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On certain continued fraction expansions of fixed period length

by

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1. Introduction. It is an interesting problem to detect infinite families of positive integers D for which one can readily describe the fundamental unit of the quadratic number field $\mathbb{Q}(\sqrt{D})$. We will discuss here a class of cases $D = F(X)$ with F a polynomial of even degree and with leading coefficient a square, for which one obtains particularly small units, essentially because the period length is independent of the integer parameter X. That of course means a particularly large class number for the field $\mathbb{Q}(\sqrt{D})$.

The context is a result of Schinzel [4], [5], who shows that if F is an integer-valued polynomial, either of odd degree, or of even degree with its leading coefficient not a square, then as the integer X varies one has $\overline{\lim}\, \mathrm{lp}(\sqrt{F(X)}) = \infty$; here $\mathrm{lp}(\delta)$ denotes the length of the period of the continued fraction expansion of the quadratic irrational δ. On the other hand, in the quadratic case Schinzel shows that $\overline{\lim}\, \mathrm{lp}(\sqrt{F(X)}) < \infty$ if and only if $F(X) = A^2X^2 + BX + C$ with $A > 0$, discriminant $\Delta = B^2 - 4A^2C \neq 0$ and $\Delta \mid 4(2A^2, B)^2$. Well known examples of such F include the Richaud–Degert types: $A^2X^2 \pm A$, $A^2X^2 \pm 2A$, and $A^2X^2 \pm 4A$, which provide periods of length at most 12. As these Richaud–Degert types have been fully investigated (see, for example, Theorem 3.2.1 of Mollin [1]), we will exclude them from our investigations here.

It will also be convenient to deal only with those $F(X)$ such that $2 \mid A$ and $2 \mid B$. There is no loss of generality in doing so as we can divide the possible values of X into even ($X = 2W$) or odd ($X = 2W + 1$) integers and

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write
$$F(X) = G(W) = A'^2W^2 + B'W + C',$$
where, if $X = 2W$,
$$A' = 2A, \quad B' = 2B, \quad C' = C,$$
or, if $X = 2W + 1$,
$$A' = 2A, \quad B' = 4A^2 + 2B, \quad C' = A^2 + B + C.$$
In either case we get $\Delta' = B'^2 - A'^2C = 4\Delta$ and $2(2A^2, B) \mid (2A'^2, B')$; hence $\Delta' \mid 4(2A'^2, B')^2$ whenever $\Delta \mid 4(2A^2, B)^2$. As we may always assume that $2 \mid B$ it will be convenient in what follows to replace B by $2B$ in $F(X)$ and rewrite it as
$$F(X) = A^2X^2 + 2BX + C.$$
In this case Schinzel's condition becomes $B^2 - A^2C \mid 4(A^2, B)^2$.

In [6], Stender determines the fundamental unit of $\mathbb{Q}(\sqrt{D}\,)$ when $D = F(X)$ with F quadratic as above, provided that D is squarefree. In this paper we consider the quadratic case only. We find that for $X > 0$, Schinzel's condition, together with $(A^2, 2B, C)$ squarefree, entails that the "approximation" $AX + B/A$ to $\sqrt{F(X)}$ usually provides the first half of a period of $\sqrt{F(X)}$. Thus, aside from some possibly degenerate cases with X small and a special case we are about to allude to, the period of $\sqrt{F(X)}$ is not just of bounded, but in fact of constant length. However, if $F(X) \equiv 1 \bmod 4$ and both the numerator and denominator, after division by the greatest common divisor of A and B, of the approximation $AX + B/A$ are odd, then the expansion of that approximation provides just the first sixth of the period. Indeed, we shall show that, under our conditions, if $C \le 0$ or C is a perfect square, then $F(X)$ is of Richaud–Degert type; but if C is positive and not a square then the continued fraction expansion of $\sqrt{F(X)}$ can usually be expressed very simply in terms of the continued fraction of $\sqrt{C}$. Furthermore, we show that no matter how large a value of N is selected there are always some A, B, C obeying Schinzel's condition such that $\mathrm{lp}(\sqrt{F(X)}) > N$. Furthermore, this value of the period length is independent of X as long as X is large enough to avoid some degenerate cases. For example, we must have X large enough that $F(X)$ cannot be a perfect square; this will certainly be the case if $2(A^2X + |B|) > |\Delta|$. We should mention that some of our results were known to Stern [7], but we will be more general than he and use different techniques.

2. Preliminary observations. To begin our investigation it is necessary to characterise those values of A, B, C such that $B^2 - A^2C \mid 4(A^2, B)^2$, and $(A^2, 2B, C)$ is squarefree.

Lemma 2.1. *Set $S = (A, B)$ and $(B/S)^2 - (A/S)^2C = G^2H$, where H is squarefree. If $B^2 - A^2C$ divides $4(A^2, B)^2$, then GH divides $2A, 2B/S$ and $2S$, and G^2H divides $4(A^2, 2B, C)$. Therefore if $(A^2, 2B, C)$ is squarefree, then $G = 1, 2$.*

Proof. Since $G^2H \mid 4(A^2/S, B/S)^2$ and $(A/S, B/S) = 1$, it follows that $GH \mid 2(S, B/S)$. Also, since $GH \mid 2B/S$ we must have $G^2H \mid 4C$; hence, $G^2H \mid 4(A^2, 2B, C)$. ∎

Theorem 2.2. *Assume that $B^2 - A^2C$ divides $4(A^2, B)^2$ and $(A^2, 2B, C)$ is squarefree. Then $F(X) = A^2X^2 + 2BX + C$ is of Richaud–Degert type when $C \le 0$ or C is a perfect square; that is $F(X) = R^2 + S$ where S divides $4R$.*

Proof. If $C = 0$, then $B \mid 2A^2$ and since $\gcd(A^2, 2B)$ is squarefree, we must have $B \mid 2A$ and $F(X) = A^2X^2 + 2BX$ where $2BX \mid 4AX$. This is of Richaud–Degert type. If $C < 0$, we see from Lemma 2.1 that

$$H(2B/(SGH))^2 + (A/S)^2 4|C|/(G^2H) = 4.$$

We must have $H > 0$ and $H(2B/(SGH))^2 \le 3$; hence $2|B| = SG|H|$ and $|H| = 1, 2, 3$. If $|H| = 2$, then $|A| = S$, $2|C| = 2G^2$ and $|B| = SG$. By Lemma 2.1 we get $G \mid A$; it follows that $F(X) = A^2X^2 \pm 2|A|GX - G^2 = (|A|X \pm G)^2 - 2G^2$, where by Lemma 2.1 we have $2G^2 \mid 4(|A|X \pm G)$. If $|H| = 1$ or 3, then $4|C| = 3G^2$, $|A| = S$, $2|B| = SG|H|$. Since $G = 1, 2$ we must get $G = 2, C = -3$, $|B| = S|H|$, $|A| = S$. Hence, $F(X) = (SX \pm H)^2 - H^2 - 3$, where, by Lemma 2.1, $H \mid S$. Since $H^2 + 3 \mid 4H$ when $H = 1, 3$ we see that $F(X)$ is of Richaud–Degert type.

Now suppose $C = K^2$. Since $G^2H \mid 4C$, we get $GH \mid 2K$ and

$$\left(\frac{2B}{SGH}\right)^2 - \left(\frac{2K}{GH}\right)^2 \left(\frac{A}{S}\right)^2 = \frac{4}{H};$$

thus $|H| = 1, 2$. Since a difference of two squares can never be 2, we must have $|H| = 1$. Since two squares can differ by 4 only when both are even, we get $2 \mid (2B/(SG))$ and $2 \mid (2K/G)$; hence,

$$|B/(SG)| + (K/G)|A/S| = 1$$

and $B = 0$, $|A| = S$, $K = G$. Since $F(X) = A^2X^2 + G^2$ and $G = 1, 2$, we get $G^2 \mid 4AX$ and $F(X)$ is of Richaud–Degert type. ∎

Note that if $X < 0$, we may write $F(X) = A^2|X|^2 - 2B|X| + C$; thus, we may always assume that $X > 0$. Also, if we put $X = W + h$, then

$$F(X) = G(W) = A^2(W + h)^2 + 2B(W + h) + C = A'^2W^2 + 2B'W + C',$$

where $A' = A$, $B' = A^2h + B$, $C' = A^2h^2 + 2Bh + C$. We get $\Delta' = 4B'^2 - 4A'^2C = 4B^2 - 4A^2C = \Delta$, $(A'^2, B') = (A^2, B)$, and $(A'^2, 2B', C') =$

$(A^2, 2B, C)$. Thus, since $B' > 0$ for $h > -B/A^2$, we may assume that $B > 0$ for X large enough. Indeed, since

$$C' - G^4H^2 = A^2(h^2 - G^4H^2/A^2) + Bh + C,$$

we see that $C' > G^4H^2$ when $h > G^2H/A \ge 2G \ge 4$. Thus, we may also assume that $C > G^4H^2$ for X large enough.

From all of these observations it is clear that if the conditions of Theorem 2.2 hold, and $F(X)$ is not of Richaud–Degert type, then we may assume with no loss of generality that $F(X) = A^2X^2 + 2BX + C$, where $X > 0$, $2 \mid A$, $A > 0$, C is not a perfect integral square, and $B > 0$, $C > G^4H^2$ for X large enough. To avoid repeating all of these conditions in the sequel, we will simply use the expression $F(W) = A^2W^2 + 2BW + C$ to represent a form satisfying all of these conditions.

3. Continued fractions. Suppose D is a positive integer, not a square, and let δ be an integer of $\mathbb{Q}(\sqrt{D})$ with trace t and norm n. In pursuing the continued fraction expansion of δ one obtains a sequence $((\delta + P_h)/Q_h)$ of complete quotients and a sequence (c_h) of partial quotients given by the formulae

$$(\delta + P_h)/Q_h = c_h - (\overline{\delta} + P_{h+1})/Q_h \quad \text{and} \quad \mathrm{Norm}(\delta + P_{h+1}) = -Q_hQ_{h+1}.$$

Here $\overline{\delta}$ denotes the conjugate of δ, and plainly $t + P_h + P_{h+1} = c_hQ_h$. The usual notation for continued fractions has us write

$$\delta = [\, c_0, c_1, \ldots, c_h,\ (\delta + P_{h+1})/Q_{h+1}\,].$$

We denote the convergents $[\, c_0, c_1, \ldots, c_h\,]$ by x_h/y_h. It is often convenient to drop subscripts, writing $x_h = x$, $x_{h-1} = x'$, and so forth. Then we have the decomposition

$$\begin{aligned}\begin{pmatrix} x & -ny \\ y & x - ty \end{pmatrix} &= \begin{pmatrix} x & x' \\ y & y' \end{pmatrix}\begin{pmatrix} 1 & P \\ 0 & Q \end{pmatrix} \\ &= \begin{pmatrix} c_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} c_h & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & P \\ 0 & Q \end{pmatrix}\end{aligned}$$

where $P = P_{h+1}$, $Q = Q_{h+1}$; hence $x^2 - txy + ny^2 = (-1)^{h+1}Q$. In particular, the case $Q = Q_{h+1} = 1$ ($P_{h+1} = P_1 = c_0$) yields a nontrivial solution

$$X^2 - tXY + nY^2 = (X - \delta Y)(X - \overline{\delta}Y) = \pm 1$$

to "Pell's equation". A central remark is that

Lemma 3.1. *A nontrivial solution $X^2 - tXY + nY^2 = \pm 1$ to "Pell's equation" formally corresponds to a period of δ in that*

$$\begin{pmatrix} X & -nY \\ Y & X - tY \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} a_{r+1} & 1 \\ 1 & 0 \end{pmatrix},$$

entails $\delta = [\, \overline{a_0, a_1, \ldots, a_{r+1}}\,]$.

For details see [2] or [3]. Note, however, that the entries in the "period" may not be *admissible*, as they might not all be positive. For example, $\delta = (\sqrt{D}+1)/2$ has a periodic expansion of the shape

$$[\,\overline{a_0, a_1, \ldots, a_r, 0}\,] = [\,a_0, \overline{a_1, \ldots, a_r, 0, a_0}\,] = [\,a_0, \overline{a_1, \ldots, a_r + a_0}\,].$$

The case $Q = 1$, signalling a complete period—and thus halfway to two such periods—is a special case of $Q \mid t + 2P$, signalling halfway to a period. We note

LEMMA 3.2. *Suppose $(\delta + P)/Q$ is a complete quotient of the quadratic integer δ with norm n and trace t. If $Q \mid t + 2P$ then $Q \mid t^2 - 4n$; and if Q is squarefree and $Q \mid t^2 - 4n$ then $Q \mid t + 2P$.*

Proof. It is easy to verify that every complete quotient has $Q \mid \mathrm{Norm}(\delta + P)$, that is, $Q \mid n + tP + P^2$. Hence $4Q \mid (t^2 - 4n) - (t + 2P)^2$ and the claims are immediate. ∎

Of course this is well known and says no more than that the $\mathbb{Z}$-module $\langle Q, \delta + P\rangle$ is equal to its conjugate essentially when its norm Q is squarefree and divides the discriminant $t^2 - 4n$. The point is that it is easy to check that such $\mathbb{Z}$-modules—to wit, with $Q \mid \mathrm{Norm}(\delta + P)$—are $\mathbb{Z}[\delta]$-modules, and thus precisely the ideals of the order $\mathbb{Z}[\delta]$. The condition just mentioned is the *ambiguity* of the ideal. These matters are discussed in extenso in [2].

We will find it useful to introduce the definitions

$$L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

the point being that

$$\begin{pmatrix} c & 1 \\ 1 & 0 \end{pmatrix} J = R^c, \qquad J \begin{pmatrix} c & 1 \\ 1 & 0 \end{pmatrix} = L^c,$$

whilst $J^2 = I$. This notation allows one the alternative of viewing a product

$$\begin{pmatrix} c_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_h & 1 \\ 1 & 0 \end{pmatrix} \cdots,$$

corresponding to a continued fraction expansion, as an R–L sequence

$$R^{c_0} L^{c_1} R^{c_2} L^{c_3} \ldots$$

LEMMA 3.3. *If $x^2 - txy + ny^2 = \pm Q$, and $Q \mid t + 2P$, then x/y yields half a period of δ.*

Proof. It is convenient to notice that a matrix

$$\begin{pmatrix} x & -ny \\ y & x - ty \end{pmatrix} R^t J = \begin{pmatrix} -ny + tx & x \\ x & y \end{pmatrix}$$

is symmetric. Hence, if $t+2P=cQ$, we have

$$\begin{aligned}\begin{pmatrix} x & -ny \\ y & x-ty \end{pmatrix}^2 &= \begin{pmatrix} x & x' \\ y & y' \end{pmatrix}\begin{pmatrix} 1 & P \\ 0 & Q \end{pmatrix} R^t J \begin{pmatrix} 1 & 0 \\ P & Q \end{pmatrix}\begin{pmatrix} x & y \\ x' & y' \end{pmatrix} JR^{-t} \\ &= \begin{pmatrix} x & x' \\ y & y' \end{pmatrix}\begin{pmatrix} t+2P & Q \\ Q & 0 \end{pmatrix}\begin{pmatrix} y & x \\ y' & x' \end{pmatrix} R^{-t} \\ &= Q\begin{pmatrix} x & x' \\ y & y' \end{pmatrix}\begin{pmatrix} c & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} y & x \\ y' & x' \end{pmatrix} R^{-t},\end{aligned}$$

yielding a product of unimodular matrices corresponding to the expansion

$$[\,c_0, c_1, \dots, c_h, c, c_h, \dots, c_1, c_0 - t, 0\,];$$

that is,

$$\delta = [\,c_0, \overline{c_1, \dots, c_h, c, c_h, \dots, c_1, 2c_0 - t}\,].$$

Here we use the observation that if the continued fraction $[\,c_0, c_1, \dots, c_h\,]$ corresponds to the matrix $\left(\begin{smallmatrix} x & x' \\ y & y' \end{smallmatrix}\right)$, then the matrix $\left(\begin{smallmatrix} y & x \\ y' & x' \end{smallmatrix}\right)$ corresponds to the expansion $[\,c_h, c_{h-1}, \dots, c_0, 0\,]$. ∎

We also point out that if some $Q \mid t+2P$, then $Q = Q_n$ where $n \equiv \mathrm{lp}(\delta)/2 \pmod{\mathrm{lp}(\delta)}$ and $P_n = P_{n+1}$.

We conclude these "rappels" by recalling that

Lemma 3.4. *If $D > 0$, and x, y are integers satisfying $x^2 - Dy^2 = K$ where $|K| < \sqrt{D}$, then x/y is a convergent in the continued fraction expansion of $\sqrt{D}$.*

4. A continued fraction expansion of $\sqrt{A^2W^2+2BW+C}$. We set out to expand $\sqrt{D}$, where $D = A^2W^2 + 2BW + C$. As in Lemma 2.1, put $S = \gcd(A, B)$; so $S \mid B$. We notice that

$$\sqrt{D} = AW + B/A - (-\sqrt{D} + AW + B/A),$$

suggesting we consider the approximation u/v of $\sqrt{D}$, where $v = A/S$ and $u = (A^2/S)W + B/S$. We compute that

$$u^2 - Dv^2 = (B^2 - A^2C)/S^2 = G^2H,$$

where H is squarefree.

We set $x = B/S$, $y = A/S$ and remark that

$$u/v - AW = B/A = [\,c_0, c_1, \dots, c_n\,] = x/y.$$

This may appear not well defined. Thus we shall insist that $c_n \ge 2$ unless $B/A = 0$ or 1, cases which are excluded by our insistence that D not be of

Richaud–Degert type. Then, writing $x'/y' = [\,c_0, c_1, \ldots, c_{n-1}\,]$, we have

$$\begin{aligned}\begin{pmatrix} u & Dv \\ v & u \end{pmatrix} &= R^{AW} \begin{pmatrix} B/S & AC/S \\ A/S & B/S \end{pmatrix} R^{AW} \\ &= R^{AW} \begin{pmatrix} x & x' \\ y & y' \end{pmatrix} \begin{pmatrix} 1 & P \\ 0 & G^2|H| \end{pmatrix} R^{AW}.\end{aligned}$$

Here $(x'y - y'x)P = xx' - yy'C$, which is $\pm P = y'AC/S - x'B/S$. The square of the matrix above is readily seen ([1]) to be

$$\begin{aligned}R^{AW} \begin{pmatrix} x & x' \\ y & y' \end{pmatrix} \begin{pmatrix} 1 & P \\ 0 & G^2|H| \end{pmatrix} R^{2AW} \begin{pmatrix} P & G^2|H| \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y & x \\ y' & x' \end{pmatrix} R^{AW} \\ = R^{AW} \begin{pmatrix} x & x' \\ y & y' \end{pmatrix} \begin{pmatrix} 2P + 2AW & G^2|H| \\ G^2|H| & 0 \end{pmatrix} \begin{pmatrix} y & x \\ y' & x' \end{pmatrix} R^{AW}.\end{aligned}$$

The equation for P and Lemma 2.1 show that always $GH \mid 2P + 2AW$, so the product has the constant divisor GH. In other words, if we "nearly" disregard a possibly unpleasant 2, we see that

$$|GH|^{-1} \begin{pmatrix} u & Dv \\ v & u \end{pmatrix}^2$$

is "nearly" a unimodular matrix and "nearly" corresponds to a period of $\sqrt{D}$.

THEOREM 4.1. *Suppose $G = 1$ in Lemma 2.1. Then, if $|H| > 1$, we get*

$$\sqrt{D} = [\,AW + c_0, \overline{\overrightarrow{w}, 2(P + AW)/|H|, \overleftarrow{w}, 2(AW + c_0)}\,],$$

where $x/y = B/A = [\,c_0, c_1, \ldots, c_n\,]$. Since we set $\overrightarrow{w} = c_1, \ldots, c_n$ for brevity, we also write $\overleftarrow{w} = c_n, c_{n-1}, \ldots, c_1$. If $|H| = 1$, then

$$\sqrt{D} = [\,AW + c_0, \overline{\overrightarrow{w}, 2(AW + c_0)}\,].$$

Proof. The claim follows easily from Lemma 3.1 and

$$\begin{pmatrix} u & Dv \\ v & u \end{pmatrix}^2 = |H| R^{AW} \begin{pmatrix} x & x' \\ y & y' \end{pmatrix} \begin{pmatrix} 2(P + AW)/|H| & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y & x \\ y' & x' \end{pmatrix} R^{AW}.$$

Note that if $|H| = 1$, then $Q = 1$, $P = P_1 = c_0$. ∎

THEOREM 4.2. *If $G = 2$ in Lemma 2.1, then $D \equiv 5 \pmod 8$. If $|H| > 1$, then*

$$\tfrac{1}{2}(\sqrt{D} + 1) = \big[\,\tfrac{1}{2}(AW + c_0 + 1), \overline{\overrightarrow{w}, (P + AW)/|H|, \overleftarrow{w}, AW + c_0}\,\big]$$

([1]) The matrix $\begin{pmatrix} B/S & AC/S \\ A/S & B/S \end{pmatrix}$ is *false symmetric*. Taking its *false transpose* we get
$$\begin{pmatrix} G^2|H| & P \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y' & x' \\ y & x \end{pmatrix} = \begin{pmatrix} G^2|H| & P \\ 0 & 1 \end{pmatrix} J \cdot J \begin{pmatrix} y' & x' \\ y & x \end{pmatrix} = \begin{pmatrix} P & G^2|H| \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y & x \\ y' & x' \end{pmatrix}.$$

displays the period of $\frac12(\sqrt{D}+1)$. *Here*, $[\,\frac12 c_0+\frac12,\overrightarrow{w}\,]=\frac12 x/y+\frac12=\frac12(B/(2A)+1)$. *If* $|H|=1$, *then*

$$\tfrac12(\sqrt{D}+1)=\big[\,\tfrac12(AW+c_0+1),\overline{\overrightarrow{w},AW+c_0}\,\big].$$

Proof. On referring to Lemma 2.1 we see that if $2\mid A/S$, then $2\mid B/S$, which is impossible. If $2\mid B/S$, then $4\mid C$ and $4\mid\gcd(A^2,2B,C)$, which is also impossible. Hence, we must have $2\nmid B/S$ and $2\nmid A/S$. It follows that C is odd and, since $H\mid C$, that H is odd; hence, $D\equiv 5\pmod 8$.

We note that $u\equiv v\pmod 2$ and

$$\begin{pmatrix}\frac12(u+v) & \frac14(D-1)v\\ v & \frac12(u-v)\end{pmatrix}=\frac14\begin{pmatrix}1&0\\0&2\end{pmatrix}R\begin{pmatrix}u&Dv\\v&u\end{pmatrix}R^{-1}\begin{pmatrix}2&0\\0&1\end{pmatrix}.$$

Also,

$$\begin{aligned}&\begin{pmatrix}1&0\\0&2\end{pmatrix}R\begin{pmatrix}x&x'\\y&y'\end{pmatrix}\begin{pmatrix}1&P\\0&4|H|\end{pmatrix}R^{-1}\begin{pmatrix}2&0\\0&1\end{pmatrix}\\&\qquad=2\begin{pmatrix}\frac12(x+y)&x'+y'\\y&2y'\end{pmatrix}\begin{pmatrix}2&P-1\\0&2|H|\end{pmatrix},\end{aligned}$$

whence

$$\begin{aligned}&\begin{pmatrix}(u+v)/2&\frac14(D-1)v\\v&(u-v)/2\end{pmatrix}^2\\&\qquad=|H|R^{AW/2}\begin{pmatrix}\frac12(x+y)&x'+y'\\y&2y'\end{pmatrix}\\&\qquad\quad\times\begin{pmatrix}(P+AW)/|H|&1\\1&0\end{pmatrix}\begin{pmatrix}y&\frac12(x+y)\\2y'&x'+y'\end{pmatrix}R^{AW/2-1}.\end{aligned}$$

Our result now follows from Lemma 3.1. ∎

As hinted at earlier, all these expansions are of the shape

$$\big[\,b_0+\tfrac12(QWy+s),\overline{b_1,\ldots,b_h,b+Wy,b_h,\ldots,b_1,2b_0+QWy}\,\big].$$

To sustain this remark we note that above $[\,c_0,c_1,\ldots,c_h\,]=(B/S)/(A/S)$ whence $y=A/S$ and $Qy=QA/S$. Thus $S=Q$ and $|H|=Q$ throughout. Finally, we observe that all the $\sqrt{A^2W^2+2BW+C}$ have the same period length for all W with W large enough to avoid some degenerate cases.

We can also produce the continued fraction expansion of

$$\sqrt{A^2W^2+2BW+C}$$

in terms of the continued fraction expansion of $\sqrt{C}$.

THEOREM 4.3. *If $G = 1$ in Lemma 2.1, let $\sqrt{C} = [\,c_0, c_1, \ldots, c_n, \ldots]$. Set $\overrightarrow{w} = c_1, \ldots, c_n$, and so $\overleftarrow{w} = c_n, \ldots, c_1$. Here $Q_{n+1} = |H|$. Then*

$$\sqrt{D} = [\,AW + c_0, \overline{\overrightarrow{w}, 2AW/Q_{n+1} + c_{n+1}, \overleftarrow{w}, 2(AW + c_0)}\,]$$

if $|H| > 1$. When $|H| = 1$,

$$\sqrt{D} = [\,AW + c_0, \overline{\overrightarrow{w}, 2(AW + c_0)}\,].$$

Proof. By Lemma 3.4, we know that if

$$(B/S)^2 - (A/S)^2 C = H$$

is soluble, then $|H|$ must be some Q_{n+1}. Further, $n + 1 = (2k + 1)\pi/2$, where $\pi = \mathrm{lp}(\sqrt{C}\,)$, $P = P_{n+1} = P_n$, and $c_{n+1} = 2P_{n+1}/Q_{n+1}$. Also, by Lemma 3.1,

$$\begin{pmatrix} x & x' \\ y & y' \end{pmatrix} = \begin{pmatrix} c_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_n & 1 \\ 1 & 0 \end{pmatrix}.$$

The result now follows in a similar fashion to that of Theorem 4.1. ∎

We also have a result connecting the continued fraction expansion of $\frac{1}{2}(\sqrt{D} + 1)$ to that of $\frac{1}{2}(\sqrt{C} + 1)$.

THEOREM 4.4. *If we take $G = 2$ in Lemma 2.1 and $\frac{1}{2}(\sqrt{C} + 1) = [\,c_0, c_1, \ldots, c_n, \ldots]$, where $Q_{n+1} = |H|$, then*

$$\tfrac{1}{2}(\sqrt{D} + 1) = [\,AW/2 + c_0, \overline{\overrightarrow{w}, AW/Q_{n+1} + c_{n+1}, \overleftarrow{w}, AW + 2c_0 - 1}\,]$$

if $|H| > 1$, and when $|H| = 1$

$$\tfrac{1}{2}(\sqrt{D} + 1) = [\,AW/2 + c_0, \overline{\overrightarrow{w}, AW + 2c_0 - 1}\,].$$

Proof. As in the proof of Theorem 4.3 we have $n + 1 = (2h + 1)\pi/2$—recall that $\pi = \mathrm{lp}\left(\frac{1}{2}(\sqrt{C} + 1)\right)$—$Q_{n+1} = |H|$, $P_{n+1} = P_{n+2}$, $Q_{n+1} \mid 2P_{n+1} + 1$, $P = 2P_{n+1} + 1$, and $c_{n+1} = P/Q_{n+1}$. Since by Lemma 3.1 we have

$$\begin{pmatrix} (x + y)/2 & x' + y' \\ y & 2y' \end{pmatrix} = \begin{pmatrix} c_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_n & 1 \\ 1 & 0 \end{pmatrix},$$

we get our result as in the proof of Theorem 4.2. ∎

Notice that in the case of Theorem 4.3 we get

$$\mathrm{lp}(\sqrt{D}\,) = \begin{cases} (2k + 1)\pi & \text{if } |H| > 1, \\ (2k + 1)\pi/2 & \text{if } |H| = 1, \end{cases}$$

where $\pi = \mathrm{lp}(\sqrt{C})$; hence $\mathrm{lp}(\sqrt{D}\,)$ can be as large as we want by selecting k large enough. There is, of course, a similar result in the case of Theorem 4.4.

5. The final case. Our task at this point is still incomplete. We undertook to produce the continued fraction expansion of $\sqrt{D}$, but we have only that of $(\sqrt{D}+1)/2$ in the case of $G=2$. However, we can get a result, like that of Theorem 4.3, in which we relate the continued fraction expansion of $\sqrt{D}$ to that of $\sqrt{C}$. First we need some results concerning the continued fraction expansion of $\sqrt{C}$.

THEOREM 5.1. *Let* $C \equiv 5 \pmod 8$ *and suppose that in the continued fraction expansion of* $\sqrt{C}$ *we get* $4 \mid Q_{m+1}$, $1 < \frac{1}{4}Q_{m+1} < C$, $\frac{1}{4}Q_{m+1}$ *square-free, and* $\frac{1}{4}Q_{m+1} \mid C$. *If* m *is the least nonnegative integer for which these conditions hold, then the fundamental unit* ε *of the order* $\mathcal{O} = \langle 1, \sqrt{C}\rangle$ *is given by*

$$\varepsilon = \delta^6/Q_{m+1}^3,$$

where $\delta = x_m + \sqrt{C}\, y_m$.

Proof. We know that $2Q_{m+1} \mid x_m(x_m^2+3Cy_m^2)$ and $2Q_{m+1} \mid y_m(Cy_m^2 + 3x_m^2)$ by the same reasoning as that used above. Hence $\nu = \lambda^3/(2Q_{m+1}) \in \mathcal{O}$. Now $|n(\nu)| = \frac{1}{4}Q_{m+1}$ and $\frac{1}{4}Q_{m+1} \mid C$. Thus, $\varepsilon \equiv 4\nu^2/Q_{m+1} \in \mathcal{O}$ and $n(\varepsilon) = 1$. It follows that $\varepsilon = \lambda^6/Q_{m+1}^3$ is a unit of $\mathcal{O}$. Also there must exist some $\theta = x_r + y_r\sqrt{C}$ such that $Q_{r+1} = Q_{m+1}$ and $\eta = 4\theta^2/Q_{m+1}$ is the fundamental unit of $\mathcal{O}$. By definition of λ we have $\lambda < \theta$; consequently, $\lambda^2 < \eta$ and $\varepsilon < \eta^3$. It follows that $\varepsilon = 1$, η, or η^2. If $\varepsilon = 1$, we get $\lambda^2/Q_{m+1} = 1$ and $\lambda^2 = Q_{m+1}$. If $\varepsilon = \eta^2$, we get $(\lambda^3/(Q_{m+1}\eta))^2 = Q_{m+1}$. In either case we find that $Q_{m+1} = \alpha^2$ where $\alpha \in \mathcal{O}$. If $\alpha = a + b\sqrt{C}$, then $ab = 0$. If $b = 0$, then $Q_{m+1} = a^2$; if $Q = 0$, then $\frac{1}{4}Q_{m+1} = (b/2)^2 C$. Thus $\varepsilon = \eta$. ■

COROLLARY 5.2. *If* $\mu = 2\lambda^2/Q_{m+1}$, *then* $\mu = x_n + \sqrt{C}y_n$, *where* n *is the least nonnegative integer such that* $Q_{n+1} = 4$. *Also*, $\nu = x_p + \sqrt{C}y_p$, *where* $p = \pi/2$ *and* $\pi = \mathrm{lp}(\sqrt{C}\,)$.

COROLLARY 5.3. *If* $Q_{k+1} = Q_{m+1}$ *and* $k < \pi$, *then* $k = m$ *or* $k = \pi - m - 2$; *if* $Q_{k+1} = 4$ *and* $k < \pi$, *then* $k = n$ *or* $k = \pi - n - 2$; *if* $Q_{k+1} = Q_{p+1}$ *and* $k < \pi$, *then* $k + 1 = \pi/2$.

COROLLARY 5.4. *Suppose* $r = m + k\pi$ (*for some* $k \ge 0$). *Set*

$$2(x_r+\sqrt{D}\,y_r)^2/Q_{m+1} = x_s+\sqrt{C}\,y_s, \qquad (x_r+\sqrt{C}\,y_r)^3/(2Q_{m+1}) = x_t+\sqrt{C}\,y_t.$$

Then $s = n + 2k\pi$ *and* $t = p + 3k\pi$. *If* $r = -m + (k+1)\pi$ $(k \ge 0)$, *then* $s = -n + 2(k+1)\pi$ *and* $t = -p + 3(k+1)\pi$.

COROLLARY 5.5. *Let* r, s, *and* t *be defined as in Corollary* 5.4. *We must have* $s > r + 1$ *and, unless* $y_r = 1$, *we must have* $t > s + 1$.

Proof. We have $|x_k - \sqrt{C}\,y_r| < 1$ as a property of the convergents in a continued fraction expansion; hence, $x_m + \sqrt{C}y_m > Q_{m+1}$, and therefore $\mu > \nu > \lambda$. It follows that $p > n > m$. Thus, we must have $s > r$. If $s = r+1$, then $4 \mid Q_{r+1}$ and $4 \mid Q_{r+2}$, which means that $P_{r+2}^2 - C \equiv 0 \pmod{16}$, which is impossible as $C \equiv 5 \pmod 8$. If $t = s+1$, then we can only have $r = m$, $s = n$, and $t = p$. Now

$$\begin{pmatrix} x_m & Cy_m \\ y_m & x_m \end{pmatrix} = Q_{n+1} \begin{pmatrix} x_n & Cy_n \\ y_n & x_n \end{pmatrix}^{-1} \begin{pmatrix} x_p & Cy_p \\ y_p & x_p \end{pmatrix}$$
$$= \begin{pmatrix} Q_{n+1} & -P_{n+1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_{n+1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & P_{p+1} \\ 0 & Q_{p+1} \end{pmatrix},$$

by our observations at the beginning of Section 3. Hence, $y_r = y_m = 1$. We note that if $t = s+1$, then $v = A/S = y_r = 1$ and $u^2 - D = 4Q_{r+1}$. Since $u = (A^2/S)W + B/S$ and $Q_{r+1} = |H|$ and H is odd, we get $4Q_{r+1} \mid 4u$ and D is of Richaud–Degert type. ∎

In the following theorem we will, as usual, assume that D is not of Richaud–Degert type and that therefore $t > s+1$. We are now able to derive the form of the continued fraction expansion of $\sqrt{D}$.

THEOREM 5.6. *If $G = 2$ in Lemma* 2.1, *then for $\sqrt{C} = [c_0, c_1, \ldots, c_k, \ldots]$, we must get $G^2|H| = Q_{r+1}$ for $r = m + k\pi$, or $r = -m + (k+1)\pi$ $(k \ge 0)$, where m is defined in Theorem* 5.1 *and π is defined in Corollary* 5.2. *Put $\overrightarrow{w_1} = c_1, \ldots, c_p$; $\overrightarrow{w_2} = c_{r+2}, \ldots, c_s$; $\overrightarrow{w_3} = c_{s+2}, \ldots, c_t$, where s and t are given by Corollary* 5.4. *The continued fraction expansion of $\sqrt{D}$ is given by*

$$\big[\,AW + c_0, \overline{\overrightarrow{w_1}, 2AW/Q_{m+1} + c_{m+1}, \overrightarrow{w_2}, \tfrac12 AW + c_{n+1}, \overrightarrow{w_3}, 2AW/Q_{p+1} + c_{p+1},}$$
$$\overline{\overleftarrow{w_3}, \tfrac12 AW + c_{n+1}, \overleftarrow{w_2}, 2AW/Q_{m+1} + c_{m+1}, \overleftarrow{w_1}, 2AW + 2c_0}\,\big],$$

when $4|H| = Q_{m+1} \ne 4$. If $4|H| = Q_{m+1} = 4$, it is given by

$$\sqrt{D} = \big[\,AW + c_0, \overline{\overrightarrow{w_1}, \tfrac12 AW + c_{m+1}, \overrightarrow{w_2}, \tfrac12 AW + c_{n+1}, \overrightarrow{w_3}, 2AW + 2c_0}\,\big].$$

P r o o f. The first part of the theorem follows from Lemma 3.4 and Corollary 5.4. We note that

$$\begin{pmatrix} x_s & Cy_s \\ y_s & x_s \end{pmatrix} = \frac{2}{Q_{m+1}} \begin{pmatrix} x_r & Cy_r \\ y_r & x_r \end{pmatrix}^2,$$
$$\begin{pmatrix} x_t & Cy_t \\ y_t & x_t \end{pmatrix} = \frac12 \begin{pmatrix} x_s & Cy_s \\ y_s & x_s \end{pmatrix} \begin{pmatrix} x_r & Cy_r \\ y_r & x_r \end{pmatrix},$$

and

$$\begin{pmatrix} 1 & P_{k+1} \\ 0 & Q_{k+1} \end{pmatrix} R^{2AW} \begin{pmatrix} 0 & Q_{k+1} \\ 1 & -P_{k+2} \end{pmatrix}^{-1} = \begin{pmatrix} 2AW/Q_{k+1} + c_{k+1} & 1 \\ 1 & 0 \end{pmatrix}.$$

For $i+1<j$, define

$$T_{i,j} = \begin{pmatrix} x_{i+1} & x_i \\ y_{i+1} & y_i \end{pmatrix}^{-1} \begin{pmatrix} x_j & x_{j-1} \\ y_j & y_{j-1} \end{pmatrix} = \begin{pmatrix} c_{i+2} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_{i+3} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_j & 1 \\ 1 & 0 \end{pmatrix}.$$

Then we have

$$T_{i,j} \begin{pmatrix} 1 & P_{j+1} \\ 0 & Q_{j+1} \end{pmatrix} = \begin{pmatrix} 0 & Q_{i+1} \\ 1 & -P_{i+2} \end{pmatrix} \begin{pmatrix} x_i & Cy_i \\ y_i & x_i \end{pmatrix}^{-1} \begin{pmatrix} x_j & Cy_j \\ y_j & x_j \end{pmatrix}.$$

Thus,

$$\begin{aligned} \begin{pmatrix} u & Dv \\ v & u \end{pmatrix}^3 &= R^{AW} \begin{pmatrix} x_r & x_{r-1} \\ y_r & y_{r-1} \end{pmatrix} \begin{pmatrix} 1 & P_{m+1} \\ 0 & Q_{m+1} \end{pmatrix} \\ &\quad \times R^{2AW} \begin{pmatrix} x_r & Cy_r \\ y_r & x_r \end{pmatrix} R^{2AW} \begin{pmatrix} x_r & Cy_r \\ y_r & y_r \end{pmatrix} R^{AW} \\ &= 2Q_{m+1} R^{AW} \begin{pmatrix} c_0 & 1 \\ 1 & 0 \end{pmatrix} T_{-1,r} \begin{pmatrix} 2AW/Q_{m+1} + c_{m+1} & 1 \\ 1 & 0 \end{pmatrix} \\ &\quad \times T_{r,s} \begin{pmatrix} AW/2 + c_{n+1} & 1 \\ 1 & 0 \end{pmatrix} T_{s,t} \begin{pmatrix} 1 & P_{p+1} \\ 0 & Q_{p+1} \end{pmatrix} R^{AW}. \end{aligned}$$

We set

$$\begin{aligned} K = {} & \begin{pmatrix} c_0 & 1 \\ 1 & 0 \end{pmatrix} T_{-1,r} \begin{pmatrix} 2AW/Q_{m+1} + c_{m+1} & 1 \\ 1 & 0 \end{pmatrix} T_{r,s} \\ & \times \begin{pmatrix} AW/2 + c_{n+1} & 1 \\ 1 & 0 \end{pmatrix} T_{s,t}, \end{aligned}$$

and note that $K\left(\begin{smallmatrix} 1 & P_{p+1} \\ 0 & Q_{p+1} \end{smallmatrix}\right)$ is false symmetric. Hence,

$$\frac{1}{Q_{m+1}^3} \begin{pmatrix} u & Dv \\ v & u \end{pmatrix}^6 = K \begin{pmatrix} 2AW/Q_{p+1} + c_{p+1} & 1 \\ 1 & 0 \end{pmatrix} K^* R^{AW},$$

where $K^* = K^t J$, with K^t the transpose of K and, as above, $J = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Then our claimed continued fraction expansion follows immediately from Lemma 3.1 and our observations at the beginning of this section. Further, if $|H| = 1$, then $Q_{m+1} = Q_{n+1} = 4$, $c_{p+1} = 2c_0$, and we have the palindrome

$$\overrightarrow{w_1}, c_{m+1}, \overrightarrow{w_2}, c_{n+1}, \overrightarrow{w_3} = \overleftarrow{w_3}, c_{n+1}, \overleftarrow{w_2}, c_{m+1}, \overleftarrow{w_1}.$$

This information yields the expansion of $\sqrt{D}$ claimed for the case $|H| = 1$. ∎

Thus, if $G = 2$, we get, for some $k \ge 0$,

$$\operatorname{lp}(\sqrt{D}\,) = \begin{cases} (6k+1)\operatorname{lp}(\sqrt{C}\,) & \text{if } |H| > 1, \\ (6k+1)\operatorname{lp}(\sqrt{C}\,)/2 & \text{if } |H| = 1. \end{cases}$$

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