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Three two-dimensional Weyl steps in the circle problem II. The logarithmic Riesz mean for a class of arithmetic functions

by

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1. Summary. In Part II we study arithmetic functions whose Dirichlet series satisfy a rather general type of functional equation. For the logarithmic Riesz mean of these functions we give a representation involving finite trigonometric sums. An essential tool here is the saddle point method. Estimation of the exponential sums in the special case of the circle problem will be the topic of Part III.

2. Introduction. One purpose of the present paper is to provide the starting point for Part III by connecting the logarithmic Riesz mean of the function $r_2(n)$ with a certain exponential sum (see Theorem 2 below). However, the circle problem is concerned with only one of a large class of interesting arithmetic functions. Therefore it seems reasonable to give our investigation a wider scope. Consider functions defined by Dirichlet series

$$Z(s) = \sum_{n=1}^{\infty} f(n) n^{-s} \quad \text{and} \quad Z_1(s) = \sum_{n=1}^{\infty} f_1(n) n^{-s},$$

$f(n), f_1(n) \in \mathbb{C}$, convergent in some right half plane, that are connected by a functional equation

$$Z(s) = \mathcal{H}(s) Z_1(\gamma - s). \tag{1}$$

Here γ is a real parameter and $\mathcal{H}$ a function of the type

$$\mathcal{H}(s) = AB^s \frac{\prod_{j=1}^{M} \Gamma(1/2 + \beta_j - b_j s)}{\prod_{j=1}^{L} \Gamma(1/2 + \delta_j + d_j s)} \tag{2}$$

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with complex A, real β_j, δ_j and positive B, b_j, d_j. We assume furthermore that one of the functions $Z(s)$ and $Z_1(s)$ has only finitely many singularities. Since b_j, d_j are positive, this carries over to the other function. Finally we assume that one of these functions, say $Z(s)$, is of finite order in the usual sense: $Z(\sigma+it) \ll |t|^C$ as $|t| \to \infty$ for any ξ uniformly in $\sigma \geq \xi$ with suitable $C = C(\xi, Z)$. The analysis of the asymptotic expansion of $\mathcal{H}$ together with well-known properties of the Lindelöf function show that these assumptions imply $\sum_j b_j = \sum_j d_j$, and this in turn shows that the other function $Z_1(s)$ is also of finite order.

In 1957 Richert [12] considered essentially this class of functional equations and developed a representation of the arithmetic Riesz mean of order κ for the corresponding function f, which contains a sum that by analogy with a well-known identity of Voronoï [14] may be called a Voronoï sum. It differs, however, from the classical sum on the one hand by being finite, not an infinite series, and on the other by the nature of its summands, which are trigonometric functions instead of Bessel functions. Both properties are of advantage if one wants to estimate these sums by exponential sum methods. Other authors ([1], [2]–[3], [4], [5]–[8]) treat functions with more special functional equations to prove identities closer to the classical one of Voronoï; see also Ivić [9].

In the present paper we follow Richert's approach with respect to the logarithmic Riesz mean, for which we prove a similar representation with a finite trigonometric Voronoï sum with the same generality. In Part III of this paper we will exploit the formula in the case of the circle problem to estimate the error term there.

In contrast to Richert (and Landau [10]–[11] before) we consistently use complex variable techniques, which lead in a very natural way to the saddle point method and hence to better error estimates. Our method can equally well be applied to the arithmetic mean, and then the results improve those of Richert.

3. Results. To develop the necessary notation we must first discuss the function $\mathcal{H}(s)$. From Stirling's formula we deduce the asymptotic formula

$$\mathcal{H}(s) = A\Lambda p^s e^{qs} (se^{-i\pi/2})^{\lambda - qs}(1 + O(1/|s|)) \quad \text{for } |s| \geq 1, \tag{3}$$

uniformly in any sector $\delta \leq \arg s \leq \pi - \delta$, with

$$p = B \prod_{j=1}^{M} b_j^{-b_j} \prod_{j=1}^{L} d_j^{-d_j}, \quad q = \sum_{j=1}^{M} b_j + \sum_{j=1}^{L} d_j = 2\sum_{j=1}^{M} b_j,$$

$$\lambda = \sum_{j=1}^{M} \beta_j - \sum_{j=1}^{L} \delta_j, \quad \Lambda = (2\pi)^{(M-L)/2} \prod_{j=1}^{M} b_j^{\beta_j} \prod_{j=1}^{L} d_j^{-\delta_j} e^{-i(\pi/2)(\sum_j \beta_j + \sum_j \delta_j)}.$$

If we write $\mathcal{H}(s) = A\mathcal{H}_0(s)$, then $\mathcal{H}_0(\overline{s}) = \overline{\mathcal{H}_0(s)}$, as the definition shows. This together with (3) implies that

$$\mathcal{H}(\sigma + it) = A\widetilde{\Lambda} p^\sigma |t|^{\lambda - q\sigma} e^{-it(q \log |t| - \log p - q)}(1 + O(1/|t|)) \tag{4}$$

as $t \to \pm\infty$ uniformly in each strip $\sigma_1 \le \sigma \le \sigma_2$, where $\widetilde{\Lambda} = \Lambda$ if $t \ge 0$ and $\widetilde{\Lambda} = \overline{\Lambda}$ otherwise.

Through the functional equations of the Γ-function, $\mathcal{H}$ has many different representations of type (2). As will be seen, however, from (4), the quantities p, q, λ, $A\Lambda$ and $A\overline{\Lambda}$ are well-defined by $Z(s)$ and $Z_1(s)$. We call such quantities *invariants* of the functional equation (1). The parameter q which, following Selberg, is called the *dimension* of the functional equation, was proved by Richert [12] to be at least 1. As usual we denote by $\sigma_{\mathrm{c}} := \sigma_{\mathrm{c}}(Z)$ and $\sigma_{\mathrm{a}} := \sigma_{\mathrm{a}}(Z)$ the abscissae of convergence and absolute convergence of Z. Let also $\sigma_{\mathrm{g}} := \sigma_{\mathrm{g}}(Z) = \inf\{\xi : f(n) \ll n^\xi \text{ as } n \to \infty\}$ be the growth index of f. We will use σ_{c}^1, σ_{a}^1 and σ_{g}^1 for the respective abscissae of Z_1.

THEOREM 1. *With the above notation and assumptions let $\kappa \ge 0$, $\varepsilon > 0$ and N be a natural number and assume $\sigma_{\mathrm{g}} \ge -1$. Then the logarithmic Riesz mean of f has the following representation*:

$$\frac{1}{\Gamma(\kappa+1)} \sum_{n<x} f(n) \log^\kappa(x/n) = H_\kappa(x) + V_\kappa(x, N) + R_\kappa(x, N)$$

with a main term H_κ, an exponential sum V_κ and an error R_κ. Here

$$H_\kappa(x) = \frac{1}{2\pi i} \int_{\mathcal{C}} Z(s) \frac{x^s}{s^{\kappa+1}}\, ds, \tag{5}$$

where the path $\mathcal{C}$ encloses all singularities of $Z(s)$ apart from those on the non-positive real axis. The Voronoï sum V_κ is given by

$$V_\kappa(x, N) = C \sum_{n \le N} \frac{f_1(n)}{n^\gamma} (pxn)^{(\lambda - \kappa - 1/2)/q} \times \cos(q(pxn)^{1/q} - (3/4 + \kappa/2)\pi + \phi), \tag{6}$$

where

$$C = \sqrt{\frac{2}{\pi q}} A|\Lambda|, \qquad \phi = \arg \Lambda = -\frac{\pi}{2}\Big(\sum_{j=1}^{M} \beta_j + \sum_{j=1}^{L} \delta_j\Big),$$

and the error term

$$R_\kappa(x, N) \ll_\varepsilon x^{\sigma_{\mathrm{g}} + 1 + \varepsilon}(xN)^{-(\kappa+1)/q} + x^{\sigma_{\mathrm{g}} + \varepsilon}(xN)^{-\kappa/q} + (xN)^{(\lambda - \kappa - 1)/q} N^{\sigma_{\mathrm{g}}^1 + 1 - \gamma + \varepsilon} + x^{(\lambda - \kappa - 3/2)/q} + x^\varepsilon, \tag{7}$$

as x tends to infinity, uniformly with respect to $N \in \mathbb{N}$.

REMARK 1. The error term x^ε may be replaced by a further main term (see (13)).

REMARK 2. One would expect the Voronoï sum to be an invariant of the functional equation in the sense explained above. This is indeed the case, since

$$2A|\Lambda|\cos(z+\phi) = A|\Lambda|(e^{iz+i\phi}+e^{-iz-i\phi}) = A\Lambda e^{iz} + A\overline{\Lambda}e^{-iz}.$$

COROLLARY 1. *If $f_1 = f$, that is, if $Z_1 = Z$, then A is real, and*

$$C = \sqrt{\frac{2}{\pi q}}p^{-\gamma/2}\operatorname{sgn} A \quad \text{and} \quad \frac{\lambda}{q} = \frac{\gamma}{2}.$$

Furthermore in this case $\gamma \geq 0$ implies $\sigma_{\mathrm{g}} \geq -1$.

An important application is to the circle problem, that is, with $f(n) = r_2(n)$, the number of representations of n as $n_1^2+n_2^2$, $n_1, n_2 \in \mathbb{Z}$. Here $Z(s) = 4\zeta(s)L(s,\chi_4)$ satisfies the functional equation (1) with $Z_1(s) = Z(s)$, $\gamma = 1$, χ_4 the non-principal character modulo 4, and

$$\mathcal{H}(s) = \pi^{2s-1}\frac{\Gamma(1-s)}{\Gamma(s)}. \tag{8}$$

Since $L(1,\chi_4) = \pi/4$, the pole of $\zeta(s)$ at $s=1$ yields $H_\kappa(x) = \pi x$. From (8) we see that $q=2$, $p=\pi^2$ and $\phi = 0$. Therefore the corollary leads to

THEOREM 2. *Let κ be a non-negative real parameter and N a natural number. Then*

$$\frac{1}{\Gamma(\kappa+1)}\sum_{n<x} r_2(n)\log^\kappa(x/n) = \pi x + V_\kappa(x,N) + R_\kappa(x,N),$$

where

$$V_\kappa(x,N) = \frac{1}{\pi^{\kappa+1}}x^{1/4-\kappa/2}\sum_{n\leq N}\frac{r_2(n)}{n^{3/4+\kappa/2}}\cos(2\pi\sqrt{xn}-(\pi/2)(\kappa+3/2)) \tag{9}$$

and

$$R_\kappa(x,N) \ll_\varepsilon x^{(1-\kappa)/2+\varepsilon}N^{-(1+\kappa)/2} + x^\varepsilon \tag{10}$$

for $x \to \infty$ and any $\varepsilon > 0$ uniformly in N.

Another classical problem to which the method applies involves the number of divisors $d_2(n)$. Here $Z(s) = Z_1(s) = \zeta^2(s)$, $\gamma = 1$ and

$$\mathcal{H}(s) = \pi^{2s-1}\frac{\Gamma^2((1-s)/2)}{\Gamma^2(s/2)},$$

from which it is easily seen that $p = 4\pi^2$, $q = 2$ and $\phi = \pi/2$. In this case the above corollary gives

THEOREM 3.

$$\frac{1}{\Gamma(\kappa+1)}\sum_{n<x} d_2(n)\log^{\kappa}(x/n) = x\log x+(2\gamma-1-\kappa)x+V_\kappa(x,N)+R_\kappa(x,N)$$

with

$$V_\kappa(x,N) = \frac{\sqrt{2}}{(2\pi)^{\kappa+1}}x^{1/4-\kappa/2}\sum_{n\le N}\frac{d_2(n)}{n^{3/4+\kappa/2}} \times\cos(4\pi\sqrt{xn}-(\pi/2)(\kappa+1/2)) \tag{11}$$

and $R_\kappa(x,N)$ *as in* (10).

4. Preliminaries. We need a form of the well-known inversion formula.

LEMMA 1. *Let* κ *and* c *be real constants*, $\kappa\ge 0$, $c>0$. *Then*, *uniformly in* x *and* $T\ge 1$,

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{e^{sx}}{s^{\kappa+1}}\,ds = \frac{1}{\Gamma(\kappa+1)}\chi_\kappa(x)+O\biggl(\frac{e^{cx}}{T^\kappa}\min\biggl(1,\frac{1}{T|x|}\biggr)\biggr),$$

where

$$\chi_\kappa(x) = \begin{cases} x^\kappa & \text{for } x>0,\\ 1/2 & \text{for } x=0 \text{ and } \kappa=0,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. First assume that $x=\kappa=0$ does not hold. To the given segment $\mathcal{C}_1$ from $c-iT$ to $c+iT$ we add two horizontal half lines $\mathcal{C}_2$, $\overline{\mathcal{C}}_2$ to form a U-shaped contour that opens to the left if $x>0$ and to the right if $x\le 0$. The integral along the right open contour ($x\le 0$) vanishes since there is no singularity inside and the integrand vanishes sufficiently fast. The other, corresponding to the case $x>0$, is reduced to Hankel's well-known representation of $1/\Gamma(\kappa+1)$ by substituting $u:=sx$. Therefore we can estimate in either of the cases $x>0$ or $x<0$:

$$\biggl|\int_{\mathcal{C}_2,\overline{\mathcal{C}}_2}\frac{e^{sx}}{s^{\kappa+1}}\,ds\biggr| \le \frac{1}{T^{\kappa+1}}\int_{\mathcal{C}_2} e^{\sigma x}\,d\sigma = \frac{e^{cx}}{T^{\kappa+1}|x|}.$$

If $\kappa>0$, we also have

$$\biggl|\int_{\mathcal{C}_2,\overline{\mathcal{C}}_2}\frac{e^{sx}}{s^{\kappa+1}}\,ds\biggr| \le e^{cx}\int_{-\infty}^{\infty}\frac{d\sigma}{|\sigma+iT|^{\kappa+1}} \ll \frac{e^{cx}}{T^\kappa}.$$

In the remaining case $x=\kappa=0$ we can integrate explicitly:

$$\int_{c-iT}^{c+iT}\frac{ds}{s} = \log\biggl(\frac{c+iT}{c-iT}\biggr) = 2i\arctan\biggl(\frac{T}{c}\biggr) = \pi i+O\biggl(\frac{c}{T}\biggr). \qquad \blacksquare$$

We state without proof

Lemma 2. *For any real number $x \geq 1$ and any ε, $0 < \varepsilon \leq 1$, we have*

$$\sum_{|n-x|\geq 1} \frac{1}{n^{1+\varepsilon}|\log(x/n)|} \ll \frac{1}{\varepsilon}. \ \blacksquare$$

5. Beginning the proof of Theorem 1. As usual we express the sum

$$F_\kappa(x) := \frac{1}{\Gamma(\kappa+1)} \sum_{n\leq x} f(n) \log^\kappa(x/n),$$

where in the case $\kappa = 0$ and $x \in \mathbb{N}$ the last term is counted only with half weight, as an integral over a segment $\mathcal{C}_1$ of a vertical line:

$$F_\kappa(x) = \frac{1}{2\pi i} \int_{\alpha-iT}^{\alpha+iT} Z(s) \frac{x^s}{s^{\kappa+1}}\, ds + R \tag{12}$$

with an error term R. Here we require $\alpha > 0$ to allow the application of Lemma 1; $\alpha > \sigma_{\mathrm{a}}$ in order to exchange summation and integration; and moreover $\alpha > \sigma_{\mathrm{g}} + 1$ to be able to estimate individual terms. T will be large, in fact larger than the imaginary part of any of the finitely many singularities of Z. Furthermore we insert $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_4$, $\overline{\mathcal{C}}_2$, $\overline{\mathcal{C}}_3$ to form a closed contour in the cut plane $\mathbb{C} \setminus (-\infty, 0]$ to the left of $\mathcal{C}_1$ as shown in Figure 1.

$\mathcal{C}_2$: the line segment from $\alpha + iT$ to $\beta + iT$, $\beta < \alpha$, β to be fixed later,

$\mathcal{C}_3$: the line segment from $\beta + iT$ to s_1, s_1 somewhere near $\beta + i$, to be fixed later,

$\mathcal{C}_4$: from s_1 to $\overline{s}_1$ encircling the non-positive real axis but no singularity that is not on this part of the real line,

and the conjugates $\overline{\mathcal{C}}_3$, $\overline{\mathcal{C}}_2$ in the lower half plane with the proper orientation.

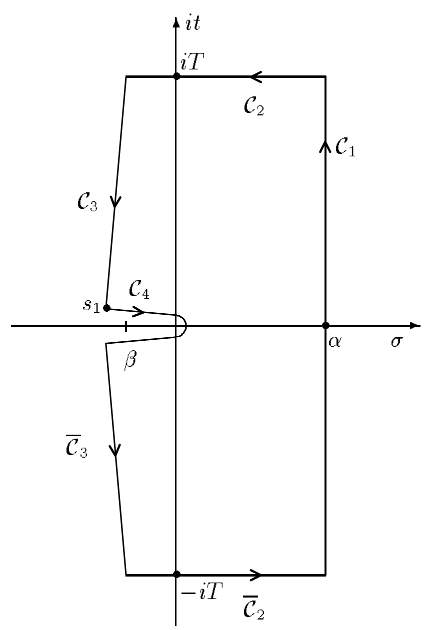

Fig. 1. Path of integration $\mathcal{C}$

An immediate consequence is the relation

$$F_\kappa(x) = H_\kappa(x) + R - \frac{1}{2\pi i} \int\limits_{\mathcal{C}_2+\ldots+\overline{\mathcal{C}}_2} Z(s)\frac{x^s}{s^{\kappa+1}}\, ds,$$

which determines our further program. H_κ is a sum of residues. The integral

$$J_4(x) := -\frac{1}{2\pi i} \int\limits_{\mathcal{C}_4} Z(s)\frac{x^s}{s^{\kappa+1}}\, ds \tag{13}$$

is of a similar nature as the residues; for $x > 0$ it is a holomorphic function of x with the properties $J_4(x) \ll x^\varepsilon$, $J_4'(x) \ll x^{\varepsilon-1}$, etc. As a matter of taste one might include it in the main term. The integrals over $\mathcal{C}_2$, $\overline{\mathcal{C}}_2$ are treated as error terms. On $\mathcal{C}_3$, $\overline{\mathcal{C}}_3$ we want to apply the functional equation (1) and then integrate term by term. Therefore we take $\beta < \gamma - \sigma_{\mathrm{g}}^1 - 1$ and $\Re(s_1) < \gamma - \sigma_{\mathrm{g}}^1 - 1$. As far as possible we use the saddle point method to extract further main terms from the individual integrals, which will then form the Voronoï sum.

Our first step is, as indicated, to verify (12). On the line $\mathcal{C}_1$ we may exchange summation and integration since $\alpha > \sigma_{\mathrm{a}}$ and then apply Lemma 1. Thus

$$\frac{1}{2\pi i} \int\limits_{\mathcal{C}_1} Z(s)\frac{x^s}{s^{\kappa+1}}\, ds = \sum_{n=1}^{\infty} f(n) \frac{1}{2\pi i} \int\limits_{\alpha-iT}^{\alpha+iT} \left(\frac{x}{n}\right)^s \frac{1}{s^{\kappa+1}}\, ds = F_\kappa(x) - R,$$

with

$$\begin{aligned} R &\ll \sum_{n=1}^{\infty} |f(n)| \left(\frac{x}{n}\right)^\alpha \frac{1}{T^\kappa} \min\left(1, \frac{1}{T|\log(x/n)|}\right) \\ &\ll \frac{x^\alpha}{T^\kappa} \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \min\left(1, \frac{1}{T|\log(x/n)|}\right), \end{aligned}$$

since $f(n) \ll n^{\sigma_{\mathrm{g}}+\varepsilon}$ and $\sigma_{\mathrm{g}} < \alpha - 1$. The second bound combined with Lemma 2 is used on all but the one or two terms with $|n-x| < 1$. Altogether we get

$$R \ll x^{\alpha-1}T^{-\kappa} + x^\alpha T^{-\kappa-1}. \tag{14}$$

In order to estimate the integrals over the horizontal lines $\mathcal{C}_2$ and $\overline{\mathcal{C}}_2$, we start from

$$Z(\sigma + it) \ll \begin{cases} 1 & \text{for } \sigma = \alpha, \\ |t|^{\lambda - q\beta} & \text{for } \sigma = \beta, \end{cases} \tag{15}$$

as $t \to \pm\infty$; the first because $\alpha > \sigma_{\mathrm{a}}$, the second by the functional equation (1), the asymptotic estimate given in (4) and $\gamma - \beta > \sigma_{\mathrm{a}}^1$. Since Z is of finite order by assumption, the Phragmén–Lindelöf principle (see [13],

§5.65) applies. It follows that

$$Z(\sigma+it)\ll |t|^{\chi(\sigma)}\qquad \text{uniformly in } \beta\le\sigma\le\alpha \text{ and } |t|\ge t_0,$$

where $\chi(\sigma)$ interpolates linearly between the values at the boundary given in (15). Therefore $x^\sigma T^{\chi(\sigma)}$ takes its maximum value at one of the endpoints of the interval $[\beta,\alpha]$. Hence

$$(16)\qquad \frac{1}{2\pi i}\int\limits_{\mathcal{C}_2+\overline{\mathcal{C}}_2} Z(s)\frac{x^s}{s^{\kappa+1}}\,ds \ll (x^\alpha T^{\chi(\alpha)}+x^\beta T^{\chi(\beta)})T^{-\kappa-1} \ll x^\alpha T^{-\kappa-1}+x^\beta T^{\lambda-q\beta-\kappa-1}.$$

The most significant contributions come from $\mathcal{C}_3$ and $\overline{\mathcal{C}}_3$. By our choice of β, after using the functional equation, we are allowed to integrate term by term, and so we obtain

$$(17)\qquad \frac{1}{2\pi i}\int\limits_{\mathcal{C}_3+\overline{\mathcal{C}}_3} Z(s)\frac{x^s}{s^{\kappa+1}}\,ds = \frac{1}{2\pi i}\int\limits_{\mathcal{C}_3+\overline{\mathcal{C}}_3} \mathcal{H}(s)Z_1(\gamma-s)\frac{x^s}{s^{\kappa+1}}\,ds$$
$$= \sum_{n=1}^\infty \frac{f_1(n)}{n^\gamma}\,\frac{1}{2\pi i}\int\limits_{\mathcal{C}_3+\overline{\mathcal{C}}_3}\mathcal{H}(s)\frac{(nx)^s}{s^{\kappa+1}}\,ds$$
$$= -A\sum_{n=1}^\infty \frac{f_1(n)}{n^\gamma}\,\Re\biggl(\frac{1}{\pi i}\int\limits_{\mathcal{C}_3}\mathcal{H}_0(s)\frac{(nx)^s}{s^{\kappa+1}}\,ds\biggr),$$

since $\mathcal{H}_0(\overline{s})=\overline{\mathcal{H}_0(s)}$, where $\mathcal{H}(s)=A\mathcal{H}_0(s)$ as before. Therefore we need to study $\mathcal{H}_0(s)(nx)^s s^{-\kappa-1}$ only in the upper half plane. We will write $\varrho:=(\lambda-\kappa-1)/q$ and $\omega=\omega_n:=(pxn)^{1/q}$. Then, noticing that

$$\left(\frac{s-\varrho}{s}\right)^{\varrho-s} = e^{(s-\varrho)(\varrho/s+O(1/|s|^2))} = e^{\varrho+O(1/|s|)},$$

we deduce from (3) that

$$\mathcal{H}_0(s)\frac{(nx)^s}{s^{\kappa+1}} = \Lambda i^{-\kappa-1}(e\omega)^{qs}\left(\frac{s}{i}\right)^{q\varrho-qs}\left(1+O\left(\frac{1}{|s|}\right)\right)$$
$$= \Lambda e^{-(\kappa+1)\pi i/2}(e\omega)^{q\varrho}\left(\frac{ei\omega}{s}\right)^{q(s-\varrho)}\left(1+O\left(\frac{1}{|s|}\right)\right)$$
$$= \Lambda e^{-(\kappa+1)\pi i/2}\omega^{q\varrho}\left(\frac{ei\omega}{s-\varrho}\right)^{q(s-\varrho)}\left(1+O\left(\frac{1}{|s|}\right)\right)$$

for $t\ge 1$, uniformly in each sector $\delta\le\arg(s-\varrho)\le\pi-\delta$. Inserting this into (17) yields

$$(18)\qquad \frac{1}{2\pi i}\int\limits_{\mathcal{C}_3+\overline{\mathcal{C}}_3} Z(s)\frac{x^s}{s^{\kappa+1}}\,ds = \frac{A}{\pi}\sum_{n=1}^\infty \frac{f_1(n)}{n^\gamma}\omega_n^{q\varrho}\,\Re(\Lambda e^{-\kappa\pi i/2}\mathfrak{I}_n)$$

with

$$\mathfrak{I}_n = \mathfrak{I}_n(-\mathcal{C}_3) = \int\limits_{-\mathcal{C}_3} \left(\frac{ei\omega_n}{s-\varrho}\right)^{q(s-\varrho)} (1+r_1(s))\,ds, \qquad r_1(s) \ll \frac{1}{|s|}.$$

Note that ω_n tends to infinity uniformly in n. We analyze the modulus of the main factor in

LEMMA 3. *Let $z = x+iy$, $y>0$. Then*

$$\left|\left(\frac{z}{ie}\right)^z\right| = y^x e^{yF(x/y)}$$

with $F(-v) = -F(v)$, $0 \le F(v) \le \frac{1}{6}v^3$ for $v \ge 0$.

Proof. We have

$$\begin{aligned}\log\left|\left(\frac{z}{ie}\right)^z\right| &= \Re\left(z\log\left(\frac{z}{ie}\right)\right) = x\log\left(\frac{|z|}{e}\right) - y\arg\left(\frac{z}{i}\right)\\ &= x\log y + \frac{1}{2}x\log\left(1+\frac{x^2}{y^2}\right) - x + y\arctan\left(\frac{x}{y}\right)\\ &= x\log y + yF\left(\frac{x}{y}\right),\end{aligned}$$

where

$$F(v) = \frac{1}{2}\int\limits_0^v \log(1+w^2)\,dw. \ \blacksquare$$

For simplicity we usually drop the subscript n, although it is to be understood that many parameters depend on n. If we apply Lemma 3 with $z = (s-\varrho)/\omega$, we find

$$\left|\left(\frac{ei\omega}{s-\varrho}\right)^{q(s-\varrho)}\right| = \left(\frac{\omega}{t}\right)^{q(\sigma-\varrho)} e^{-qtF((\sigma-\varrho)/t)}. \tag{19}$$

Since this equals

$$\left(\frac{\omega}{t}\right)^{q(\sigma-\varrho)}\left(1+O\left(\frac{|\sigma-\varrho|^3}{t^2}\right)\right)$$

as long as $\sigma-\varrho \ll t^{2/3}$, the formula exhibits a saddle point $s_0 = \varrho + i\omega$ ($\omega = \omega_n$) with valleys to the north-east and south-west of it.

6. Applying the saddle point method. For given n we are no longer restricted to the half plane $\beta < \gamma - \sigma_{\mathrm{a}}^1$. Following the idea of the saddle point method, we choose a path $\widetilde{\mathcal{C}} = \widetilde{\mathcal{C}}_n$ across the saddle which stays in the valleys as far as possible but allows reasonably easy computation. In the neighborhood of the saddle point s_0 the method calls for the Taylor expansion of

$$h(s) = q(s-\varrho)\log\left(\frac{ei\omega}{s-\varrho}\right).$$

With $s = s_0 + \omega z$ and $z = (1+i)v$ we obtain

$$\begin{aligned}(20)\qquad h(s) &= iq\omega(1-iz)(1-\log(1-iz)) = iq\omega\left(1+\frac{z^2}{2}+\frac{iz^3}{6}+O(z^4)\right)\\ &= iq\omega - q\omega v^2 + \frac{1-i}{3}q\omega v^3 + O(\omega v^4).\end{aligned}$$

As a first step consider a preliminary path $\widehat{\mathcal{C}}$ leading up to $i\infty$ as shown in Figure 2. With a suitable constant K and

$$u = u_n := \sqrt{K\frac{\log\omega}{\omega}},$$

take $s_\nu = \sigma_\nu + it_\nu$ as follows:

$$\begin{aligned} &s_0 = \varrho + i\omega, && s_1 = \varrho + K(i-1), && s_2 = \varrho + u\omega(i-1),\\ &s_3 = s_0 - u\omega(1+i), && s_4 = s_0 + u\omega(1+i). \end{aligned}$$

The choice of u implies that

$$\Re h(s_3), \Re h(s_4) = -qK\log\omega + O(1), \qquad |e^{h(s_3)}|, |e^{h(s_4)}| \ll \omega^{-qK}.$$

$$\mathfrak{I}_n(\mathcal{C}) = \int_{\mathcal{C}} e^{h(s)}(1+r_1(s))\,ds,$$

$$e^{h(s)} = \left(\frac{ei\omega}{s-\varrho}\right)^{q(s-\varrho)},$$

$$r_1(s) = O\left(\frac{1}{|s|}\right),$$

$$\varrho = \frac{1}{q}(\lambda-\kappa-1),$$

$$\omega = (pxn)^{1/q},$$

$$u = \sqrt{K\frac{\log\omega}{\omega}}.$$

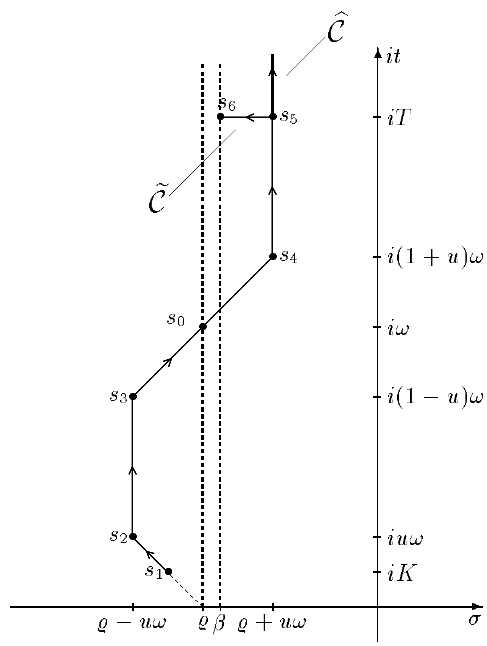

Fig. 2. Paths $\widetilde{\mathcal{C}}$ and $\widehat{\mathcal{C}}$

All segments of the path except $[s_3, s_4]$ yield error terms; we begin with this segment $[s_3, s_4]$. Using $s = s_0 + \omega(1+i)v$, $-u \le v \le u$ and (20) we get

$$\begin{aligned} e^{h(s)} &= e^{iq\omega} e^{-q\omega v^2}\left(1 + \frac{1-i}{3} q\omega v^3 + O(\omega^2 v^6)\right)(1 + O(\omega v^4)) \\ &= e^{iq\omega} e^{-q\omega v^2}\left(1 + \frac{1-i}{3} q\omega v^3 + O(\omega^2 v^6) + O(\omega v^4)\right). \end{aligned}$$

Since

$$\int_u^\infty e^{-q\omega v^2}\,dv \le \frac{1}{u}\int_u^\infty e^{-q\omega v^2} v\,dv \ll \frac{1}{\omega u} e^{-q\omega u^2} \ll \omega^{-qK-1/2} \ll \omega^{-3/2},$$
$$\int_{-u}^{u} v^3 e^{-q\omega v^2}\,dv = 0, \qquad \int_{-\infty}^{\infty} \omega v^4 e^{-q\omega v^2}\,dv,\ \int_{-\infty}^{\infty} \omega^2 v^6 e^{-q\omega v^2}\,dv \ll \omega^{-3/2}$$

and furthermore $r_1(s) \ll 1/t \ll 1/\omega$ on this segment, we have

$$\begin{aligned} \int_{s_3}^{s_4} e^{h(s)}(1 + r_1(s))\,ds &= (1+i)\omega e^{iq\omega}\left(\sqrt{\frac{\pi}{q\omega}} + O\left(\frac{1}{\omega^{3/2}}\right)\right) \\ &= \sqrt{\frac{2\pi\omega}{q}}\, e^{i(q\omega + \pi/4)} + O\left(\frac{1}{\omega^{1/2}}\right). \end{aligned} \tag{21}$$

On the rest of the path we estimate only the order of magnitude, so the term $r_1(s)$ becomes irrelevant. On the vertical line $[s_4, i\infty)$, where $s - \varrho = u\omega + it$ and $t \ge (1+u)\omega$, formula (19) yields

$$|e^{h(s)}| = \left|\left(\frac{ei\omega}{s-\varrho}\right)^{q(s-\varrho)}\right| \le \left(\frac{\omega}{t}\right)^{q(\sigma_4 - \varrho)} = \left(\frac{\omega}{t}\right)^{qu\omega},$$

$$\int_{s_4}^{i\infty} |e^{h(s)}|\,dt \ll \left(\frac{\omega}{t_4}\right)^{qu\omega} \frac{t_4}{u\omega} \ll \frac{1}{\omega^{qK} u} \ll \omega^{-qK+1/2} \ll \omega^{-1/2}. \tag{22}$$

Estimating $\int_{s_2}^{s_3}$ is a little harder. Here, because we have

$$-tF\left(\frac{\sigma_2 - \varrho}{t}\right) = tF\left(\frac{\varrho - \sigma_2}{t}\right) \le \frac{(\varrho - \sigma_2)^3}{6t^2} = \frac{u^3\omega^3}{6t^2},$$

formula (19) provides

$$|e^{h(s)}| \le \left(\frac{\omega}{t}\right)^{-qu\omega} e^{qu^3\omega^3/(6t^2)}.$$

This function is increasing in t, since

$$\frac{d}{dt}\left(u\omega \log t + \frac{u^3\omega^3}{6t^2}\right) = \frac{u\omega}{t} - \frac{u^3\omega^3}{3t^3} > 0.$$

Hence $|e^{h(s)}| \le (\omega/t_3)^{-qu\omega} e^{O(u^3\omega)} \ll \omega^{-qK}$ and

$$\int_{s_2}^{s_3} |e^{h(s)}|\,dt \ll \omega^{-qK+1}. \tag{23}$$

Finally, on $[s_1, s_2]$ we have $s = \varrho + t(i-1)$, $K \le t \le u\omega$,

$$|e^{h(s)}| \le (\omega/t)^{-qt} e^{qt/6} \le (ue)^{qt}.$$

Thus

$$\int_{s_1}^{s_2} |e^{h(s)}|\,|ds| \le \int_K^\infty (ue)^{qt}\,dt \le (ue)^{qK} \frac{1}{q|\log(ue)|} \ll \omega^{-qK/2+\varepsilon}. \tag{24}$$

The actual path $\widetilde{\mathcal{C}}$, which is homotopic to $-\mathcal{C}_3$, is derived from $\widehat{\mathcal{C}}$ by joining the point s_5 on $\widehat{\mathcal{C}}$ where $t = T$ by a horizontal line segment to $s_6 = \beta + iT$, and discarding the portion of $\widehat{\mathcal{C}}$ above s_5. Depending on n, the point s_5 will belong to different segments of $\widehat{\mathcal{C}}$. The estimates (22)–(24) obviously remain valid for any part of the respective segment, but the change also causes one or two more error terms, one from the horizontal line

$$P = P_n := \int_{s_5}^{s_6} |e^{h(s)}|\,|ds|, \tag{25}$$

and another, which occurs only if $s_5 \in [s_3, s_4]$, from truncating $\int_{s_3}^{s_4}$:

$$Q = Q_n := \begin{cases} \int_{s_5}^{s_4} |e^{h(s)}|\,|ds| & \text{if } \omega \le T \le (1+u)\omega, \\ \int_{s_3}^{s_5} |e^{h(s)}|\,|ds| & \text{if } (1-u)\omega \le T < \omega; \end{cases} \tag{26}$$

for convenience we consider $Q = 0$ outside the above range.

If $\omega \le T \le (1+u)\omega$, we write $\int_{s_3}^{s_5} = \int_{s_3}^{s_4} - \int_{s_5}^{s_4}$. This allows the use of (21) with an extra error term Q_n. If $(1-u)\omega \le T < \omega$, we have only Q_n instead of $\int_{s_3}^{s_4}$. Thus, in effect, we keep the main term together with the error $O(\omega^{-1/2})$, as in (21), only as long as $\omega \le T$.

Assigning T a value

$$T := (px(N+1/2))^{1/q}, \qquad \text{where } N \in \mathbb{N},$$

we translate the two cases $T < \omega$, $T \ge \omega$ into $n > N$, $n \le N$, and we also avoid the situation where $\omega = T$. Note that $T \gg x^{1/q}$ tends to infinity as x does.

For the error term Q we have $|v| \le u$ and therefore

$$\Re h(s) = -q\omega v^2 + O(\omega v^3) = -q\omega v^2 + o(1), \qquad |e^{h(s)}| \ll e^{-q\omega v^2}.$$

Thus

$$Q \ll \omega \int_{T/\omega-1}^{u} e^{-q\omega v^2}\,dv \ll \omega \int_{T/\omega-1}^{\infty} e^{-q\omega v^2}\,dv \quad \text{for } \omega \le T \le (1+u)\omega, \tag{27}$$

and similarly

$$Q \ll \omega \int_{-\infty}^{T/\omega-1} e^{-q\omega v^2}\,dv \quad \text{for } (1-u)\omega \le T < \omega. \tag{28}$$

To estimate $|e^{h(s)}|$ on $[s_5, s_6]$, apply Lemma 3 in two slightly different ways. This yields

$$|e^{h(s)}| \ll \left(\frac{\omega}{t}\right)^{q(\sigma-\varrho)} \qquad \text{for } t \ge \omega^{4/5},\ |\sigma-\varrho| \le u\omega, \tag{29}$$

$$|e^{h(s)}| \ll \left(\frac{\omega}{et}\right)^{q(\sigma-\varrho)} \qquad \text{for } t \le \omega^{4/5},\ \varrho - t \le \sigma \le \max(\varrho,\beta). \tag{30}$$

For the first case notice that

$$-tF\left(\frac{\sigma-\varrho}{t}\right) \le \frac{u^3\omega^3}{t^2} \ll \omega^{3/2+\varepsilon-8/5} = \omega^{\varepsilon-1/10};$$

for the second

$$-tF\left(\frac{\sigma-\varrho}{t}\right) \le \frac{(\varrho-\sigma)^3}{6t^2} \le \varrho - \sigma = -(\sigma-\varrho)$$

if $-t \le \sigma - \varrho \le 0$, and

$$-tF\left(\frac{\sigma-\varrho}{t}\right) \le 0 \le \beta - \sigma = -(\sigma-\varrho) + O(1)$$

if $\varrho \le \sigma \le \beta$ (in case $\beta > \varrho$).

In the situation where s_5 is downhill from s_6, that is, if $T > \omega$, $\beta \le \sigma_5$ or $T < \omega$, $\sigma_5 \le \beta$, we have

$$P \ll \left|\int_{\beta}^{\sigma_5} \left(\frac{\omega}{T}\right)^{q(\sigma-\varrho)} d\sigma\right| \le \left(\frac{\omega}{T}\right)^{q(\beta-\varrho)} \frac{1}{q|\log(T/\omega)|}, \tag{31}$$

provided that $T \ge \omega^{4/5}$, by (29). The same result is obtained for $T \le \omega^{4/5}$ with formula (30), since

$$P \ll \left(\frac{\omega}{eT}\right)^{q(\beta-\varrho)} \frac{1}{\log(\omega/(eT))} \ll \left(\frac{\omega}{T}\right)^{q(\beta-\varrho)} \frac{1}{\log(\omega/T)}.$$

If, however, s_5 is uphill from s_6, we have $|T-\omega| \le |\varrho-\beta| \ll 1$ and $|\sigma-\beta| \le |\varrho-\beta|$. Hence, by (19) we see that

$$|e^{h(s)}| \ll \left(\frac{\omega}{T}\right)^{q(\beta-\varrho)}\left(\frac{\omega}{T}\right)^{q(\sigma-\beta)} e^{q|\sigma-\varrho|^3/(6T^2)}$$
$$\ll \left(\frac{\omega}{T}\right)^{q(\beta-\varrho)}\left(1+O\left(\frac{1}{T}\right)\right)^{O(1)} e^{O(1/T^2)} \ll \left(\frac{\omega}{T}\right)^{q(\beta-\varrho)}.$$

Thus

$$P \ll \left(\frac{\omega}{T}\right)^{q(\beta-\varrho)}|\beta-\varrho| \ll \left(\frac{\omega}{T}\right)^{q(\beta-\varrho)} \ll \left(\frac{\omega}{T}\right)^{q(\beta-\varrho)}\left|\log\left(\frac{T}{\omega}\right)\right|^{-1},$$

and (31) holds in this case too.

We summarize our results concerning $\mathfrak{I}_n$:

$$\mathfrak{I}_n(-\mathcal{C}_3) = \mathfrak{I}_n(\widetilde{\mathcal{C}}_n) = \chi_{[1,N]}(n)\left(\sqrt{\frac{2\pi\omega_n}{q}}e^{i(q\omega_n+\pi/4)} + O(\omega_n^{-1/2})\right) + O(\omega_n^{-qK'}) + P_n + Q_n, \tag{32}$$

where χ denotes the indicator function, P_n, Q_n are given in (25), (26) and are bounded by (31), (27), (28) respectively, and K' may be chosen suitably large.

7. Summing the $\mathfrak{I}_n$. Inserting the main term of (32) into (18) we obtain the Voronoï sum given in (6):

$$A\sqrt{\frac{2}{\pi q}}\sum_{n\le N}\frac{f_1(n)}{n^\gamma}\omega_n^{q\varrho+1/2}\Re(\Lambda e^{i(q\omega_n-\kappa\pi/2+\pi/4)})$$
$$= -C\sum_{n\le N}\frac{f_1(n)}{n^\gamma}(pxn)^{(\lambda-\kappa-1/2)/q}\cos(q(pxn)^{1/q}-(3/4+\kappa/2)\pi+\phi)$$
$$= -V_\kappa(x,N).$$

The contribution from the error terms $O(\omega^{-1/2})$ is

$$\sum_{n\le N} n^{\sigma_{\mathrm{g}}^1+\varepsilon-\gamma}\omega_n^{q\varrho}O(\omega_n^{-1/2}) \ll x^{\varrho-1/(2q)}\sum_{n\le N} n^{\sigma_{\mathrm{g}}^1-\gamma+\varrho-1/(2q)+\varepsilon}$$
$$\ll x^{\varrho-1/(2q)}\max(1, N^{\sigma_{\mathrm{g}}^1-\gamma+\varrho-1/(2q)+1+\varepsilon})$$
$$\ll x^{(\lambda-\kappa-3/2)/q} + (xN)^{(\lambda-\kappa-3/2)/q}N^{\sigma_{\mathrm{g}}^1+1-\gamma+\varepsilon}.$$

The $\omega^{-qK'}$ are negligible. If K' is large, then the following sum converges and we get

$$\sum_{n>N} n^{\sigma_{\mathrm{g}}^1+\varepsilon-\gamma}\omega_n^{q\varrho}O(\omega_n^{-qK'}) \ll x^{\cdots-K'}\sum_{n>N} n^{\cdots-K'} \ll x^{\cdots-K'}N^{\cdots-K'+1},$$

which is smaller than any negative power of xN.

Next we treat the sum with the P_n. Here it becomes necessary to fix our choice of β. We needed to have $\beta < \gamma - \sigma_{\mathrm{g}}^1 - 1$, so we take $\beta = \gamma - \sigma_{\mathrm{g}}^1 - 1 - 2\varepsilon$. We use (31) and Lemma 2 and find

$$\begin{aligned}(33)\quad \sum_n n^{\sigma_{\mathrm{g}}^1+\varepsilon-\gamma}\omega_n^{q\varrho}P_n &\ll \sum_n n^{\sigma_{\mathrm{g}}^1+\varepsilon-\gamma}\omega_n^{q\beta}T^{q(\varrho-\beta)}\left|\log\frac{n}{N+1/2}\right|^{-1}\\ &\ll x^\varrho N^{\varrho-\beta}\sum_n n^{-1-\varepsilon}\left|\log\frac{n}{N+1/2}\right|^{-1}\\ &\ll (xN)^\varrho N^{-\beta} = (xN)^{(\lambda-\kappa-1)/q}N^{\sigma_{\mathrm{g}}^1+1-\gamma+\varepsilon}.\end{aligned}$$

Only the Q_n are left. If $Q_n \neq 0$, then $(1-u)\omega \le T \le (1+u)\omega$. Thus $T \sim \omega$ and therefore $n \sim N$. If we put $\eta := \sqrt{\omega}v$ then

$$Q_n \ll \begin{cases} \sqrt{\omega_n}\displaystyle\int_{(T-\omega_n)/\sqrt{\omega_n}}^{\infty} e^{-q\eta^2}\,d\eta & \text{for } \omega_n < T \le (1+u_n)\omega_n,\\ \sqrt{\omega_n}\displaystyle\int_{-\infty}^{(T-\omega_n)/\sqrt{\omega_n}} e^{-q\eta^2}\,d\eta & \text{for } (1-u_n)\omega_n \le T < \omega_n.\end{cases}$$

Let us write

$$(34)\qquad A(\eta) := \begin{cases} \#\{n\in\mathbb{N} : 0 < (T-\omega_n)/\sqrt{\omega_n} \le \eta\} & \text{if } \eta > 0,\\ \#\{n\in\mathbb{N} : \eta \le (T-\omega_n)/\sqrt{\omega_n} < 0\} & \text{if } \eta < 0.\end{cases}$$

Then

$$\sum_{n\le N} Q_n \ll \sum_{n\le N}\sqrt{\omega_n}\int_{\eta\ge(T-\omega_n)/\sqrt{\omega_n}} e^{-q\eta^2}\,d\eta \ll \sqrt{T}\int_0^\infty e^{-q\eta^2}A(\eta)\,d\eta$$

and similarly for $n > N$. Therefore

$$(35)\qquad \sum_{n=1}^{\infty} Q_n \ll \sqrt{T}\int_{-\infty}^{\infty} e^{-q\eta^2}A(\eta)\,d\eta.$$

Let $\eta > 0$. Note that $|\eta| \le \sqrt{\omega}\,u \ll \sqrt{\log\omega} \ll \sqrt{\log T}$ and that $T \to \infty$. Then (34) implies $\omega < T$ and $\omega + \sqrt{\omega}\,\eta - T \ge 0$ or, after solving for $\sqrt{\omega}$,

$$\begin{gathered}\sqrt{\omega} \ge \sqrt{T+\frac{\eta^2}{4}} - \frac{\eta}{2} = \sqrt{T}\left(1+O\left(\frac{\eta^2}{T}\right)\right) - \frac{\eta}{2},\\ \sqrt{\frac{\omega}{T}} \ge 1 - \frac{\eta}{2\sqrt{T}} + O\left(\frac{\eta^2}{T}\right).\end{gathered}$$

Therefore $n \le N$ and

$$\frac{n}{N+1/2} = \left(\frac{\omega}{T}\right)^q \ge \left(1 - \frac{\eta}{2\sqrt{T}} + O\left(\frac{\eta^2}{T}\right)\right)^{2q}$$
$$\ge 1 - \frac{q\eta}{\sqrt{T}} + O\left(\frac{\eta^2}{T}\right) \ge 1 - \frac{2q\eta}{\sqrt{T}},$$
$$N \ge n \ge \left(N + \frac{1}{2}\right)\left(1 - \frac{2q\eta}{\sqrt{T}}\right) \ge N + \frac{1}{2} - \frac{3Nq\eta}{\sqrt{T}}.$$

Hence, using $[x-1/2]+1 \le 2x$, the number of solutions n is seen to be

$$A(\eta) \ll \frac{N}{\sqrt{T}}|\eta|,$$

and a similar computation shows that this also holds for $\eta < 0$. Together with (35) it implies that

$$\sum_{n=1}^{\infty} Q_n \ll N,$$

and, since $n \sim N$ if $Q_n \ne 0$, the terms Q_n contribute to (18) at most

$$\sum_n \frac{|f_1(n)|}{n^\gamma}(pxn)^\varrho Q_n \ll x^\varrho N^{\sigma_{\mathrm{g}}^1+\varepsilon-\gamma+\varrho+1} \ll (xN)^{(\lambda-\kappa-1)/q} N^{\sigma_{\mathrm{g}}^1+1-\gamma+\varepsilon}.$$

Thus $-\int_{\mathcal{C}_3+\overline{\mathcal{C}}_3}$ yields the Voronoï sum $V_\kappa(x,N)$ plus errors as given in (7).

8. End of the proof. There are still the errors from (14) and (16). Inserting $\alpha = \sigma_{\mathrm{g}} + 1 + \varepsilon$ and $\beta = \gamma - \sigma_{\mathrm{g}}^1 - 1 - \varepsilon$ and $T \asymp (xN)^{1/q}$ they are found to be of order no more than

$$x^{\sigma_{\mathrm{g}}+\varepsilon}(xN)^{-\kappa/q}, \qquad x^{\sigma_{\mathrm{g}}+1+\varepsilon}(xN)^{-(\kappa+1)/q},$$

and again $(xN)^\varrho N^{-\beta}$ as in (33). Finally $\int_{\mathcal{C}_4} \ll x^\varepsilon$ as said at the beginning. ∎

9. Proof of the Corollary. Evaluating both sides of $\mathcal{H}(\gamma - s) = \mathcal{H}(s)^{-1}$ through (4) shows $A^2|\Lambda|^2 = p^{-\gamma}$ and $2\lambda = q\gamma$.

Since $Z(\sigma + it) \ll 1$ as $|t| \to \infty$ if $\sigma > \sigma_{\mathrm{a}}$, formula (4) implies $\sigma_{\mathrm{a}} \ge \lambda/q$, which is $\gamma/2$ here. Therefore $\sigma_{\mathrm{g}} + 1 \ge \sigma_{\mathrm{a}} \ge \gamma/2$. ∎

To any who feels arithmetic,
sing'larities are truly aesthetic:
To the POLES if we travel,
all secrets unravel!
Oh, Poland! Oh, POLE-land prophetic!

Eduard Wirsing

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