ANNALES
POLONICI MATHEMATICI
LXXI.2 (1999)

Topology of families of affine plane curves

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Abstract. We determine bifurcation sets of families of affine curves and study the topology of such families.

1. Introduction. Let $f_\alpha(x,y)$ be a family of polynomials of two complex variables $(x,y) \in \mathbb{C}^2$ whose coefficients are polynomial functions of $\alpha \in \mathbb{C}^n$. We consider the family of affine curves $\{(x,y) \in \mathbb{C}^2 \mid f_\alpha(x,y) = 0\}$.

In this paper, we first determine the bifurcation set B_f, i.e., the smallest set of parameters α such that the family is equisingular outside this set. Then applying this result, we introduce the notions of semi-cycles vanishing at infinity and study the topology of the family. Finally, we show that our results imply some well-known facts on the topology of polynomial functions ([2], [3], [6], [7]).

2. Bifurcation set of families of affine plane curves. Let $f_\alpha(x,y) := P(x,y,\alpha)$, $\alpha \in \mathbb{C}^n$, be a family of polynomials of two variables whose coefficients are polynomials of α.

2.1. DEFINITION. The family of affine curves $\{(x,y) \in \mathbb{C}^2 \mid f_\alpha(x,y) = 0\}$, $\alpha \in \mathbb{C}^n$, is said to be *equisingular* outside a set $B \subset \mathbb{C}^n$ if for all $\alpha^0 \notin B$ there exist a neighborhood U_{α^0} of α^0 and a diffeomorphism h such that the diagram

$$\begin{tikzcd} \{(x,y) \mid f_{\alpha^0}(x,y) = 0\} \times U_{\alpha^0} \arrow[r, "h"] \arrow[d, "\pi"] & \{(x,y,\alpha) \mid f_\alpha(x,y) = 0\} \cap \pi^{-1}(U_{\alpha^0}) \arrow[d, "\pi"] \\ U_{\alpha^0} \arrow[r, "\mathrm{id}"] & U_{\alpha^0} \end{tikzcd}$$

is commutative, where π is the second projection.

1991 *Mathematics Subject Classification*: 32S55, 57M25, 57Q45.

Key words and phrases: affine curves, singularity, equisingularity, Milnor number.

Supported in part by the National Basic Research Program in Natural Sciences, Vietnam. The second author would like to thank the AUBONNE for financial support.

Let B_f be the smallest set of parameters α such that the family $\{f_\alpha(x,y)=0\}$ is equisingular ouside B_f. We call B_f the *bifurcation set* of the family.

2.2. The following assumptions will be needed throughout the paper:

- $\deg(f_\alpha) = \deg_y(f_\alpha) = d = \text{const}$;
- the curves $\{f_\alpha(x,y) = 0\}$ are all reduced.

Let l be the linear function defined by $l(x,y) = x$. By assumption, the restriction map

$$l_\alpha : f_\alpha^{-1}(0) \to \mathbb{C}, \qquad (x,y) \mapsto x,$$

is proper for each $\alpha \in \mathbb{C}^n$.

Let $\delta(x,\alpha) := \operatorname{disc}_y(f_\alpha(x,y))$ be the discriminant of f_α with respect to y. Then we may write

$$\delta(x,\alpha) = q_k(\alpha)x^k + q_{k-1}(\alpha)x^{k-1} + \dots$$

where $q_i(\alpha)$, $i = 0,\dots,k$, are polynomials of α. Put

$$B_\infty := \{\alpha \mid q_k(\alpha) = 0\}.$$

Denote by $C_f(f_\alpha)$ the set of critical values of f_α.

2.3. THEOREM. *Assume that $0 \notin C_f(f_\alpha)$ for generic α. Then the bifurcation set of the family of affine curves $\{f_\alpha(x,y) = 0\}$ is precisely the set*

$$B = \{\alpha \mid 0 \in C_f(f_\alpha)\} \cup B_\infty.$$

Proof. We first prove that the family of affine curves $\{f_\alpha(x,y) = 0\}$ is equisingular outside B. For each polynomial f_α, define

$$\operatorname{grad} f_\alpha := (\overline{\partial f_\alpha/\partial x, \partial f_\alpha/\partial y}).$$

Assume that $\alpha^0 \notin B$. Let $U_{\alpha^0} := \{\alpha \in \mathbb{C}^n \mid \|\alpha - \alpha^0\| < \delta\}$ so that $\mathbb{C}^n \setminus B$ contains the closure of U_{α^0}.

By the definition of B, $q_k(\alpha) \neq 0$ for all $\alpha \in U_{\alpha^0}$; hence there exists $c_0 > 0$ such that if $\alpha \in U_{\alpha^0}$, then $\delta(x,\alpha) \neq 0$ on the set $\{x \in \mathbb{C} \mid |x| \geq c_0\}$. By the properties of resultants, the system of equations

$$\begin{cases} f_\alpha(x,y) = 0, \\ \partial f_\alpha/\partial y = 0, \end{cases}$$

has no solution on the set $\{(x,y) \in \mathbb{C}^2 \mid |x| \geq c_0\}$ for any fixed $\alpha \in U_{\alpha^0}$. Thus

$$\text{(1)} \qquad \partial f_\alpha/\partial y \neq 0 \quad \text{for all } (x,y,\alpha) \in V_{\alpha^0} \cup \Omega,$$

where

$$V_{\alpha^0} := \{(x, y, \alpha) \in \mathbb{C}^2 \times \mathbb{C}^n \mid f_\alpha(x, y) = 0,\ \alpha \in U_{\alpha^0},\ |x| \geq c_0\}$$

and Ω is an open neighborhood of the set

$$\{(x, y, \alpha) \in \mathbb{C}^2 \times \mathbb{C}^n \mid f_\alpha(x, y) = 0,\ \alpha \in U_{\alpha^0},\ |x| = c_0\}.$$

On the other hand, by the definition of U_{α^0}, one has

$$\text{(2)} \qquad \operatorname{grad} f_\alpha \neq 0 \quad \text{for all } (x, y, \alpha) \in \mathbb{C}^2 \times U_{\alpha^0} \text{ with } f_\alpha(x, y) = 0,\ |x| \leq c_0.$$

From (1) and (2) we conclude that there exist smooth vector fields

$$\xi^j(x, y, \alpha) = (\xi_1^j(x, y, \alpha), \xi_2^j(x, y, \alpha)),$$
$$\eta^j(x, y, \alpha) = (\eta_1^j(x, y, \alpha), \eta_2^j(x, y, \alpha)), \qquad j = 1, \ldots, n,$$

such that

$$\text{(3)} \qquad \begin{cases} \langle \xi^j(x, y, \alpha), \operatorname{grad} f_\alpha(x, y)\rangle + \dfrac{\partial f_\alpha}{\partial \alpha_j}(x, y) = 0, \\ \langle \eta^j(x, y, \alpha), \operatorname{grad} f_\alpha(x, y)\rangle + \sqrt{-1}\,\dfrac{\partial f_\alpha}{\partial \alpha_j}(x, y) = 0, \end{cases}$$

on the set $X := \{(x, y, \alpha) \in \mathbb{C}^2 \times \mathbb{C}^n \mid f_\alpha(x, y) = 0,\ \alpha \in U_{\alpha^0}\}$, and

$$\text{(4)} \qquad \begin{cases} \xi_1^j(x, y, \alpha) = 0, \\ \eta_1^j(x, y, \alpha) = 0, \end{cases}$$

for all $(x, y, \alpha) \in V_{\alpha^0} \cup \Omega$. (We can construct such vector fields locally and then extend them over X by a smooth partition of unity.)

To shorten notation, we write $e^j = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in the jth place.

Let $\varphi^j(x^j, y^j, \alpha^j, \tau) := (\varphi_1^j(x^j, y^j, \alpha^j, \tau), \varphi_2^j(x^j, y^j, \alpha^j, \tau), \alpha(x^j, y^j, \alpha^j, \tau))$, $j = 1, \ldots, n$, be solutions of the system

$$\text{(5)} \qquad \begin{cases} \dfrac{dx(\tau)}{d\tau} = \xi_1^j(x(\tau), y(\tau), \alpha(\tau)), \\ \dfrac{dy(\tau)}{d\tau} = \xi_2^j(x(\tau), y(\tau), \alpha(\tau)), \\ \dfrac{d\alpha(\tau)}{d\tau} = e^j, \\ x(0) = x^j, \quad y(0) = y^j, \quad \alpha(0) = \alpha^j, \end{cases}$$

and $\psi^j(u^j, v^j, \beta^j, \tau) := (\psi_1^j(u^j, v^j, \beta^j, \tau), \psi_2^j(u^j, v^j, \beta^j, \tau), \beta(u^j, v^j, \beta^j, \tau))$, $j = 1, \ldots, n$, be solutions of the system

$$(6)\qquad \begin{cases} \dfrac{du(\tau)}{d\tau} = \eta_1^j(u(\tau), v(\tau), \beta(\tau)), \\ \dfrac{dv(\tau)}{d\tau} = \eta_2^j(u(\tau), v(\tau), \beta(\tau)), \\ \dfrac{d\beta(\tau)}{d\tau} = \sqrt{-1}e^j, \\ u(0) = u^j, \quad v(0) = v^j, \quad \beta(0) = \beta^j, \end{cases}$$

where $(x^j, y^j, \alpha^j), (u^j, v^j, \beta^j) \in X$. We conclude from (3) and (5) that

$$\begin{aligned} \frac{d}{d\tau} P(\varphi^j(x^j, y^j, \alpha^j, \tau)) &= \frac{\partial P}{\partial x}\frac{\partial \varphi_1^j}{\partial \tau} + \frac{\partial P}{\partial y}\frac{\partial \varphi_2^j}{\partial \tau} + \sum_{k=1}^{n} \frac{\partial P}{\partial \alpha_k}\frac{\partial \alpha_k}{\partial \tau} \\ &= \frac{\partial P}{\partial x}\frac{\partial \varphi_1^j}{\partial \tau} + \frac{\partial P}{\partial y}\frac{\partial \varphi_2^j}{\partial \tau} + \frac{\partial P}{\partial \alpha_j}\frac{\partial \alpha_j}{\partial \tau} = 0, \end{aligned}$$

hence that

$$P(\varphi_1^j(x^j, y^j, \alpha^j, \tau), \varphi_2^j(x^j, y^j, \alpha^j, \tau), e^j\tau + \alpha^j) = 0.$$

Further, it follows from (4) and (5) that

$$\begin{aligned} \frac{d}{d\tau}|\varphi_1^j(x^j, y^j, \alpha^j, \tau)|^2 &= \frac{d}{d\tau}\langle \varphi_1^j(x^j, y^j, \alpha^j, \tau), \varphi_1^j(x^j, y^j, \alpha^j, \tau)\rangle \\ &= \left\langle \frac{d\varphi_1^j(x^j, y^j, \alpha^j, \tau)}{d\tau}, \varphi_1^j(x^j, y^j, \alpha^j, \tau) \right\rangle \\ &\quad + \left\langle \varphi_1^j(x^j, y^j, \alpha^j, \tau), \frac{d\varphi_1^j(x^j, y^j, \alpha^j, \tau)}{d\tau} \right\rangle \\ &= 0, \end{aligned}$$

hence

$$(7)\qquad |\varphi_1^j(x^j, y^j, \alpha^j, \tau)| = \text{const} \quad \text{for each } (x^j, y^j, \alpha^j) \in V_{\alpha^0}.$$

Analogously,

$$P(\psi_1^j(u^j, v^j, \beta^j, \tau), \psi_2^j(u^j, v^j, \beta^j, \tau), \sqrt{-1}\, e^j\tau + \beta^j) = 0;$$

moreover,

$$(8)\qquad |\psi_1^j(u^j, v^j, \beta^j, \tau)| = \text{const} \quad \text{for each } (u^j, v^j, \beta^j) \in V_{\alpha^0}.$$

Since $\deg(f_\alpha) = \deg_y(f_\alpha) = d = \text{const}$, the restriction map

$$l_{U_{\alpha^0}} : \{(x, y) \mid f_\alpha(x, y) = 0,\ \alpha \in U_{\alpha^0}\} \to \mathbb{C}, \quad (x, y) \mapsto x,$$

is proper. In addition, it is clear from (7) and (8) that the solutions $\varphi^j(x^j, y^j, \alpha^j, \tau)$ and $\psi^j(u^j, v^j, \beta^j, \tau)$, $j = 1, \dots, n$, can be extended over their maximal intervals.

Now define

$$h:\{(x,y)\mid f_{\alpha^0}(x,y)=0\}\times U_{\alpha^0}\to\{(x,y,\alpha)\mid f_\alpha(x,y)=0\}\cap\pi^{-1}(U_{\alpha^0})$$

by

$$h(x,y,\alpha)=\psi^n(\varphi^n(\dots(\psi^1(\varphi^1(x,y,\alpha^0,\operatorname{Re}\alpha_1-\operatorname{Re}\alpha_1^0),\operatorname{Im}\alpha_1-\operatorname{Im}\alpha_1^0),\dots,\\ \operatorname{Re}\alpha_n-\operatorname{Re}\alpha_n^0),\operatorname{Im}\alpha_n-\operatorname{Im}\alpha_n^0)$$

for $(x,y)\in f_{\alpha^0}^{-1}(0)$ and $\alpha\in U_{\alpha^0}$. We can easily check that the map h is a diffeomorphism, and that $\pi\circ h(x,y,\alpha)=\pi(x,y,\alpha)=\alpha$. This gives a trivialization of the fibration over the set U_{α^0}.

We next prove that the set B is smallest. By contradiction, assume that for some $\alpha^0\in B$ there exist a neighborhood U_{α^0} of α^0 and a diffeomorphism

$$h:\{(x,y)\mid f_{\alpha^0}(x,y)=0\}\times U_{\alpha^0}\to\{(x,y,\alpha)\mid f_\alpha(x,y)=0\}\cap\pi^{-1}(U_{\alpha^0}).$$

Consider the following two cases:

Case 1: $0\in C_f(f_{\alpha^0})$. Since h is a diffeomorphism of $\{(x,y)\mid f_{\alpha^0}(x,y)=0\}\times U_{\alpha^0}$ onto $\{(x,y,\alpha)\mid f_\alpha(x,y)=0\}\cap\pi^{-1}(U_{\alpha^0})$, it follows that $0\in C_f(f_\alpha)$ for all $\alpha\in U_{\alpha^0}$, a contradiction.

Case 2: $\alpha^0\in B_\infty$. In this case, the next lemma is needed.

2.4. Lemma. *Let F be a polynomial of two complex variables such that the restriction map $l|_V, V:=F^{-1}(0)$, is proper where*

$$l:\mathbb{C}^2\to\mathbb{C},\qquad (x,y)\mapsto x.$$

Suppose that the curve V is reduced. Then

$$\chi(F^{-1}(0))=d-\deg\operatorname{disc}_y F(x,y).$$

Proof. Let x_β, $\beta=1,\dots,p$, be the critical values of $l|_V$ and (x_β,y_β^j), $j=1,\dots,i_\beta$, be the corresponding critical points of $l|_V$ with multiplicity l_β^j. We may write

$$\operatorname{disc}_y F(x,y)=a(x-x_1)^{\gamma_1}\cdots(x-x_p)^{\gamma_p},$$

where $a\neq0$ and $\gamma_\beta=\sum_{j=1}^{i_\beta}l_\beta^j$, $\beta=1,\dots,p$.

Take $x_*\in\mathbb{C}\setminus\{x_1,\dots,x_p\}$. Then $l|_V^{-1}(x_*)$ consists of $d:=\deg(F)$ distinct points. In the x-plane, we consider a system of paths $T_1,\dots,T_p$ connecting $x_1,\dots,x_p$ to x_* such that

(i) no path T_j has self-intersection points;
(ii) $T_i\cap T_j=\{x_*\}$ $(i\neq j)$.

Put

$$\widehat{\mathcal{S}}(V):=l|_V^{-1}\Big(\bigcup_{i=1}^p T_i\Big).$$

Then $\widehat{\mathcal{S}}(V)$ is a union of 1-dimensional curves. Let $\check{\mathcal{S}}(V)$ be the set of all curves in $\widehat{\mathcal{S}}(V) \setminus l|_V^{-1}(x_*)$ which contain a point of $\Sigma := \{(x_\beta, y_\beta^j) \mid j = 1, \ldots, i_\beta,\ \beta = 1, \ldots, p\}$.

Since $\bigcup_{i=1}^p T_i$ is a deformation retract of $\mathbb{C}$ and the restriction map

$$l|_V : F^{-1}(0) \setminus \Sigma \to \mathbb{C} \setminus \{x_1, \ldots, x_p\}, \qquad (x, y) \mapsto x,$$

is a locally trivial fibration, it follows that $\widehat{\mathcal{S}}(V)$ is a deformation retract of V.

It is not hard to see that

$$\mathcal{S}(V) := \check{\mathcal{S}}(V) \cup l|_V^{-1}(x_*)$$

is a deformation retract of $\widehat{\mathcal{S}}(V)$ and also of $V = F^{-1}(0)$. (The set $\mathcal{S}(V)$ is called the *skeleton* of the curve V ([2]).) Hence,

$$\chi(F^{-1}(0)) = \chi(\mathcal{S}(V)).$$

The set $\mathcal{S}(V)$ can be identified with a 1-dimensional graph of $d + \sum_{\beta=1}^p i_\beta$ vertices and $\sum_{\beta=1}^p \sum_{j=1}^{i_\beta} (l_\beta^j + 1)$ edges. Thus,

$$\begin{aligned}
\chi(\mathcal{S}(V)) &= \Big(d + \sum_{\beta=1}^p i_\beta\Big) - \sum_{\beta=1}^p \sum_{j=1}^{i_\beta} (l_\beta^j + 1) \\
&= d - \sum_{\beta=1}^p \sum_{j=1}^{i_\beta} l_\beta^j = d - \sum_{\beta=1}^p \gamma_\beta \\
&= d - \deg \operatorname{disc}_y F(x, y).
\end{aligned}$$

We now return to Case 2.

For each $\alpha \in \mathbb{C}^n$, let $k(\alpha) := \max\{j \in \{0, \ldots, k\} \mid q_j(\alpha) \neq 0\}$. By Lemma 2.4, we conclude that $\chi(f_\alpha^{-1}(0)) = d - k(\alpha)$. Since $\alpha^0 \in B_\infty = q_k^{-1}(0)$ and $\alpha \notin B_\infty$ for generic α, one gets

$$\chi(f_\alpha^{-1}(0)) = d - k < \chi(f_{\alpha^0}^{-1}(0)),$$

a contradiction, which ends the proof of Theorem 2.3. ∎

2.5. Remark. From the construction of B_∞, it is reasonable to call each $\alpha^0 \in B_\infty$ a *bifurcation value* corresponding to the singularity at infinity of the family $\{f_\alpha(x, y) = 0\}$.

3. Topology of families of affine plane curves. From now on, we assume that the curves $\{f_\alpha = 0\}$ are smooth for generic α.

Consider the family of affine curves $\{f_\alpha = 0\}$. By Lemma 2.4, $\chi(f_\alpha^{-1}(0)) = d - k(\alpha)$. On the other hand, by the definition of the set B_∞, we see that $\alpha^0 \in B_\infty$ iff $k(\alpha^0) < k = k(\alpha)$ for generic α. Hence $\chi(f_\alpha^{-1}(0)) < \chi(f_{\alpha^0}^{-1}(0))$ for generic α.

Thus we can reformulate Theorem 2.3 as follows.

3.1. Theorem. *$\alpha^0 \in \mathbb{C}^n$ is a bifurcation value of the family of affine curves $\{f_\alpha(x,y)=0\}$ if and only if either*

(i) *the curve $\{(x,y) \mid f_{\alpha^0}(x,y)=0\}$ is singular*, *or*

(ii) *$\chi(f_\alpha^{-1}(0)) < \chi(f_{\alpha^0}^{-1}(0))$ for generic α.*

3.2. Remark. In the general case, it is not sufficient to use the group $H_1(f_\alpha^{-1}(0))$ to distinguish the generic curve from the special curve. For example, consider the family of polynomials $f_\alpha(x,y) = y^3 + xy^2 + y - \alpha$, $\alpha \in \mathbb{C}$. Then $B_f = \{0\}$ and $\operatorname{rank} H_1(f_\alpha^{-1}(0)) = \operatorname{rank} H_1(f_0^{-1}(0)) = 1$.

There is no loss of generality in assuming that the map $l_\alpha = x$ is simple for each α near a given α^0 (l_α is said to be *simple* iff $l_\alpha^{-1}(x)$ consists of $d-1$ distinguished points for every critical value x of l_α). Then the number of singular points of l_α is exactly $k(\alpha)$. Let

$$(x_1(\alpha), y_1(\alpha)), \ldots, (x_k(\alpha), y_k(\alpha))$$

be the critical points of the map l_α, $\alpha \notin B_\infty$. Now we use the notations as in the proof of Lemma 2.4. Suppose that $x_* \in \mathbb{C}$ is a common regular value of l_{α^0} and l_α for all α near α^0 and let $e_j(\alpha) := l_\alpha^{-1}(T_j) \cap S(f_\alpha^{-1}(0))$, $j = 1, \ldots, k$, be the cycles of the group $H_1(f_\alpha^{-1}(0), l_\alpha^{-1}(x_*))$ corresponding to the singular points $(x_j(\alpha), y_j(\alpha))$. These elements define a basis of the group $H_1(f_\alpha^{-1}(0), l_\alpha^{-1}(x_*))$. By definition, $\alpha^0 \in B_\infty$ iff there exist critical points $(x_j(\alpha), y_j(\alpha))$ of l_α such that $\|(x_j(\alpha), y_j(\alpha))\| \to \infty$ as $\alpha \to \alpha^0$. Moreover, since the map l_α is simple, the number of singular points of l_α tending to infinity as $\alpha \to \alpha^0$ is $r := k - k(\alpha^0)$. Therefore, we may assume without loss of generality that such critical points are

$$(x_1(\alpha), y_1(\alpha)), \ldots, (x_r(\alpha), y_r(\alpha)).$$

3.3. Definition. We call $e_j(\alpha)$, $j = 1, \ldots, r$, the *semi-cycles vanishing at infinity* as $\alpha \to \alpha^0$.

From Theorem 3.1 and the above definition we easily obtain the following.

3.4. Theorem. *α^0 is a bifurcation value of the family of affine curves $\{f_\alpha(x,y)=0\}$ if and only if either*

(i) *the curve $\{(x,y) \mid f_{\alpha^0}(x,y)=0\}$ is singular*, *or*

(ii) *there exist semi-cycles $e_j(\alpha)$ vanishing at infinity as $\alpha \to \alpha^0$. The number of such semi-cycles is exactly $k - k(\alpha^0)$.*

3.5. Remark. The number of semi-cycles $e_j(\alpha)$ vanishing at infinity can be given in other ways as follows.

(i) Let (F, G) be the intersection number of two curves $\{F = 0\}$ and $\{G = 0\}$. Then

$$\begin{aligned}\chi(f_{\alpha^0}^{-1}(0)) - \chi(f_\alpha^{-1}(0)) &= k - k(\alpha^0)\\ &= (f_\alpha, \partial f_\alpha/\partial y) - (f_{\alpha^0}, \partial f_{\alpha^0}/\partial y)\\ &= \dim_{\mathbb{C}} \mathbb{C}[x,y]/(f_\alpha, \partial f_\alpha/\partial y)\\ &\quad - \dim_{\mathbb{C}} \mathbb{C}[x,y]/(f_{\alpha^0}, \partial f_{\alpha^0}/\partial y).\end{aligned}$$

(ii) In $\mathbb{CP}^2$ we consider the family of curves

$$\bar{\varGamma}_\alpha := \{(x : y : z) \mid z^d f_\alpha\big(x/z, y/z\big) = 0\}.$$

Clearly, $\bar{\varGamma}_\alpha$ is the compactification of $\varGamma_\alpha := f_\alpha^{-1}(0)$. In addition, we assume that the homogeneous part of degree d of f_α does not depend on α. Then the curves $\bar{\varGamma}_\alpha$ intersect the line $z = 0$ at the same points $A_1, \dots, A_s$ for any α. Let $\mu_{A_i}(\bar{\varGamma}_\alpha)$ be the Milnor number of the germ of the analytic curve $\bar{\varGamma}_\alpha$ at A_i. By arguments of [3], we can show that

$$\begin{aligned}k - k(\alpha^0) &= \chi(f_{\alpha^0}^{-1}(0)) - \chi(f_\alpha^{-1}(0))\\ &= \sum_{i=1}^{s}[\mu_{A_i}(\bar{\varGamma}_{\alpha^0}) - \mu_{A_i}(\bar{\varGamma}_\alpha)].\end{aligned}$$

Next, we describe the change in the homotopy type of the curve $\{f_\alpha(x, y) = 0\}$ as $\alpha \to \alpha^0$, $\alpha^0 \in B_\infty$.

3.6. Definition. An operation of *attaching a* 1*-dimensional cell* to an affine curve V is a map

$$\varphi : I := [0, 1] \to \mathbb{C}^2$$

such that

(i) $\varphi(I)$ is diffeomorphic to I;
(ii) $\varphi(I) \cap V = \{\varphi(0), \varphi(1)\}$.

The set $V' = V \cup \varphi(I)$ is called V *with a* 1*-cell attached.*

3.7. Theorem. *Let α^0 be a bifurcation value corresponding to the singularity at infinity of the family $\{f_\alpha(x, y) = 0\}$ such that the curve $\{f_{\alpha^0}(x, y) = 0\}$ is smooth. Then a generic curve $f_\alpha^{-1}(0)$ may be obtained from $f_{\alpha^0}^{-1}(0)$, up to homotopy type, by attaching exactly $k - k(\alpha^0)$ 1-dimensional cells.*

Proof. Let $\mathcal{S}(f_\alpha^{-1}(0))$ (resp. $\mathcal{S}(f_{\alpha^0}^{-1}(0))$) be the skeleton of the affine plane curve $\{f_\alpha(x, y) = 0\}$ (resp. $\{f_{\alpha^0}(x, y) = 0\}$) as in the proof of Lemma 2.4. According to the construction of skeletons, the set $\mathcal{S}(f_\alpha^{-1}(0))$ (resp. $\mathcal{S}(f_{\alpha^0}^{-1}(0))$) is a graph with $d + k$ (resp. $d + k(\alpha^0)$) vertices and $2k$ (resp. $2k(\alpha^0)$) edges.

Furthermore, $\mathcal{S}(f_{\alpha^0}^{-1}(0))$ is obtained from $\mathcal{S}(f_\alpha^{-1}(0))$ by deleting $k-k(\alpha^0)$ vertices $(x_\beta(\alpha), y_\beta(\alpha))$, $\beta=1,\ldots,r$, and $k-k(\alpha^0)$ pairs of edges. These pairs of edges connect a point $(x_\beta(\alpha), y_\beta(\alpha))$ tending to infinity to two distinct points in $l_\alpha^{-1}(x_*)$. In other words, the set $\mathcal{S}(f_\alpha^{-1}(0))$ is $\mathcal{S}(f_{\alpha^0}^{-1}(0))$ with $k-k(\alpha^0)$ 1-cells attached.

On the other hand, the graph $\mathcal{S}(f_\alpha^{-1}(0))$ (resp. $\mathcal{S}(f_{\alpha_0}^{-1}(0))$) is a deformation retract of the curve $f_\alpha^{-1}(0)$ (resp. $f_{\alpha_0}^{-1}(0)$). This proves the theorem. ∎

3.8. Corollary. *If α^0 is a bifurcation value at infinity of the family of curves $\{f_\alpha = 0\}$, then the number of connected components of the curve $f_{\alpha^0}^{-1}(0)$ is greater than or equal to the one for $f_\alpha^{-1}(0)$.*

Moreover, we can describe a change mechanism of the number of connected components when passing from the general curves $f_\alpha^{-1}(0)$ to the special curve $f_{\alpha^0}^{-1}(0)$. For that, we need:

3.9. Definition. A subgraph $B(\alpha)$ of the graph $S(f_\alpha^{-1}(0))$ is said to be a *block vanishing at infinity* as $\alpha \to \alpha^0$ if the following four conditions are satisfied.

(i) $B(\alpha)$ is connected;

(ii) each vertex of $B(\alpha)$ either belongs to $l_\alpha^{-1}(x_*)$ or tends to infinity as $\alpha \to \alpha^0$;

(iii) the number of connected components of $S(f_\alpha^{-1}(0))$ is different from that of $S(f_\alpha^{-1}(0)) \setminus B(\alpha)$;

(iv) $B(\alpha)$ is minimal in the sense that there exists no subgraph $B'(\alpha) \supsetneq B(\alpha)$ of $S(f_\alpha^{-1}(0))$ satisfying (i)–(iii).

Let $v(\alpha^0)$ be the number of blocks vanishing at infinity as $\alpha \to \alpha^0$, and let $b_0(\alpha)$ and $b_0(\alpha^0)$ be the numbers of connected components of $f_\alpha^{-1}(0)$ and $f_{\alpha^0}^{-1}(0)$, respectively. By Theorem 3.7, we obtain the following.

3.10. Theorem. $b_0(\alpha^0) - b_0(\alpha) = v(\alpha^0)$.

4. Corollaries

4.1. We begin by recalling some facts on the topology of polynomials of two variables.

Let $F : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial function. It is well known that there exists a finite set $C(F) \subset \mathbb{C}$, called the *bifurcation set* of F, such that the restriction

$$F : \mathbb{C}^2 \setminus F^{-1}(C(F)) \to \mathbb{C} \setminus C(F)$$

is a locally trivial C^∞-fibration (see, for example, [5], [6], [7], [3]).

We say that a value $t_0 \in \mathbb{C}$ is *regular at infinity* if there exist a small $\delta > 0$ and a compact $K \subset \mathbb{C}^2$ such that the restriction

$$F : F^{-1}(D_\delta) \setminus K \to D_\delta, \qquad D_\delta := \{t \mid |t - t_0| < \delta\},$$

is a trivial C^∞-fibration ([4]).

If t_0 is not regular at infinity, it is called a *critical value at infinity* of F. Denote by $C_\infty(F)$ the set of critical values at infinity of F. It is known ([3]) that $C(F) = C_f(F) \cup C_\infty(F)$.

Let F be a polynomial with isolated critical points only. Denote by $\mu^c(F)$ the fibre Milnor number of F at c.

Let $f_\alpha(x, y) := F(x, y) - \alpha$, $\alpha \in \mathbb{C}$. Using the notation of Remark 3.5, we put

$$\lambda^{\alpha^0}(F) := \sum_{i=1}^{s} [\mu_{A_i}(\bar{\Gamma}_\alpha) - \mu_{A_i}(\bar{\Gamma}_{\alpha^0})],$$

for $\alpha^0 \in \mathbb{C}$ and generic α.

Now, let us mention an important consequence of the above results.

4.2. Corollary. *The following statements are equivalent.*

(i) $\alpha^0 \in B_f$;
(ii) $0 \in C(f_{\alpha^0})$;
(iii) $\mu^0(f_{\alpha^0}) + \lambda^0(f_{\alpha^0}) > 0$.

Proof. We first show that

$$B_\infty = \{\alpha \mid 0 \in C_\infty(f_\alpha)\}. \tag{9}$$

In fact, let $\Delta(x, \alpha, t) := \operatorname{disc}_y(f_\alpha(x, y) - t)$. Then we may write

$$\Delta(x, \alpha, t) = Q_{m(\alpha)}(\alpha, t)x^{m(\alpha)} + Q_{m(\alpha)-1}(\alpha, t)x^{m(\alpha)-1} + \dots$$

According to [1],

$$C_\infty(f_\alpha) = \{t \in \mathbb{C} \mid Q_{m(\alpha)}(\alpha, t) = 0\}.$$

On the other hand, since $0 \notin C_\infty(f_\alpha)$ for generic α,

$$Q_{m(\alpha)}(\alpha, 0) \not\equiv 0.$$

Furthermore, because $\delta(x, \alpha) = \Delta(x, \alpha, 0)$, we have

$$Q_{m(\alpha)}(\alpha, 0) \equiv q_k(\alpha) \quad \text{and} \quad m(\alpha) \equiv k.$$

Therefore,

$$B_\infty = \{\alpha \mid q_k(\alpha) = 0\} = \{\alpha \mid Q_{m(\alpha)}(\alpha, 0) = 0\} = \{\alpha \mid 0 \in C_\infty(f_\alpha)\}.$$

We now prove the theorem. By [1], $t^0 \in C_\infty(f_\alpha)$ iff

$$\lambda^{t^0}(f_\alpha) > 0. \tag{10}$$

By (9), (10) and the definition of B_f, one has $B_f = \{\alpha \mid 0 \in C(f_\alpha)\}$, from which the assertion easily follows. ∎

In a special case, the following corollary is well known.

4.3. COROLLARY ([3], [6]). *Suppose that $F \in \mathbb{C}[x, y]$ is a polynomial of two complex variables. Then $\alpha^0 \in C(F)$ if and only if either*

(i) *α^0 is a singular value of F, or*
(ii) *$\chi(F^{-1}(\alpha)) < \chi(F^{-1}(\alpha^0))$ for generic α.*

Proof. Let $f_\alpha(x, y) := F(x, y) - \alpha$, $\alpha \in \mathbb{C}$. Then the conclusion follows from Theorem 3.1 and Corollary 4.2. ∎

4.4. REMARK. Let $f_\alpha(x, y) = F(x, y) - \alpha$. Then the results of §3 also give us the corresponding results of [2] on the semi-cycles vanishing at infinity and on the construction of the homotopy type of the generic fiber for a global Milnor fibration.

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Reçu par la Rédaction le 17.11.1997
Révisé le 11.5.1998