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Non-zero constant Jacobian polynomial maps of $\mathbb{C}^2$

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Abstract. We study the behavior at infinity of non-zero constant Jacobian polynomial maps $f = (P, Q)$ in $\mathbb{C}^2$ by analyzing the influence of the Jacobian condition on the structure of Newton–Puiseux expansions of branches at infinity of level sets of the components. One of the results obtained states that the Jacobian conjecture in $\mathbb{C}^2$ is true if the Jacobian condition ensures that the restriction of Q to the curve $P = 0$ has only one pole.

1. Introduction. Let $f = (P, Q)$ be a polynomial mapping of $\mathbb{C}^2$ into itself, $P, Q \in \mathbb{C}[x, y]$, and denote by $J(P, Q) := P_x Q_y - P_y Q_x$ the Jacobian of f. The *Jacobian conjecture* in $\mathbb{C}^2$ (JC_2), first posed by Keller [K] in 1939 and still open, asserts that *a polynomial map $f = (P, Q)$ is an automorphism of $\mathbb{C}^2$ if $J(P, Q) \equiv \mathrm{const} \neq 0$.* We refer the readers to [BCW] and [D2] for the history of the conjecture and related topics, and to [Ka], [H], [LW], [O1–O3], [P] and [St] for some recent partial results on (JC_2).

In this paper we study the behavior at infinity of polynomial maps $f = (P, Q)$ of $\mathbb{C}^2$ satisfying the *Jacobian condition* $J(P, Q) \equiv \mathrm{const} \neq 0$. First we try to analyse the influence of the Jacobian condition on the structure of Newton–Puiseux expansions of branches at infinity of level sets of P and Q. Then, by applying standard results on the topology of plane curves we give some estimates on the geometric degree and branched value set of f and the topology of the generic fibers of the components P and Q.

Denote by $\deg_{\mathrm{geo}} f := \max\{\# f^{-1}(a) : a \in \mathbb{C}^2\}$ the geometric degree of f, and by χ_P and χ_Q the Euler–Poincaré characteristics of the generic fibers of P and Q, respectively.

Our main results can be stated as follows.

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THEOREM A. *The Jacobian conjecture in* $\mathbb{C}^2$ *is equivalent to the following statement*:

(∗) *If* $J(P,Q) \equiv \mathrm{const} \neq 0$, *then the restriction of* Q *to the curve* $P=0$ *has only one pole*, *that is*, *there is only one irreducible branch at infinity of the curve* $P=0$ *such that* Q *tends to infinity along this branch.*

THEOREM B. *Let* $f=(P,Q)$ *be a non-zero constant Jacobian polynomial map of* $\mathbb{C}^2$ *with* $\deg P = kd \geq \deg Q = ke$, $\gcd(d,e)=1$. *Then either* $e=1$ *or*

$$\deg_{\mathrm{geo}} f = rd + se \geq \min\{2e, d\},$$

where $r \geq 0$, $s \geq 0$ *and* $r+s \geq 1$.

THEOREM C. *For every non-zero constant Jacobian polynomial map* $f = (P,Q)$ *of* $\mathbb{C}^2$,

$$\deg P(\chi_Q - \deg_{\mathrm{geo}} f) = \deg Q(\chi_P - \deg_{\mathrm{geo}} f).$$

In particular, *if* $\deg P \neq \deg Q$ *then* f *is an automorphism if* $\chi_P = \chi_Q$.

In order to prove Theorem A we show that the statement (∗) is equivalent to the implication

$$J(P,Q) \equiv \mathrm{const} \neq 0 \Rightarrow \deg P \mid \deg Q \text{ or } \deg Q \mid \deg P.$$

So, Theorems A and B each enable us to recover Jung's theorem [J] on the tameness of automorphisms of $\mathbb{C}^2$ (see also [CK] and [Kul]). In view of Theorem C the conjecture (JC_2) is equivalent to the implication

$$J(P,Q) \equiv \mathrm{const} \neq 0 \Rightarrow \chi_P = \chi_Q.$$

Indeed, under the Jacobian condition the generic fibers of the non-constant polynomials $aP+bQ$ with $\deg(aP+bQ) = \max(\deg P, \deg Q)$ have the same topological type (g,n) with $n \neq 2,3,4,5$ (Theorem 4.8). Moveover, the number of punctures of the generic fiber of $aP+bQ$ with $\deg(aP+bQ) < \max(\deg P, \deg Q)$ must be different from 2 and 3 (Theorem 4.6). The last statement is an improvement of earlier results due to Abhyankar [A] and Drużkowski [D1]. In particular, we show that if f is not an automorphism, then the branched value set of f must be composed of the images of some polynomial maps $(p_i(\xi), q_i(\xi))$ with $\deg p_i / \deg q_i = \deg P/\deg Q$ (Theorem 4.4). These results and Theorem B could be useful to check (JC_2) in special cases.

The proofs of these results, presented in Sections 3 and 4, are based on Main Lemma (Lemma 3.3) and its corollary (Theorem 3.6). In these we describe the influence of the Jacobian condition on the structure of Newton–Puiseux types of the polynomials P and Q. At this stage the polynomials P and Q are only considered as polynomials in y with coefficients meromorphic in x, but not really as polynomials in x and y. The notion of Newton–Puiseux type of a polynomial together with its properties are introduced in Section 2.

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2. Preliminaries. We work with the complex plane $\mathbb{C}^2$ endowed with affine coordinates (x, y). A polynomial $g(x, y)$ is *monic in* y if $g(x, y) = y^{\deg g} +$ lower order terms in y.

2.1. *Newton–Puiseux expansion at infinity.* Let $g(x, y)$ be a non-constant polynomial in $\mathbb{C}^2$, monic in y. Let $B_g(c)$ denote the collection of all irreducible branches at infinity of the curve $g = c$. Let $\gamma \in B_g(c)$. Since g is monic in y, γ intersects the line at infinity $z = 0$ of the compactification $\mathbb{CP}^2$ of $\mathbb{C}^2$ at a point $(1 : a : 0)$. Newton's algorithm allows us to find a *Newton–Puiseux expansion* of γ, a fractional power series

$$y(z) = \sum_{k=0}^{\infty} a_k z^{k/m}, \qquad \gcd(\{k : a_k \neq 0\} \cup \{m\}) = 1,\ a_0 = a, \tag{2.1}$$

for which the map $\tau \mapsto (1 : y(\tau^m) : \tau^m)$ gives a holomorphic parametrization of γ for τ small enough (see [BK]). Translating the series $y(z)$ into the standard coordinates (x, y) of $\mathbb{C}^2$, we obtain a fractional power series

$$u(x) = xy(x^{-1/m}) = x\sum_{k=0}^{\infty} a_k x^{-k/m}, \qquad \gcd(\{k : a_k \neq 0\} \cup \{m\}) = 1, \tag{2.2}$$

for which $g(x, u(x)) \equiv c$ and the map $t \mapsto (t^m, u(t^m))$ is a meromorphic parametrization of γ for t large enough. Such a series $u(x)$ is called a *Newton–Puiseux expansion at infinity* of γ. We call the natural number $\operatorname{mult}(u) := m$ the *multiplicity* of the series $u(x)$. The *equivalence class* of u, denoted by $[u]$ or $[\gamma]$, consists of m distinct Newton–Puiseux expansions at infinity $u_\nu(x)$ of the branch γ,

$$\begin{aligned} u_\nu(x) &:= xy(\varepsilon^\nu x^{1/m}) \\ &= x\sum_{k=0}^{\infty} a_k \varepsilon^{\nu k} x^{-k/m}, \qquad \nu = 0, 1, \ldots, m-1, \end{aligned} \tag{2.3}$$

where ε is a primitive mth root of 1, $\varepsilon = \exp(2\pi i/m)$. The branch γ then becomes the zero-set of the meromorphic function $\prod_{\nu=0}^{m-1}(y - u_\nu(x))$ and $\operatorname{mult}(u)$ is the intersection number of γ and the line at infinity.

The most important facts on Newton–Puiseux expansions at infinity of a plane curve are the following:

1) (Newton's Theorem, see [A]) Every reduced polynomial $g(x, y) \in \mathbb{C}[x, y]$ monic in y can be factorized with respect to Newton–Puiseux ex-

pansions at infinity of the curve $g = 0$, i.e.

$$g(x, y) = \prod_{\gamma \in B_g(0)} \prod_{u \in [\gamma]} (y - u(x)). \tag{2.4}$$

2) The Euler–Poincaré characteristic of a smooth reduced plane curve (and hence, the topological type of a smooth irreducible plane curve) can be completely determined by the data of its Newton–Puiseux expansions at infinity (see [BK]).

2.2. *Newton–Puiseux types of a polynomial.* By a *π-series* we mean a finite fractional power series of the form

$$\begin{gathered} \varphi(x, \xi) = \sum_{k=0}^{n_\varphi - 1} a_k x^{1-k/m_\varphi} + \xi x^{1-n_\varphi/m_\varphi}, \\ \gcd(\{n_\varphi, m_\varphi\} \cup \{k : a_k \neq 0\}) = 1, \end{gathered} \tag{2.5}$$

where ξ is a complex parameter. For such a π-series φ we put $\operatorname{mult}(\varphi) := m_\varphi$ and $\operatorname{ind}(\varphi) := i_\varphi := m_\varphi / \operatorname{mult}(\varphi(x, 0))$. Note that by definition

$$\operatorname{mult}(\varphi(x, 0)) = m_\varphi i_\varphi^{-1} \quad \text{and} \quad \gcd(n_\varphi, i_\varphi) = 1. \tag{2.6}$$

Two π-series $\varphi(x, \xi)$ and $\psi(x, \xi)$ are *equivalent* if $m_\varphi = m_\psi$, $n_\varphi = n_\psi$ and $\psi(x, 0) \in [\varphi(x, 0)]$. We denote by $[\varphi]$ the equivalence class of a π-series φ. It consists of $m_\varphi i_\varphi^{-1}$ distinct π-series,

$$\begin{gathered} \varphi_\nu(x, \xi) = \sum_{k < n_\varphi} a_k \varepsilon^{\nu i_\varphi k} x^{1-k/m_\varphi} + \xi x^{1-n_\varphi/m_\varphi}, \\ \nu = 0, 1, \ldots, \operatorname{mult}(\varphi(x, 0)) - 1, \end{gathered} \tag{2.7}$$

where ε is a primitive m_φth root of 1, $\varepsilon = \exp(2\pi i / m_\varphi)$.

For each π-series φ we can write

$$g(x, \varphi(x, \xi)) = g_\varphi(\xi) x^{a_\varphi / m_\varphi} + \text{lower order terms in } x^{1/m_\varphi}, \tag{2.8}$$

where $0 \neq g_\varphi \in \mathbb{C}[\xi]$. The number a_φ is an integer and depends only on the equivalence class of φ. The polynomial g_φ depends on φ. But, for $\varphi_\nu \in [\varphi]$ in (2.7),

$$g_{\varphi_\nu}(\xi) = g_\varphi(\varepsilon^{\nu i_\varphi} \xi) \varepsilon^{\nu i_\varphi a_\varphi}. \tag{2.9}$$

Notation. For each $c \in \mathbb{C}$ and a π-series φ we denote by $B_g([\varphi], c)$ the collection of all $\gamma \in B_g(c)$ such that γ has a Newton–Puiseux expansion of the form $\varphi(x, a)$ + lower order terms in x for some $a \in \mathbb{C}$. The collection $B_g([\varphi], c)$ depends only on the equivalence class $[\varphi]$.

From now on, sometimes we will use the lower index "$_{[\varphi]}$" instead of "$_\varphi$" to indicate characteristics of an equivalence class $[\varphi]$.

PROPOSITION 2.1. *Let* $g\in\mathbb{C}[x,y]$ *be monic in* y. *For a given* π*-series* φ:

(i) *A value* c *is a zero of* g_φ *if and only if there exists an irreducible branch* $\gamma\in B_g(0)$ *having a Newton–Puiseux expansion at infinity of the form* $\varphi(x,c)+$ *lower order terms in* x. *Such a branch* γ *is unique if* c *is a simple zero of* g_φ.

(ii) $g_\varphi(\xi)=\xi^k g^*(\xi^{i_\varphi})$ *for some* $k\ge 0$ *and a polynomial* g^* *with* $g^*(0)\neq 0$.

Proof. Write

$$g(t^{-m_\varphi},\varphi(t^{-m_\varphi},\xi))=t^{-a_\varphi}(g_\varphi(\xi)+\text{higher order terms in } t)=:t^{-a_\varphi}h(t,\xi).$$

Obviously, $h(t,\xi)$ is a non-zero polynomial in (t,ξ). The conclusions (i) and (ii) result from the following simple observations:

1) The curve $h=0$ has an irreducible branch β at a point $(0,c)$ if and only if c is a zero of g_φ. Such a branch β is unique if c is a simple zero of g_φ.

2) For each Newton–Puiseux expansion $\beta(t)$ of such a branch β, the branch γ determined by the series

$$u(x):=\varphi(x,c+\beta(x^{-1/m_\varphi}))$$

is a branch at infinity of the curve $g=0$.

3) By using the representation of g in (2.4), we can see that each such branch γ contributes to $g_\varphi(\xi)$ a factor $(\xi^{i_\varphi}-c^{i_\varphi})^r$ for $c\neq 0$ and ξ^r for $c=0$, where $r:=\mathrm{mult}(u)\mathrm{mult}(\varphi(x,c))^{-1}$. ■

DEFINITION 2.2. A π-series φ is a *Newton–Puiseux type* of g if $a_\varphi=0$ and $\deg g_\varphi>0$. Denote by Π_g the collection of all equivalence classes of Newton–Puiseux types of g.

REMARK 2.3. (i) Let φ be a Newton–Puiseux type of g and $\varphi_\nu\in[\varphi]$ be as in (2.7). By Lemma 2.1(ii), (2.8) and (2.9),

$$g_\varphi(\xi)\in\mathbb{C}[\xi^{i_\varphi}]\quad\text{and}\quad g_{\varphi_\nu}(\xi)=g_\varphi(\varepsilon^{\nu i_\varphi}\xi). \tag{2.10}$$

(ii) For each given Newton–Puiseux expansion at infinity $u(x)$ of $\gamma\in B_g(0)$ we can construct a unique Newton–Puiseux type φ so that $u(x)=\varphi(x,c)+$ lower order terms in x, $c\in\mathbb{C}$. Indeed, assume that $u(x)=\sum_{k=0}^{\infty}a_kx^{1-k/m}$. By using the representation of g in (2.4) we can see that the greatest power of x in $g(x,\sum_{k=1}^{i}a_kx^{1-k/m})$, viewed as a function of $i\in\mathbb{N}$ with values in $\mathbb{Q}\cup\{-\infty\}$, is decreasing and tends to $-\infty$ as i tends to ∞. This ensures that there exists a unique rational number α and an index n such that $1-(n-1)/m<\alpha\le 1-n/m$ and the π-series $\varphi(x,\xi)=\sum_{k=0}^{n-1}a_kx^{1-k/m}+\xi x^\alpha$ is a Newton–Puiseux type of g. In certain cases the series $u(x)$ and the Newton–Puiseux type $\varphi(x,\xi)$ have the same Puiseux pairs.

(iii) There is a natural one-to-one correspondence $\phi:[\varphi]\mapsto l$ between Π_g and the collection of all dicritical (horizontal) components l of the divisor curve D in a regular extension g^* of g, $g^* : M = \mathbb{C}^2 \sqcup D \to \mathbb{CP}$, which can be obtained by resolution of singularities (cf. [LW] and [O1]). The meaning of the correspondence ϕ is the following: *If an analytic irreducible branch at infinity $\gamma \subset \mathbb{C}^2$ is a branch at a point of a dicritical component $l = \phi([\varphi])$, then γ has a Newton–Puiseux expansion at infinity of the form $\varphi(x,c)+$ lower order terms in x.* This observation is not used in this paper. However, it should be useful to view Main Lemma and Theorem 3.6, presented in Section 3, from the viewpoint of regular extensions.

It is well known from [Ve] that for every non-constant polynomial g on $\mathbb{C}^2$ there is a finite set E such that the map $g : \mathbb{C}^2 - g^{-1}(E) \to \mathbb{C} - E$ determines a locally trivial fibration. The fiber of this fibration is called the *generic fiber* of g. The smallest E_g among such sets E is said to be the *exceptional value set* of g.

Denote by χ_g the Euler–Poincaré characteristic of the generic fiber of g, by C_g the critical value set of g and by C_g^∞ the union of all critical value sets of $g_\varphi(\xi)$, $[\varphi] \in \Pi_g$.

THEOREM 2.4. *Let $g(x,y)$ be a primitive polynomial and monic in y. Then*

$$B_g(c) = \bigsqcup_{[\varphi]\in \Pi_g} B_g([\varphi],c) \quad \text{for all } c \in \mathbb{C},$$

$$\#B_g(c) = \sum_{[\varphi]\in\Pi_g} \frac{\deg g_{[\varphi]}}{i_{[\varphi]}} \quad \text{for all } c \in \mathbb{C} - C_g^\infty,$$

$$E_g = C_g \cup C_g^\infty.$$

Proof. The first formula can be obtained from the definition and Remark 2.3(ii). Then the second results from Lemma 2.1(ii) (and (2.10)). We prove the last conclusion. Recall from [HL] that for a primitive polynomial g we have $c \in E_g$ if and only if the Euler–Poincaré characteristic of the curve $g = c$ is greater than χ_g. We can consider $\{B_g(c) = \bigcup_{[\varphi]} B_g([\varphi],c) : c \in \mathbb{C}\}$ as a family of analytic curve germs. Let φ be a Newton–Puiseux type of g. For regular values c of g_φ the curve germs $B_g([\varphi],c)$ have the same Puiseux data: the number of branches; Puiseux pairs; the intersection number of each of these branches and other branches in $B_g(c)$. As in the proof of Lemma 2.1, we can see that such Puiseux data must change when c is a critical value of g_φ. Therefore, by the standard results on the topology of plane curves (see, for example, [BK] and [HL]) the Euler–Poincaré characteristic of the curves $g = c$ must change at each of the critical values of g and the values in C_g^∞. ∎

In view of Theorem 2.4, the topology of the generic fiber as well as the "critical values at infinity" of a primitive polynomial $g(x,y)$, monic in y, can be completely determined by the Newton–Puiseux types φ of g and the corresponding polynomials g_φ.

3. The Jacobian condition and structure of Newton–Puiseux types of P and Q. From now on our object of study is a given polynomial map f of $\mathbb{C}^2$, $f = (P,Q)$, satisfying the Jacobian condition, i.e. $P_xQ_y - P_yQ_x \equiv C \neq 0$. To avoid trivial cases, we also assume that $\max(\deg P, \deg Q) > 1$. Under the Jacobian condition P and Q are primitive polynomials. By choosing suitable affine coordinates (x,y) in $\mathbb{C}^2$, we can assume that P and Q are monic in y. Then we can work with the collections Π_P and Π_Q of equivalence classes of Newton–Puiseux types of P and Q.

In this section we describe the structure of $\Pi_P \cup \Pi_Q$ and give a proof of Theorem A.

3.1. *Jacobian condition.* For any π-series φ,

$$\varphi(x,\xi) = \sum_{k=0}^{n_\varphi - 1} a_k x^{1-k/m_\varphi} + \xi x^{1-n_\varphi/m_\varphi}, \tag{3.1}$$

we write

$$P(x,\varphi(x,\xi)) = P_\varphi(\xi)x^{a_\varphi/m_\varphi} + \text{lower order terms in } x^{1/m_\varphi},$$
$$Q(x,\varphi(x,\xi)) = Q_\varphi(\xi)x^{b_\varphi/m_\varphi} + \text{lower order terms in } x^{1/m_\varphi},$$

where $P_\varphi, Q_\varphi \in \mathbb{C}[\xi] - \{0\}$, and define

$$J_\varphi := a_\varphi P_\varphi \frac{d}{d\xi} Q_\varphi - b_\varphi Q_\varphi \frac{d}{d\xi} P_\varphi, \tag{3.2}$$

$$\Delta_\varphi := \frac{a_\varphi}{m_\varphi} + \frac{b_\varphi}{m_\varphi} + \frac{n_\varphi}{m_\varphi}. \tag{3.3}$$

Our starting point is the following.

LEMMA 3.1. *Let φ be a π-series and assume that*

$$\max\{\deg P_\varphi, \deg Q_\varphi\} > 0, \quad a_\varphi \ge 0, \quad b_\varphi \ge 0, \quad a_\varphi + b_\varphi > 0. \tag{3.4}$$

Then

$$\Delta_\varphi \ge 2 \tag{3.5}$$

and

$$J_\varphi \equiv \begin{cases} m_\varphi J(P,Q) & \text{if } \Delta_\varphi = 2, \\ 0 & \text{if } \Delta_\varphi > 2. \end{cases} \tag{3.6}$$

Proof. Defining the holomorphic maps Φ and F_φ from $\mathbb{C}\setminus\{0\}\times\mathbb{C}$ into $\mathbb{C}^2$ by $\Phi(t,\xi) := (t^{-m_\varphi}, \varphi(t^{-m_\varphi},\xi))$ and $F_\varphi(t,\xi) := (P(\Phi(t,\xi)), Q(\Phi(t,\xi)))$ and writing

$$P(\Phi(t,\xi)) = P_\varphi(\xi)t^{-a_\varphi} + \text{higher order terms in } t,$$

$$Q(\Phi(t,\xi)) = Q_\varphi(\xi)t^{-b_\varphi} + \text{higher order terms in } t,$$

by (3.4) we can differentiate Φ and F_φ to obtain

$$\det DF_\varphi(t,\xi) = J(P,Q)\det D\Phi(t,\xi) = -m_\varphi J(P,Q)t^{n_\varphi-2m_\varphi-1}$$

and

$$\det DF_\varphi(t,\xi) = -\left[a_\varphi P_\varphi \frac{d}{d\xi}Q_\varphi - b_\varphi Q_\varphi \frac{d}{d\xi}P_\varphi\right]t^{-a_\varphi-b_\varphi-1} + \text{higher order terms in } t.$$

It follows that

$$-m_\varphi J(P,Q)t^{n_\varphi-2m_\varphi-1} = -J_\varphi t^{-a_\varphi-b_\varphi-1} + \text{higher order terms in } t.$$

Since $J(P,Q) \equiv \text{const} \neq 0$, by comparing the first terms of the two sides in this equality we get the conclusion. ■

Considering the equation $J_\varphi \equiv 0$ and $J_\varphi \equiv m_\varphi J(P,Q)$ in detail, we can easily obtain the following.

LEMMA 3.2. *Let φ be a π-series and assume that the condition* (3.4) *of Lemma* 3.3 *holds. Then*

(a) *If $J_\varphi \equiv m_\varphi J(P,Q)$, then the polynomials P_φ and Q_φ have only simple zeros and they have no common zero.*

(b) *If $J_\varphi \equiv 0$, then $P_\varphi^{b_\varphi} = cQ_\varphi^{a_\varphi}$ for some number $c \neq 0$.*

(c) *If* $P_\varphi \equiv \text{const} \neq 0$, *then* $J_\varphi \equiv m_\varphi J(P,Q)$ *and* $\deg Q_\varphi = 1$, $Q_\varphi(0) \neq 0$ *in the case $a_\varphi > 0$ and $b_\varphi > 0$.*

(d) *If $a_\varphi = 0$ then* $J_\varphi \equiv m_\varphi J(P,Q)$, $\deg P_\varphi = 1$ *and* $Q_\varphi \equiv \text{const} \neq 0$.

REMARK. Lemma 3.1 is still true if condition (3.4) is replaced by

$$(3.4)' \qquad \max\{\deg P_\varphi, \deg Q_\varphi\} > 0, \quad a_\varphi \leq 0, \quad b_\varphi \leq 0, \quad a_\varphi + b_\varphi < 0.$$

3.2. *Main Lemma and the structure of $\Pi_P \cup \Pi_Q$.* To analyze the structure and relationship of Newton–Puiseux types of P and Q we will use the following object, which is constructed for each Newton–Puiseux type φ of P and Q.

Associate sequence. Given a Newton–Puiseux type φ of P, we can write

$$\varphi(x,\xi) = \sum_{k=0}^{K-1} c_k x^{\alpha_k} + \xi x^{1-n_\varphi/m_\varphi}, \qquad c_k \in \mathbb{C}, \tag{3.7}$$

where c_k may be zero, so that the sequence $\{\varphi_i\}_{i=0}^K$ of π-series defined by

$$\varphi_i(x,\xi) := \sum_{k=0}^{i-1} c_k x^{\alpha_k} + \xi x^{\alpha_i}, \qquad i = 0, 1, \ldots, K-1,$$

$\varphi_K := \varphi$ and $\alpha_K := 1 - n_\varphi/m_\varphi$, has the following properties:

(S1) For every $i < K$ either P_{φ_i} or Q_{φ_i} has a zero different from the origin.

(S2) For every $\psi(x,\xi) = \varphi_i(x, c_i) + \xi x^{\alpha}$, $\alpha_i > \alpha > \alpha_{i+1}$, each of P_ψ and Q_ψ is either a constant or a monomial in ξ.

The representation (3.7) and the sequence $\{\varphi_i\}_{i=0}^K$ are unique. This sequence is called the *associate sequence* of φ. For simplicity of notation, we put

$$P_k := P_{\varphi_k}, \quad Q_k := Q_{\varphi_k}, \quad a_k := a_{\varphi_k}, \quad b_k := b_{\varphi_k},$$
$$m_k := \mathrm{mult}(\varphi_k), \quad n_k := n_{\varphi_k}, \quad i_k := \mathrm{ind}(\varphi_k), \quad J_k := J_{\varphi_k}.$$

One can easily verify the following properties of the associate sequence $\{\varphi_i\}_{i=0}^K$.

(S3) $a_0/m_0 > a_1/m_1 > \ldots > a_K/m_K$ and $P_i(c_i) = 0$ for all $0 \le i < K$.

(S4) If $Q_k(c_k) \ne 0$ for some k, then

$$b_i/m_i = b_k/m_k, \quad Q_i \equiv \mathrm{const} \ne 0 \quad \text{for all } k < i \le K.$$

Note that $\varphi_0(x,\xi) = \xi x$. As in our first assumption, $\max(\deg P, \deg Q) > 1$, by the Jacobian condition we can verify that

$$J_0 \equiv 0, \quad m_0 = 1, \quad a_0 = \deg P_0 = \deg P, \quad b_0 = \deg Q_0 = \deg Q. \tag{3.8}$$

Therefore, we can determine the greatest index S such that

$$J_i \equiv 0, \quad a_i \ge 0, \quad b_i \ge 0, \quad a_i + b_i > 0 \quad \text{for all } 0 \le i \le S-1.$$

The *marked series* φ_S plays a special role in the associate sequence $\{\varphi_i\}_{i=0}^K$ of φ.

Now, we can state our main lemma on the influence of the Jacobian condition on associate sequences of Newton–Puiseux types of P and Q. The proof of this lemma will be given in Section 3.4.

LEMMA 3.3 (Main Lemma). *Let φ be a Newton–Puiseux type of P, $\{\varphi_i\}_{i=0}^K$ be the associate sequence of φ and φ_S be the corresponding marked series. Then*:

(i) *For all indices* $0 \le i \le S-1$,

$$\frac{a_i}{b_i} = \frac{\deg P_i}{\deg Q_i} = \frac{\deg P}{\deg Q} \quad \textit{and} \quad P_i^{\deg Q} = C_i Q_i^{\deg P}, \ C_i \ne 0.$$

(ii) *Case* $S = K$:

$$a_K = b_K = 0 \quad \textit{and} \quad \frac{\deg P_K}{\deg Q_K} = \frac{\deg P}{\deg Q}.$$

(iii) *Case* $S < K$:

$$J_S \equiv m_S J(P,Q) \quad \textit{and} \quad \frac{a_S}{b_S} = \frac{\deg P_S}{\deg Q_S} = \frac{\deg P}{\deg Q}$$

and for all $S < i \le K$,

$$J_i \equiv m_i J(P,Q), \quad \frac{b_i}{m_i} = \frac{b_S}{m_S} > 0, \quad \deg P_i = 1, \quad Q_i \equiv \mathrm{const} \neq 0.$$

This lemma contains more information than what we will use to describe the structure of $\Pi_P \cup \Pi_Q$ below.

Definition 3.4. A π-series φ is a *dicritical series* of f if either $[\varphi] \in \Pi_P$ and $b_\varphi \le 0$ or $[\varphi] \in \Pi_Q$ and $a_\varphi \le 0$. A π-series φ is a *separative series* of the pair (P,Q) if $a_\varphi > 0$, $b_\varphi > 0$, $J_\varphi \neq 0$ and $\min(\deg P_\varphi, \deg Q_\varphi) > 0$. Denote by Π_f and $S_{(P,Q)}$ the collection of all equivalence classes of dicritical series of f and the collection of all equivalence classes of separative series of (P,Q), respectively.

Remark 3.5. By definitions, Main Lemma and Proposition 2.1 we can see the following.

(i) $[\varphi] \in \Pi_f \cup S_{(P,Q)}$ if and only if φ is a marked series in the associate sequence of a Newton–Puiseux type of P or Q. Corresponding to each Newton–Puiseux type φ there is at most one separative series. Some Newton–Puiseux types of P and of Q may have the same separative series. Furthermore, $\Pi_f \cap S_{(P,Q)} = \emptyset$.

(ii) For $c \in \mathbb{C}$ and every Newton–Puiseux type φ of P such that $[\varphi] \in \Pi_P - \Pi_f$, the map f tends to infinity along each branch in $B_P([\varphi], c)$.

(iii) Π_f is a subset of $\Pi_P \cup \Pi_Q$, but it is independent of the coordinate presentation (P,Q) of f. In other words, $\Pi_f = \Pi_{A.f}$ for all $A \in \mathrm{Aut}(\mathbb{C}^2)$.

(iv) The collection Π_f is empty if and only if f is a proper map, and then f is an automorphism.

Convention and notation. We call a π-series φ such that $[\varphi] \in (\Pi_P \cup \Pi_Q) - \Pi_f$ a *polar series* of (P,Q) and define

$$\mathrm{Pol}(f,P) := \Pi_P - \Pi_f, \qquad \mathrm{Pol}(f,Q) := \Pi_Q - \Pi_f.$$

For each class $[\varphi]$ we put

$$\begin{aligned} N(P,[\varphi]) &:= \{[\psi] \in \Pi_P : \exists \varphi' \in [\varphi] \text{ such that} \\ &\qquad\qquad \psi(x,\xi) = \varphi'(x,c) + \text{lower order terms in } x\}, \\ N(Q,[\varphi]) &:= \{[\psi] \in \Pi_Q : \exists \varphi' \in [\varphi] \text{ such that} \\ &\qquad\qquad \psi(x,\xi) = \varphi'(x,c) + \text{lower order terms in } x\}. \end{aligned}$$

Now, we can give a description of $\Pi_P \cup \Pi_Q$.

Theorem 3.6. *Suppose $f = (P,Q)$ is a non-zero constant Jacobian polynomial map of $\mathbb{C}^2$, where $P, Q \in \mathbb{C}[x,y]$ are monic in y. Then*

$$\Pi_P \cup \Pi_Q = \mathrm{Pol}(f,P) \sqcup \mathrm{Pol}(f,Q) \sqcup \Pi_f, \qquad \Pi_P \cap \Pi_Q = \Pi_f,$$

$$\mathrm{Pol}(f,P)=\bigsqcup_{[\varphi]\in S_{(P,Q)}} N(P,[\varphi]),\qquad \mathrm{Pol}(f,Q)=\bigsqcup_{[\varphi]\in S_{(P,Q)}} N(Q,[\varphi]).$$

and the following properties hold:

(i) *For a separative series* φ *of* (P,Q),

$$J_\varphi\equiv n_\varphi J(P,Q)\quad \textit{and}\quad \frac{a_\varphi}{b_\varphi}=\frac{\deg P_\varphi}{\deg Q_\varphi}=\frac{\deg P}{\deg Q}.$$

(ii) *For a dicritical series* φ *of* f,

$$a_\varphi=0,\qquad b_\varphi=0,\qquad \frac{\deg P_\varphi}{\deg Q_\varphi}=\frac{\deg P}{\deg Q}.$$

(iii) *For a polar series* φ,

$$\begin{cases}\deg P_\varphi=1,\ Q_\varphi\equiv\mathrm{const}\neq 0 & \textit{if } [\varphi]\in\mathrm{Pol}(f,P),\\ P_\varphi\equiv\mathrm{const}\neq 0,\ \deg Q_\varphi=1 & \textit{if } [\varphi]\in\mathrm{Pol}(f,Q).\end{cases}$$

Proof. This results immediately from Main Lemma and definitions. ■

3.3. *Proof of Theorem A.* First, as an immediate consequence of Theorem 3.6(iii) and Theorem 2.4, we have

Corollary 3.7. *For all* $c\in\mathbb{C}$ *the number of poles of* Q *on the curve* $P=c$ *is equal to* $\#\mathrm{Pol}(f,P)$.

We need to know more about separative series of (P,Q).

Lemma 3.8. *Let* φ *be a separative series of* (P,Q).

(i) *Case* $i_\varphi=1$: $\#N(P,[\varphi])=\deg P_\varphi$ *and* $\#N(Q,[\varphi])=\deg Q_\varphi$.

(ii) *Case* $i_\varphi>1$: *then either* (a) *or* (b) *below holds*:

(a) $P_\varphi(\xi)=\xi p^*_\varphi(\xi^{i_\varphi})$ *and* $Q_\varphi(\xi)=q^*_\varphi(\xi^{i_\varphi})$, *where* $p^*_\varphi(0)q^*_\varphi(0)\neq 0$, *and*

$$\#N(P,[\varphi])=\frac{\deg P_\varphi-1}{i_\varphi}+1,\qquad \#N(Q,[\varphi])=\frac{\deg Q_\varphi}{i_\varphi}.$$

(b) $P_\varphi(\xi)=p^*_\varphi(\xi^{i_\varphi})$ *and* $Q_\varphi(\xi)=\xi q^*_\varphi(\xi^{i_\varphi})$, *where* $p^*_\varphi(0)q^*_\varphi(0)\neq 0$, *and*

$$\#N(P,[\varphi])=\frac{\deg P_\varphi}{i_\varphi},\qquad \#N(Q,[\varphi])=\frac{\deg Q_\varphi-1}{i_\varphi}+1.$$

Proof. Since φ is a separative series of (P,Q), P_φ and Q_φ have only simple zeros. Let $d\in\mathbb{C}$ be a zero of P_φ. Then, by Main Lemma (iii) there exists a unique Newton–Puiseux type ψ of P such that $\psi_d(x,\xi)=\varphi(x,d)+$ lower order terms in x. This series is a polar series and $[\psi_d]\in N(P,[\varphi])$. If $i_\varphi=1$ or if $d=0$, then there is no other zero $d^*\neq d$ of P_φ such that $\psi_{d^*}\in[\psi_d]$. In case $i_\varphi>1$ and $d\neq 0$, by Remark 2.3(i), $d_\nu=\varepsilon^\nu d$, $\nu=$

$0, 1, \ldots, i_\varphi - 1$, are also zeros of P_φ and $\psi_{d_\nu} \in [\psi_d]$, where $\varepsilon := \exp(2\pi i/i_\varphi)$. By these observations we can easily verify the conclusions of the lemma. ∎

The following is a variation of the Lemma on Automorphisms, a crucial step in some proofs of Jung's theorem on automorphisms of $\mathbb{C}^2$ ([J], see also [Kul] and [MK]).

Lemma 3.9 (Lemma on Divisibility).

$$\#\mathrm{Pol}(f, P) = \#\mathrm{Pol}(f, Q) = 1 \Rightarrow \deg P \mid \deg Q \ \textit{or} \ \deg Q \mid \deg P.$$

Proof. If $\#\mathrm{Pol}(f, P) = \#\mathrm{Pol}(f, Q) = 1$, then Theorem 3.6 shows that $S_{(P,Q)} = \{[\varphi]\}$ and $\#N(P, [\varphi]) = N(Q, [\varphi]) = 1$. Then, by Lemma 3.8,

$$\begin{cases} \deg P_\varphi = \deg Q_\varphi = 1 & \text{for } i_\varphi = 1, \\ \deg P_\varphi = 1, \ \deg Q_\varphi = i_\varphi & \text{for } i_\varphi > 1 \text{ and } P_\varphi(0) = 0, \\ \deg P_\varphi = i_\varphi, \ \deg Q_\varphi = 1 & \text{for } i_\varphi > 1 \text{ and } Q_\varphi(0) = 0. \end{cases}$$

This together with the equality $\deg P_\varphi / \deg Q_\varphi = \deg P / \deg Q$ of Theorem 3.6(i) concludes the proof. ∎

Proof of Theorem A. Consider the following three statements:

(J_1) (JC_2) is true;

(J_2) $J(P, Q) \equiv \text{const} \neq 0 \Rightarrow \#\mathrm{Pol}(f, P) = \#\mathrm{Pol}(f, Q) = 1$;

(J_3) $J(P, Q) \equiv \text{const} \neq 0 \Rightarrow \deg P \mid \deg Q$ or $\deg Q \mid \deg P$.

In view of Corollary 3.7, Theorem A can be restated as $(J_1) \Leftrightarrow (J_2)$. The equivalence $(J_1) \Leftrightarrow (J_3)$ is well known (see, e.g., [A], [AM] and [BCW]). The implication $(J_1) \Rightarrow (J_2)$ is obvious, and $(J_2) \Rightarrow (J_3)$ by Lemma 3.9. So, we get $(J_1) \Leftrightarrow (J_2)$. ∎

3.4. *Proof of Main Lemma.* Let φ be a given Newton–Puiseux type of P, $\{\varphi_i\}_{i=0}^K$ be the associate sequence of φ and φ_S be the corresponding marked series.

First, consider the subsequence $\{\varphi_i\}_{i=0}^S$. From the definition and property (S1), we have $\max(\deg P, \deg Q) > 0$, $J_i \equiv 0$, $a_i > 0$ and $b_i > 0$ for all $0 \leq i < S$. Therefore, by Lemma 3.1 (and Lemma 3.2(b))

$$P_i^{b_i} = C_i Q_i^{a_i}, \quad C_i \neq 0, \tag{3.9}$$

and hence

$$\frac{a_i}{b_i} = \frac{\deg P_i}{\deg Q_i} \tag{3.10}$$

for all $0 \leq i < S$.

Assertion 1. *For every π-series*

$$\psi(x, \xi) = \varphi_k(x, c_k) + \xi x^\alpha, \quad \alpha \in \mathbb{Q}, \ \alpha_k > \alpha > \alpha_{k+1}, \ 0 \leq k < S,$$

the polynomials P_ψ and Q_ψ are monomials in ξ and $J_\psi \equiv 0$.

Proof. Use the expansions $P(x,\psi(x,\xi)) = P(x,\varphi_k(x,c_k)+\xi x^\alpha)$ and $Q(x,\psi(x,\xi)) = Q(x,\varphi_k(x,c_k)+\xi x^\alpha)$, property (S2) and Lemma 3.2(b). ∎

Assertion 2.

$$\frac{\deg P_k}{\deg Q_k} = \frac{\deg P_{k+1}}{\deg Q_{k+1}} \quad \textit{for } k = 0,\ldots,S-1.$$

Proof. Let $0 \le k \le S-1$. By (S3) and (3.9) the value c_k is a common zero of P_k and Q_k. Write

$$P(x,\varphi_k(x,\xi)) = \sum_{i=0}^{M} (\xi - c_k)^{u_i} p_i(\xi) x^{a_{ki}/m_k}, \tag{3.11}$$

$$Q(x,\varphi_k(x,\xi)) = \sum_{j=0}^{N} (\xi - c_k)^{v_j} q_j x^{b_{kj}/m_k}, \tag{3.12}$$

where $p_i(0) \ne 0$, $q_j(0) \ne 0$, $a_{k0} > a_{k1} > \ldots > a_{kM}$ and $b_{k0} > b_{k1} > \ldots > b_{kN}$. Note that

$$a_{k0} = a_k, \quad b_{k0} = b_k \tag{3.13}$$

and u_i, v_j are non-negative integers. The natural numbers u_0 and v_0 are just the multiplicity of the zero c_k of P_k and Q_k, respectively. Therefore, by (3.9) and (3.10) we have

$$\frac{u_0}{v_0} = \frac{\deg P_k}{\deg Q_k}. \tag{3.14}$$

Let us construct a kind of Newton polygon relative to the representations (3.11) and (3.12) as follows:

$$V := \mathrm{Conv}(\{(0,0)\} \cup \{(a_{ki},u_i) : i = 0,1,\ldots,M\} + \mathbb{R}^- \times \mathbb{R}^+),$$
$$W := \mathrm{Conv}(\{(0,0)\} \cup \{(b_{kj},v_j) : j = 0,1,\ldots,N\} + \mathbb{R}^- \times \mathbb{R}^+).$$

Here, the notation $\mathrm{Conv}(\cdot)$ indicates the convex hull.

Let S_V and S_W denote the steepest slope segments of V and W. Observe that (a_{k0},u_0) (resp. (b_{k0},v_0)) is the highest vertex of the polygon V and of the segment S_V (resp. of the polygon W and of the segment S_W). The line containing S_V (resp. S_W) intersects the first axis at the point $(\delta_P,0)$ (resp. $(\delta_Q,0)$). Since the point $(0,0)$ lies in each of these polygons, we have $\delta_P \ge 0$ and $\delta_Q \ge 0$.

Now, rewriting $\varphi_{k+1}(x,\xi) = \varphi_k(x,c_k+\xi x^{\alpha_{k+1}-\alpha_k})$ and substituting it into (3.11) and (3.12), by the construction of φ_k and the properties (S1–S2) we can verify the following:

1) Case where P_{k+1} has a zero different from the origin:

$$P_{k+1}(\xi) = \sum_{(a_{ki},u_i)\in S_V} \xi^{u_i} p_{ki}(c_k), \quad \deg P_{k+1} = u_0,$$

$$\begin{aligned}\frac{a_{k+1}}{m_{k+1}} &= \frac{a_k}{m_k} + (\alpha_k - \alpha_{k+1})u_0,\\ Q_{k+1}(\xi) &= \xi^{v_0} q_{k0}(c_k) + \ldots, \qquad \deg Q_{k+1} = v_0,\\ \frac{b_{k+1}}{m_{k+1}} &= \frac{b_k}{m_k} + (\alpha_k - \alpha_{k+1})v_0.\end{aligned}$$

2) Case where Q_{k+1} has a zero different from the origin:

$$\begin{aligned}Q_{k+1}(\xi) = \sum_{(b_{kj},v_j)\in S_W} &\xi^{v_j} q_{kj}(c_k), \qquad \deg Q_{k+1} = v_0,\\ \frac{b_{k+1}}{m_{k+1}} &= \frac{b_k}{m_k} + (\alpha_k - \alpha_{k+1})v_0,\\ P_{k+1}(\xi) &= \xi^{u_0} p_{k0}(c_k) + \ldots, \qquad \deg P_{k+1} = u_0,\\ \frac{a_{k+1}}{m_{k+1}} &= \frac{a_k}{m_k} + (\alpha_k - \alpha_{k+1})u_0.\end{aligned}$$

In both cases we have $\deg P_{k+1} = u_0$, $\deg Q_{k+1} = v_0$. This together with (3.14) concludes the proof. ■

Assertion 3.

$$\begin{aligned}\frac{\deg P_k}{\deg Q_k} &= \frac{\deg P}{\deg Q} \qquad \textit{for } k = 0, \ldots, S,\\ \frac{a_k}{b_k} &= \frac{\deg P}{\deg Q} \qquad \textit{for } k = 0, \ldots, S-1.\end{aligned}$$

Proof. By (3.9) and (3.10),

$$\frac{\deg P_0}{\deg Q_0} = \frac{a_0}{b_0} = \frac{\deg P}{\deg Q}, \qquad \frac{\deg P_i}{\deg Q_i} = \frac{a_i}{b_i} \qquad \text{for } 0 \le i < S.$$

The conclusion, then, is clear by Assertion 2. ■

Proof of Main Lemma. (i) Combine Assertion 3 and (3.10).

(ii) By definition $a_K = 0$ and $b_K \le 0$. By Assertion 3 we need only show that $b_K = 0$. Otherwise, assuming $b_K < 0$, we can determine a unique π-series $\psi(x, \xi)$ of the form

$$\psi(x, \xi) = \varphi_{k-1}(x, \xi)$$

or

$$\psi(x, \xi) = \varphi_{k-1}(x, c_{k-1}) + \xi x^{\alpha}, \qquad \alpha_{k-1} > \alpha > \alpha_k,$$

such that $b_\psi = 0$. It is obvious that

$$a_\psi / m_\psi > a_K / m_K = 0.$$

Then, applying Lemma 3.2(c) to ψ we obtain $J_\psi \ne 0$ and $P_\psi \equiv \text{const} \ne 0$, which is impossible by Assertion 1. This proves $b_K = 0$.

(iii) Suppose $S < K$. By Property (S3), $a_S/m_S > a_{S+1}/m_{S+1} \geq 0$. Analogously to the proof of (ii) we can show that $b_S \geq 0$. Therefore, $J_S \neq 0$ by the definition of φ_S. Hence, $J_S \equiv m_S J(P,Q)$ by Lemma 3.1.

Put $P_K(\xi) := A_K p_K(\xi)$ and $Q_K(\xi) := B_K q_K(\xi)$, where $p_K, q_K \in \mathbb{C}[\xi]$ with leading coefficient 1. Then

$$a_K p_K \dot{q}_K - b_K \dot{p}_K q_K \equiv \text{const} \neq 0,$$

since $J_S \equiv m_S J(P,Q)$. Comparing the two sides of this equality, we get

$$\frac{a_S}{b_S} = \frac{\deg P_S}{\deg Q_S}.$$

Hence, by Assertion 3,

$$\frac{a_S}{b_S} = \frac{\deg P_S}{\deg Q_S} = \frac{\deg P}{\deg Q}. \ \blacksquare$$

4. Geometric degree and branched value set of f and topology of generic fibers of P and Q. In this section, by applying Theorem 3.6 and the Riemann–Hurwitz relation we will get information on the geometry of (P,Q) from the data of $\Pi_P \cup \Pi_Q$ and give the proofs of Theorems B and C. We will consider the level sets of P and Q as punctured Riemann surfaces and view each of their branches β at infinity as a small enough punctured disk D_β centered at the corresponding puncture. By the degree $\deg_\beta f$ of f on β we mean the topological degree of the restriction $f : D_\beta \to f(D_\beta)$.

From now on, for simplicity, we set $\deg P := kd$ and $\deg Q := ke$, where $k := \gcd(\deg P, \deg Q)$. Sometimes we use the lower index $_{[\varphi]}$ instead of "$_\varphi$" to indicate characteristics of an equivalence class $[\varphi]$.

4.1. *Geometric degree of f and proof of Theorem B.* Recall that the geometric degree $\deg_{\rm geo} f$ of f is the number of solutions of the equation $f = a$ for generic values $a \in \mathbb{C}^2$. This is a topological invariant of f under the action of homeomorphisms of $\mathbb{C}^2$.

Theorem 4.1. (i) *For* $[\varphi] \in S_{(P,Q)}$,

$$\sum_{\beta \in B_P([\varphi],0)} \deg_\beta f = \gcd(a_\varphi, b_\varphi) \gcd(\deg P_\varphi, \deg Q_\varphi) \frac{de}{i_\varphi}.$$

Furthermore, $\gcd(a_\varphi, b_\varphi, i_\varphi) = 1$, $\gcd(\deg P_\varphi, \deg Q_\varphi, i_\varphi) = 1$ *and either* i_φ *divides* d *or* i_φ *divides* e.

(ii) *The geometric degree* $\deg_{\rm geo} f$ *is equal to*

$$\sum_{[\varphi] \in S_{(P,Q)}} \gcd(a_{[\varphi]}, b_{[\varphi]}) \gcd(\deg P_{[\varphi]}, \deg Q_{[\varphi]}) \frac{de}{i_{[\varphi]}}.$$

Proof. (i) Given a separative series φ, $[\varphi] \in S_{(P,Q)}$, let $\beta \in B_P([\varphi], 0)$ be given by a Newton–Puiseux expansion at infinity $\beta(x) = \varphi(x, d) +$ lower order terms in x. Then $\deg_\beta f$ can be determined from the expansion

$$Q(x, \beta(x)) = Q_\varphi(d) x^{\deg_\beta f/\mathrm{mult}(\beta)} + \text{lower order terms in } x, \qquad Q_\varphi(d) \neq 0, \tag{4.1}$$

and hence,

$$\deg_\beta f = \begin{cases} b_\varphi & \text{for } d \neq 0, \\ b_\varphi / i_\varphi & \text{for } d = 0. \end{cases}$$

This together with Lemma 3.8 implies that

$$\sum_{\beta \in B_P([\varphi],0)} \deg_\beta f = \frac{b_\varphi \deg P_\varphi}{i_\varphi}. \tag{4.2}$$

On the other hand, according to Theorem 3.6(i),

$$b_\varphi = \gcd(a_\varphi, b_\varphi) e \quad \text{and} \quad \deg P_\varphi = \gcd(\deg P_\varphi, \deg Q_\varphi) d.$$

Then by (4.2) we get the desired formula for the total degree of f on $B_P([\varphi], 0)$.

To prove the rest, we need only consider the case of $i_\varphi > 1$. The conclusion

$$\gcd(P_\varphi, Q_\varphi, i_\varphi) = 1 \quad \text{and} \quad \text{either } i_\varphi \mid d \text{ or } i_\varphi \mid e$$

follows from Lemma 3.8. Since $J_\varphi \equiv m_\varphi J(P, Q)$ by Lemma 3.1, we have

$$a_\varphi + b_\varphi + n_\varphi = 2m_\varphi.$$

So, if $\gcd(a_\varphi, b_\varphi, i_\varphi) > 1$, then $\gcd(n_\varphi, i_\varphi) > 1$. This is impossible by definition of m_φ and n_φ. Thus, we get $\gcd(a_\varphi, b_\varphi, i_\varphi) = 1$.

(ii) It is a well known elementary fact that

$$\deg_{\mathrm{geo}} f = \sum_{\beta \in B_P(0) \text{ a pole of } Q \text{ on } P=0} \deg_\beta f.$$

Then, in view of Theorem 3.6,

$$\deg_{\mathrm{geo}} f = \sum_{[\psi] \in \mathrm{Pol}(f,P)} \sum_{\beta \in B_P([\psi],0)} \deg_\beta f = \sum_{[\varphi] \in S_{(P,Q)}} \sum_{\beta \in B_P([\varphi],0)} \deg_\beta f.$$

Therefore, we get the desired formula by (i). ∎

Proof of Theorem B. We need only consider the situation $d > e > 1$ and show that

$$\deg_{\mathrm{geo}} f = rd + se, \qquad r \geq 0,\ s \geq 0,\ r + s \geq 1$$

and

$$\deg_{\mathrm{geo}} f \geq \min(2e, d).$$

The first conclusion is immediate from Theorem 4.1. For the second, it is sufficient to prove that for $\varphi \in S_{(P,Q)}$,

$$\gcd(a_\varphi, b_\varphi) \gcd(\deg P_\varphi, \deg Q_\varphi) \frac{de}{i_\varphi} \geq \min(2e, d).$$

Following Lemma 3.8, we consider three cases.

1) Case $i_\varphi = 1$: By Theorem 4.1(i),

$$\sum_{\beta \in B_P([\varphi],0)} \deg_\beta f = \gcd(a_\varphi, b_\varphi) \gcd(\deg P_\varphi, \deg Q_\varphi) de \geq de.$$

2) Case $i_\varphi > 1$ and $P_\varphi(0) = 0$: By Lemma 3.8, i_φ divides $\deg Q_\varphi$, and hence, by Theorem 4.1(i), i_φ divides e. It follows that

$$\sum_{\beta \in B_P([\varphi],0)} \deg_\beta f = \gcd(a_\varphi, b_\varphi) \gcd(\deg P_\varphi, \deg Q_\varphi) \frac{e}{i_\varphi} d \geq d.$$

3) Case $i_\varphi > 1$ and $Q_\varphi(0) = 0$: By Lemma 3.8, i_φ divides $\deg P_\varphi$ and i_φ divides $\deg Q_\varphi - 1$, and hence, by Theorem 4.1(i), i_φ divides d. In particular, either $i_\varphi < d$ or $\gcd(P_\varphi, \deg Q_\varphi) > 1$, since $d > e > 1$. Hence,

$$\sum_{\beta \in B_P([\varphi],0)} \deg_\beta f = \gcd(a_\varphi, b_\varphi) \gcd(\deg P_\varphi, \deg Q_\varphi) \frac{d}{i_\varphi} e \geq 2e. \blacksquare$$

4.2. *Dicritical series and branched value set of f.* Here, we are concerned with the situation $\Pi_f \neq \emptyset$. As in Remark 3.5(iv), in this situation the *branched value set* E_f of f, $E_f := \{a \in \mathbb{C}^2 : \# f^{-1}(a) < \deg_{\mathrm{geo}} f\}$, is not empty. As in the proof of Proposition 2.1, we can verify that

$$E_f = \bigcup_{[\varphi] \in \Pi_f} C_{[\varphi]},$$

where $C_{[\varphi]} = \{(P_\varphi(\xi), Q_\varphi(\xi)) : \xi \in \mathbb{C}\}$. The curve $C_{[\varphi]}$ depends only on the class $[\varphi] \in \Pi_f$ by (2.10). For each class $[\varphi] \in \Pi_f$ we introduce the map

$$F_\varphi(t, \xi) := (P(t^{-m_\varphi}, \varphi(t^{-m_\varphi}, \xi)), Q(t^{-m_\varphi}, \varphi(t^{-m_\varphi}, \xi))).$$

This is a polynomial in (t, ξ) and

$$\det DF_\varphi(t, \xi) = -m_\varphi J(P, Q) t^{n_\varphi - 2m_\varphi - 1}.$$

The singular set of F_φ is either empty or the line $t = 0$. This allows us to determine $\deg_{(0,d)} F_\varphi$ by the formula

$$\deg_{(0,d)} F_\varphi = \sum_\beta \deg_\beta F_\varphi, \tag{4.3}$$

where the summation is taken over all irreducible branches β at $(0, d)$ of the curve $P(t^{-m_\varphi}, \varphi(t^{-m_\varphi}, \xi)) = P_\varphi(d)$.

Using (4.3) it is easy to verify the following.

LEMMA 4.2. *Let* $[\varphi] \in \Pi_f$, $c \in \mathbb{C}$ *and* $d \in \mathbb{C}$ *so that* $P_\varphi(d) = c$. *Then*

$$\sum_\gamma \deg_\gamma f = \begin{cases} \deg_{(0,d)} F_\varphi & \text{for } d \neq 0, \\ \deg_{(0,0)} F_\varphi i_\varphi^{-1} & \text{for } d = 0, \end{cases}$$

where the summation is taken over all branches $\gamma \in B_P(c)$ *having a Newton–Puiseux expansion at infinity of the form* $\varphi(x,d)$ + *lower order terms in* x.

It is well known (see for example Lemma 3.1 of [O1]) that $\deg_{(0,d)} F_\varphi$ is the same natural number μ_φ for almost $d \in \mathbb{C}$, except at most a finite number of values d for which $\deg_{(0,d)} F_\varphi > \mu_\varphi$. Moreover, the natural number μ_φ depends only on the class $[\varphi]$.

Denote by E_φ the singular set of the map $(P_\varphi, Q_\varphi)(\xi)$. By Remark 2.3(i), E_φ is a finite set depending only on the class $[\varphi]$.

LEMMA 4.3. *Let* $[\varphi] \in \Pi_f$. *Then*

$$\deg_{(0,d)} F_\varphi \begin{cases} \equiv \mu_\varphi & \text{for } d \in \mathbb{C} - E_\varphi, \\ > \mu_\varphi & \text{for } d \in E_\varphi. \end{cases} \tag{4.4}$$

Proof. Let $d \in \mathbb{C}$. Consider the map $F_\varphi(t,\xi)$ in a small enough ball B around $(0,d)$. Put $D := B \cap (\{0\}\times\mathbb{C})$ and $V := F_\varphi(D)$, which can be viewed as an irreducible curve germ at $F_\varphi(0,d)$. If $d \in E_\varphi$, then V is smooth. In this case we know from Lemma 3.1 of [O1] that $\deg_{(0,\xi)} F_\varphi \equiv \mu_\varphi$ for all $\xi \in D$. If $d \in E_\varphi$, then the germ V is singular and the germ $F_\varphi^{-1}V$ at $(0,d)$ contains D and at least one other branch. Then for every point $a \in V$ close enough to $F_\varphi(0,d)$ we have

$$\deg_{(0,d)} F_\varphi = \sum_{u \in B,\, F_\varphi(u) = a} \deg_u F_\varphi > k\mu_\varphi,$$

where k is the degree of the restriction $F_\varphi : D \to V$. ∎

REMARK. In the case $i_\varphi > 1$ we always have $0 \in E_\varphi$ and $\deg_{(0,0)} F_\varphi > i_\varphi \mu_\varphi$ (by Lemma 4.3).

THEOREM 4.4. *Let* $f = (P,Q)$ *be a non-zero constant Jacobian polynomial map of* $\mathbb{C}^2$, *monic in* y. *Then*

$$E_f = \bigcup_{[\varphi] \in \Pi_f} C_{[\varphi]},$$

where:

(E1) $\dfrac{\deg P_\varphi}{\deg Q_\varphi} = \dfrac{\deg P}{\deg Q}$.

(E2) *If* $i_\varphi > 1$, *then* $0 \in E_\varphi$ *and* $(P_\varphi, Q_\varphi)(0)$ *is always a singular point of* $(P_\varphi, Q_\varphi)(D)$ *for every neighborhood* D *of zero.*

(E3) *Every curve* $C_{[\varphi]}$ *has a singularity.*

Proof. The representation of E_f is obtained by definition of Π_f and Proposition 2.1. Properties (E1) and (E2) are stated in Theorem 3.6(ii) and the remark after Lemma 4.3. To see (E3), if $E_f \neq \emptyset$, in view of Jung's theorem on automorphisms of $\mathbb{C}^2$, in suitable coordinates f can be represented as $f = (p, q)$ with $\deg p = kd$, $\deg q = ke$, $d > e > 1$ and $\gcd(d, e) = 1$. If a curve $C_{[\varphi]}$ is smooth, the corresponding map (p_φ, q_φ) is a regular embedding of $\mathbb{C}$ into $\mathbb{C}^2$. Therefore, by the Abhyankar–Moh–Suzuki theorem ([AM], [Su]) either $\deg p_\varphi$ divides $\deg q_\varphi$ or vice versa. This contradicts (E1), since $d > e > 1$. ∎

4.3. *Generic fibers of components of* f. Remember that the generic fiber of P and of Q is a connected Riemann surface with a finite number of punctures, since P and Q are primitive. Denote by g_P and n_P the genus and the number of punctures of F_P. We will determine the Euler–Poincaré characteristic χ_P and topological type (g_P, n_P) of the generic fiber of P by calculating those numbers for a generic level $P = c$.

Consider the curve $P = c$. Set

$$b_P^\infty(c) := \{\gamma \in B_P(c) : \lim_{\gamma \ni (x,y) \to \infty} f(x, y) = \infty\},$$

$$b_P(c) := \{\gamma \in B_P(c) : \lim_{\gamma \ni (x,y) \to \infty} f(x, y) \text{ is finite}\}.$$

The number of punctures of the curve $P = c$ is equal to $b_P^\infty(c) + b_P(c)$. Let V be the normalization of $P = c$, which is a smooth compact Riemann surface, and let $q : V \to \mathbb{CP}^1$ be the regular extension of the restriction of Q to the curve $P = c$ over V. Remember that the curve $P = c$ is embedded into V so that $V - \{P = c\}$ consists of $\#B_P(c)$ distinct points and each branch $\gamma \in B_P(c)$ can be viewed as a punctured disk centered at a point z_γ of $V - \{P = c\}$. The branched set of q is contained in $V - \{P = c\}$ and the local degree $\deg_{z_\gamma} q$ is equal to $\deg_\gamma f$. Furthermore, the topological degree $\deg_{\mathrm{geo}} q$ is equal to $\deg_{\mathrm{geo}} f$. Then, by the Riemann–Hurwitz relation (see [BK]),

$$\begin{aligned}\chi(V) &= 2\deg_{\mathrm{geo}} q - \sum_{z \in V} (\deg_z q - 1)\\ &= 2\deg_{\mathrm{geo}} f - \sum_{\gamma \in b_P^\infty(c)} (\deg_\gamma f - 1) - \sum_{\gamma \in b_P(c)} (\deg_\gamma f - 1)\\ &= \deg_{\mathrm{geo}} f + \#b_P^\infty(c) + \#b_P(c) - \sum_{\gamma \in b_P(c)} \deg_\gamma f,\end{aligned}$$

where $\chi(\cdot)$ indicates the Euler–Poincaré characteristic. Thus, we obtain the formula

$$\chi(\{P = c\}) = \deg_{\mathrm{geo}} f - \sum_{\gamma \in b_P(c)} \deg_\gamma f. \tag{4.4}$$

For the number $\#b_P^\infty(c)$, by Corollary 3.7 we have

$$\#b_P^\infty(c) \equiv \#\mathrm{Pol}(f,P) \quad \text{for all } c \in \mathbb{C}. \tag{4.5}$$

Now, we assume that $\{P = c\}$ is a generic level, i.e. $c \notin E_P$. For the number $b_P(c)$, by Theorems 3.4(ii) and 2.4 we have

$$\#b_P(c) = d \sum_{[\varphi]\in\Pi_f} \frac{\gcd(\deg P_{[\varphi]}, \deg Q_{[\varphi]})}{i_{[\varphi]}} \quad \text{for } c \notin E_P. \tag{4.6}$$

Furthermore, by definition, Theorem 2.4 and Lemma 4.2 we can verify that

$$\sum_{\gamma\in b_P(c)} \deg_\gamma f = \sum_{[\varphi]\in\Pi_f} \sum_{\gamma\in B_P([\varphi],c)} \deg_\gamma f = \sum_{[\varphi]\in\Pi_f} \frac{\deg P_{[\varphi]}}{i_{[\varphi]}} \mu_{[\varphi]}.$$

Therefore, we get

$$\sum_{\gamma\in b_P(c)} \deg_\gamma f = d \sum_{[\varphi]\in\Pi_f} \frac{\gcd(\deg P_{[\varphi]}, \deg Q_{[\varphi]})}{i_{[\varphi]}} \mu_{[\varphi]} \quad \text{for } c \notin E_P \tag{4.7}$$

since $\deg P_\varphi = d \gcd(\deg P_\varphi, \deg Q_\varphi)$ by Theorem 3.6(ii).

Thus, from (4.4)–(4.7) we obtain

Theorem 4.5.

$$n_P = \#\mathrm{Pol}(f,P) + d \sum_{[\varphi]\in\Pi_f} \frac{\gcd(\deg P_{[\varphi]}, \deg Q_{[\varphi]})}{i_{[\varphi]}},$$

$$\chi_P = \deg_{\mathrm{geo}} f - d \sum_{[\varphi]\in\Pi_f} \frac{\gcd(\deg P_{[\varphi]}, \deg Q_{[\varphi]})}{i_{[\varphi]}} \mu_{[\varphi]},$$

$$2 - 2g_P = \deg_{\mathrm{geo}} f + \#\mathrm{Pol}(f,P) - d \sum_{[\varphi]\in\Pi_f} \frac{\gcd(\deg P_{[\varphi]}, \deg Q_{[\varphi]})}{i_{[\varphi]}} (\mu_{[\varphi]} - 1).$$

Proof of Theorem C. By applying Theorem 4.5 to P and Q we get the desired formula

$$\deg P(\deg_{\mathrm{geo}} f - \chi_Q) = \deg Q(\deg_{\mathrm{geo}} f - \chi_P).$$

If $\chi_P = \chi_Q$ but $\deg P \neq \deg Q$, by the above formula we get $\deg_{\mathrm{geo}} f = \chi_P \leq 1$. Therefore, since $\chi_P = 2 - 2g_P - n_P \leq 1$, we conclude that $\chi_P = \deg_{\mathrm{geo}} f = 1$ and f is an automorphism. ∎

Abhyankar [A] proved that a non-zero constant Jacobian polynomial map $f = (P,Q)$ is an automorphism if some level set of P has only one irreducible branch at infinity. Later, Drużkowski [D1] proved that $f = (P,Q)$ is an automorphism if $n_P \leq 2$. In view of Theorem 4.5 either $n_P = 1$ or $n_P \geq 1 + d$. In fact, we have

THEOREM 4.6. *A non-zero constant Jacobian polynomial map $f = (P, Q)$ of $\mathbb{C}^2$ is an automorphism if one of the following holds*:

(i) *The curve $Q = 0$ has at most two punctures.*

(ii) *The generic fiber of Q has at most three punctures.*

Proof. Let $c \in \mathbb{C}$. By the above-mentioned results in [A] and [D1], Theorem 4.5 and the fact that $b_Q^\infty(c) \geq 1$ it is sufficient to consider the following situations: (1) $\#b_Q(c) = 1$; (2) $\#b_Q(c) = 2$ and $c \notin E_Q$. In both, by Theorems 4.4 and 4.5, we can verify that $\Pi_f = \{[\varphi]\}$ and Q_φ has only one critical point. It follows that Q has only one exceptional value. Such a situation is impossible by the following.

PROPOSITION 4.7. *Let $f = (P, Q)$ be a non-zero constant Jacobian polynomial map of $\mathbb{C}^2$. Then the number of exceptional values of Q must be different from one.*

Proof. It is well known from [Su] that for every primitive polynomial g in $\mathbb{C}^2$,

$$\sum_{c \in E_g} (\chi\{g = c\} - \chi_g) = 1 - \chi_g. \tag{4.8}$$

If $E_Q = \{c\}$, then f is not bijective and $\chi\{Q = c\} = 1$. Since $\{Q = c\}$ is smooth, it has a unique component diffeomorphic to $\mathbb{C}$. Then, by applying the Abhyankar–Moh–Suzuki Theorem ([AM], [Su]) on embeddings of $\mathbb{C}$ into $\mathbb{C}^2$ one can verify that the curve $\{P = c\}$ is isomorphic to $\mathbb{C}$ and f is an automorphism (see, for example, Theorem 5.6 of [Ca] or Lemma 2.2 of [C1]). Hence, we get a contradiction. ∎

Concerning the topological type of the generic fiber of the polynomials $aP + bQ$ we have

THEOREM 4.8. *For a non-zero constant Jacobian polynomial map $f = (P, Q)$ of $\mathbb{C}^2$ the generic fibers of the polynomials $aP+bQ$ with $\deg(aP + bQ) = \max\{\deg P, \deg Q\}$ have the same topological type (g, n) with $n \neq 2, 3, 4, 5$.*

Proof. First, we prove that the generic fibers of the polynomials $aP+bQ$ with $\deg(aP + bQ) = \max\{\deg P, \deg Q\}$ have the same topological type. To do it we need only prove that the generic fibers of P and Q have the same topological type if $\deg P = \deg Q$. Assume that $\deg P = \deg Q$. From Lemma 3.8(ii) and Theorem 3.6(i) we have $i_\varphi = 1$ and $\deg P_\varphi = \deg Q_\varphi$ for all $[\varphi] \in S_{(P,Q)}$. Then by Theorem 3.4 and Lemma 3.8(i) we get

$$\#\mathrm{Pol}(f, P) = \sum_{[\varphi] \in S_{(P,Q)}} \# \deg P_{[\varphi]} = \sum_{[\varphi] \in S_{(P,Q)}} \# \deg Q_{[\varphi]} = \#\mathrm{Pol}(f, Q).$$

This together with the formulas of Theorem 4.5 proves that $(g_P, n_P) = (g_Q, n_Q)$.

Now, assume that $\deg P > \deg Q$ and (g_P, n_P) is the topological type of the generic fiber of P. Assuming $n_P \neq 1$, we want to show that $n_P > 5$. From Theorem 4.5 we know that

$$n_P = \#\mathrm{Pol}(f,P) + dD, \qquad n_Q = \#\mathrm{Pol}(f,Q) + eD,$$

where

$$D := \sum_{[\varphi]\in \Pi_f} \frac{\gcd(\deg P_{[\varphi]}, \deg Q_{[\varphi]})}{i_{[\varphi]}}.$$

As in the proof of Proposition 4.7, $dD > 2$ and $eD > 2$. Therefore, we get $dD > 3$ and $n_P > 4$, since $d > e$. If $n_P = 5$, we have to consider only the case $dD = 4$. In this case $\#\mathrm{Pol}(f,P) = 1$, $d = 4$, $D = 1$ and $e = 3$, since $eD > 2$. Lemma 3.8 shows that $\#\mathrm{Pol}(f,Q) = 1$. Then by Lemma 3.9 we get the contradiction that $e = 1$. Thus, we proved $n_P > 5$. ∎

Let us conclude the paper with a remark on the geometric degree of f.

Remark 4.9. In [O1] Orevkov constructed a a reduction f^* of a regular extension of f, $f^* : (\mathbb{C}^2 \sqcup D \sqcup \{\infty\}, D, \infty) \to (\mathbb{C}^2 \sqcup \{\infty\}, E_f, \infty)$, where D is the union of a finite number of curves homeomorphic to $\mathbb{C}$. From this Orevkov obtained a nice formula for the geometric degree of f:

$$\deg_{\mathrm{geo}} f - 1 = \sum_{l\subset D} \Big[\mu_l + \sum_{u\in l} (\deg_u f^* - \mu_l)\Big] \tag{4.9}$$

(Lemma 4.2 of [O1]), where μ_l is the degree $\deg_u f^*$ for generic $u \in l$. Using this formula, he checked (JC_2) for the cases $\deg_{\mathrm{geo}} f \le 5$ ([O1, O2]). It is possible to rewrite Orevkov's formula in terms of the data of Π_f and to obtain the following formula:

$$\begin{aligned}\deg_{\mathrm{geo}} f - 1 \\ = \sum_{[\varphi]\in \Pi_f} i_{[\varphi]}^{-1} \Big[\deg_{(0,0)} F_{[\varphi]} + \sum_{d\in E_{[\varphi]},\, d\neq 0} (\deg_{(0,d)} F_{[\varphi]} - \mu_{[\varphi]})\Big],\end{aligned} \tag{4.10}$$

where in the summation $F_{[\varphi]} := F_\varphi$ for a series $\varphi \in [\varphi]$. This formula can also be obtained in the way used for Theorem 4.5 and the equalities (4.4) and (4.8) on Euler–Poincaré characteristics. By using (4.10) and Theorem 4.4 we can claim that *the branched value set E_f can never be the image of an imbedding of $\mathbb{C}$ into $\mathbb{C}^2$*. It is worth noting here that the curve E_f can never be the image of a polynomial injection from $\mathbb{C}$ into $\mathbb{C}^2$ ([C1]) and that there exists the so-called Vitushkin covering, a branched finite covering between two complex manifolds homeomorphic to $\mathbb{R}^4$ with branched value set diffeomorphic to $\mathbb{R}^2$ ([Vi] and [O3]). As part of (JC_2), it is conjectured that *the exceptional value set E_f of a non-zero constant Jacobian polynomial map f in $\mathbb{C}^2$ can never be an irreducible curve.*

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