2}\int_0^1\int_{1-y}^1\left(\frac{1-tx-sy}{xy}\right)dx\,dy=1-\frac{6}{\pi^2}(s+t).$$

2. For $0 \le s,t \le 1$, $s+t>1$:

$$\begin{aligned}
\widetilde{\Phi}(s,t) &= \frac{6}{\pi^2} \int\limits_0^{(1-s)/t} \int\limits_{1-y}^{1} \left(\frac{1-tx-sy}{xy}\right) dx\, dy \\
&\quad + \frac{6}{\pi^2} \int\limits_{(1-s)/t}^{1} \int\limits_{1-y}^{(1-yt)/s} \left(\frac{1-tx-sy}{xy}\right) dx\, dy \\
&= -\frac{12}{\pi^2} + \frac{6}{\pi^2}\bigg(s+t+(1+\log s - s)\log\frac{1-s}{t} \\
&\quad + (1+\log t - t)\log\frac{1-t}{s} + \log s \log t + \operatorname{dilog} s + \operatorname{dilog} t\bigg).
\end{aligned}$$

3. For $s>1$, $t \le 1$:

$$\begin{aligned}
\widetilde{\Phi}(s,t) &= \frac{6}{\pi^2} \int\limits_{(s-1)/(s-t)}^{1} \int\limits_{1-y}^{(1-yt)/s} \left(\frac{1-tx-sy}{xy}\right) dx\, dy \\
&= 1 - \frac{6}{\pi^2} + \frac{6}{\pi^2}\bigg(t + (s - \log s - t)\log\frac{s-t}{s-1} + (1-t)\log\frac{s-1}{s} \\
&\quad - \operatorname{dilog}(1-t) + \operatorname{dilog}\frac{s(1-t)}{s-t} - \operatorname{dilog}\frac{1-t}{s-t}\bigg).
\end{aligned}$$

4. For $s \le 1$, $t>1$: Analogously to the previous case with the replacement $s \leftrightarrow t$.

5. For $s>1$, $t>1$: $\widetilde{\Phi}(s,t)=0$.

Using formula (12) we can find the density

$$\phi(s,t) = \frac{d\Phi(s,t)}{dsdt} = \frac{d\widetilde{\Phi}(s,t)}{dsdt}$$

of the joint asymptotic distribution. Calculation then gives (2). ∎

3.2. *Proof of Corollary 2.* The function $\alpha_n(\theta) = \min\{\delta_n(\theta), \Delta_n(\theta)\}$ is measurable with respect to $\mathcal{B}$, the σ-algebra of Borel subsets of $[0,1)$, and it can be associated with the probability measure $d\Phi_n^{\alpha}(t)$, $0 \le t < \infty$, where

$$\begin{aligned}
\Phi_n^{\alpha}(t) &= \operatorname{meas}\{\theta \in [0,1) : n\alpha_n(\theta) \le t\} \\
&= 1 - \operatorname{meas}\{\theta \in [0,1) : n\min(\delta_n(\theta), \Delta_n(\theta)) > t\} \\
&= 1 - \operatorname{meas}\{\theta \in [0,1) : n\delta_n(\theta) > t,\ n\Delta_n(\theta) > t\}.
\end{aligned}$$

Therefore, for all $0 \le t < \infty$,

$$\Phi_n^{\alpha}(t) \to \Phi^{\alpha}(t) = 1 - \widetilde{\Phi}(t,t), \qquad n \to \infty.$$

Calculation gives

$$\Phi^{\alpha}(t)=\begin{cases}\dfrac{12}{\pi^2}t & \text{for } 0\le t<1/2,\\ \dfrac{12}{\pi^2}\left(-t+\log\dfrac{1-t}{t}(t-\log t-1)+\operatorname{dilog}\dfrac{1}{t}\right)+\dfrac{12}{\pi^2}+1 & \\ & \text{for } 1/2\le t<1,\\ 1 & \text{for } t\ge 1.\end{cases}$$

Differentiation gives the expression (4) for the density $\phi_\alpha(t)=d\Phi^\alpha(t)/dt$. ■

3.3. *Proof of Corollary 3.* The function $\beta_n(\theta)=\max\{\delta_n(\theta),\Delta_n(\theta)\}$ is $\mathcal{B}$-measurable. We then have, for all $0\le t<\infty$,

$$\begin{aligned}\Phi_n^{\beta}(t)&=\operatorname{meas}\{\theta\in[0,1): n\beta_n(\theta)\le t\}\\ &=\operatorname{meas}\{\theta\in[0,1): n\delta_n(\theta)\le t,\ n\Delta_n(\theta)\le t\}.\end{aligned}$$

Therefore, for all $0\le t<\infty$,

$$\Phi_n^{\beta}(t)\to\Phi^{\beta}(t)=\Phi(t,t),\qquad n\to\infty.$$

Calculation gives

$$\Phi^{\beta}(t)=\begin{cases}0 & \text{for } 0\le t<1/2,\\ \dfrac{12}{\pi^2}\left(2t-\log\dfrac{1-t}{t}(t-\log t-1)-\operatorname{dilog}\dfrac{1}{t}\right)-\dfrac{12}{\pi^2}-1 & \\ & \text{for } 1/2\le t<1,\\ \dfrac{12}{\pi^2}\left(\log\dfrac{t-1}{t}(t-\log t-1)+\operatorname{dilog}\dfrac{1}{t}\right)+\dfrac{12}{\pi^2}-1 & \\ & \text{for } t\ge 1.\end{cases}$$

Differentiation gives the expression (5) for the density $\phi_\beta(t)=d\Phi^\beta(t)/dt$. ■

3.4. *Proof of Corollary 4.* Analogously to the proofs of Corollaries 2 and 3, the sequence of c.d.f.

$$\Phi_n^{\gamma}(t)=\operatorname{meas}\{\theta\in[0,1): n\gamma_n(\theta)\le t\},\qquad 0\le t<\infty,$$

pointwise converges to the c.d.f.

$$\Phi^{\gamma}(t)=\iint\limits_{S(t)}\phi(x,y)\,dx\,dy,\qquad 0\le t<\infty,$$

where

$$S(t)=\{(x,y): 0\le x,y\le 1,\ 1\le x+y\le t\}.$$

Calculation yields (6).

The convergence of the sequence $nA_n(\theta)$ to the asymptotic distribution with density ϕ_γ follows from the just proved convergence of the sequence $n\gamma_n(\theta)$ to the same distribution and the fact that $A_n(\theta)=\gamma_{n+1}(\theta)$ for all $\theta\in(0,1)$ and all $n\ge n(\theta)=\max\{1/\theta,1/(1-\theta)\}$. ■

3.5. *Proof of Theorem 2.* The function $\xi_n(\theta)$ is $\mathcal{B}$-measurable. Define

$$\Phi_n^{\xi}(t) = \mathrm{meas}\{\theta \in [0,1) : \xi_n(\theta) \le t\}, \quad 0 \le t \le 1.$$

Let p/q, p'/q' be consecutive fractions in $\mathcal{F}_n$. Consider the behaviour of $\xi_n(\theta)$ in the interval $[p/q, p'/q']$. Define the mediant $m = (p+p')/(q+q')$. Then for θ in $[p/q, m)$ we have $\delta_n(\theta) < \Delta_n(\theta)$, and for θ in $(m, p/q]$ we have $\delta_n(\theta) > \Delta_n(\theta)$ and $\delta_n(m) = \Delta_n(m)$, that is, $\xi_n(m) = 1$.

If $t \in [0,1]$ is fixed then there is a unique point θ_t in $[p/q, m]$ such that

$$\xi_n(\theta_t) = \frac{\delta_n(\theta_t)}{\Delta_n(\theta_t)} = t. \tag{14}$$

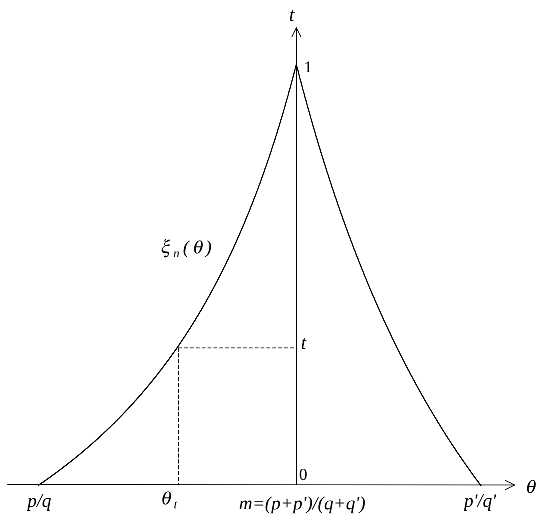

Fig. 3. Behaviour of the function $\xi_n(\theta)$ in the interval $[p/q, p'/q']$

An easy observation shows (see Fig. 3) that

$$\mathrm{meas}\{\theta \in [p/q, m] : \xi_n(\theta) \le t\} = \theta_t - p/q.$$

Formula (14) implies

$$q\theta_t - p = t(p' - q'\theta_t), \quad \theta_t = \frac{p + tp'}{q + tq'}.$$

Therefore,

$$\mathrm{meas}\{\theta \in [p/q, m] : \xi_n(\theta) \le t\} = \frac{p + tp'}{q + tq'} - \frac{p}{q} = \frac{t}{q(q + tq')}.$$

We then get, for all $0 \le t \le 1$,

$$
\begin{aligned}
&\operatorname{meas}\{\theta \in [0,1) : \xi_n(\theta) \le t\} \\
&\qquad = 2 \sum_{(q,q')\in Q_n} \frac{t}{q(q+tq')} = 2 \sum_{(q,q')\in Q_n} \int_0^t \frac{d\tau}{(q+\tau q')^2} \\
&\qquad = 2 \int_0^t \Big(\sum_{(q,q')\in Q_n} \frac{1}{(q+\tau q')^2} \Big)\, d\tau,
\end{aligned}
$$

where the factor 2 is due to the cases when $\delta_n(\theta) > \Delta_n(\theta)$.

Therefore, we can write, for all $0 \le t \le 1$,

$$
\Phi_n^{\xi}(t) = \int_0^t \phi_n^{\xi}(\tau)\, d\tau, \quad \text{where} \quad \phi_n^{\xi}(\tau) = 2 \sum_{(q,q')\in Q_n} \frac{1}{(q+\tau q')^2}.
$$

Using formula (8) write

$$
\phi_n^{\xi}(\tau) = \frac{6}{\pi^2 N(n)} \sum_{(q,q')\in Q_n} \frac{n^2}{(q+\tau q')^2} + O(n^{-1}\log n), \qquad n \to \infty.
$$

Applying Lemma 2 we get

$$
\phi_n^{\xi}(\tau) \to \phi_{\xi}(\tau) = \frac{12}{\pi^2} \iint\limits_{\substack{0\le x,y\le 1\\ x+y>1}} \frac{1}{(x+\tau y)^2}\, dx\, dy, \qquad n \to \infty.
$$

Calculation of the integral gives the expression (7) for the density. ∎

References

[1] P. P. Kargaev and A. A. Zhigljavsky, *Asymptotic distribution of the distance function to the Farey points*, J. Number Theory 65 (1997), 130–149.

[2] H. Kesten, *Some probabilistic theorems on Diophantine approximations*, Trans. Amer. Math. Soc. 103 (1962), 189–217.

[3] N. B. Slater, *Gaps and steps for the sequence* $n\theta$ mod 1, Proc. Cambridge Philos. Soc. 63 (1967), 1115–1123.

[4] V. T. Sós, *On the distribution* mod 1 *of the sequence* $n\alpha$, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. 1 (1958), 127–134.

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