ANNALES
POLONICI MATHEMATICI
LXXII.1 (1999)

On the existence of curves in $\mathbb{P}^n$ with stable normal bundle

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Abstract. We prove that for integers n, d, g such that $n \geq 4$, $g \geq 2n$ and $d \geq 2g + 3n + 1$, the general (smooth) curve C in $\mathbb{P}^n$ with degree d and genus g has a stable normal bundle N_C.

Introduction. Let C be a smooth projective curve. It is natural to ask for which triples (n, d, g) of integers there exist smooth curves C in $\mathbb{P}^n$ of degree d and genus g with a stable normal bundle N_C.

For $n = 3$, Ellingsrud and Hirschowitz proved in [9] that there exist a lot of space smooth curves having a stable normal bundle.

Here, for $n \geq 4$, we will prove the following result:

Theorem 1. *Let n, d, g be integers with $n \geq 4$, $g \geq 2n$ and $d \geq 2g + 3n + 1$. Then the general* (*smooth*) *curve $C \subset \mathbb{P}^n$ of degree d and genus g has a stable normal bundle N_C.*

As in [9], we use smoothable reducible nodal curves X having a stable normal bundle N_X. Of course for $n > 3$ the study of stability of the normal bundle N_X is more complicated than in the case $n = 3$.

The normal bundle of a general rational curve $D \subset \mathbb{P}^n$ of degree $d \geq n$ and the normal bundle of a linearly normal elliptic curve $Y \subset \mathbb{P}^n$ of degree $n + 1$ are well known (see e.g. [15] and [5]). Therefore we use a nodal curve X whose irreducible components are linearly normal elliptic curves and rational curves. Bundles on rational and elliptic curves are rather familiar. For bundles on elliptic curves we also use a recent result obtained in [2]. To check the stability of a vector bundle on a reducible nodal curve X we use a result of [3].

We work over an algebraically closed field $\mathbf{k}$ with $\mathrm{char}(\mathbf{k}) = 0$.

1991 *Mathematics Subject Classification*: 14H10, 14H60, 14C05.

Key words and phrases: bundles on smooth curves, nodal curves, bundles on nodal curves, normal bundle, elementary transformations of bundles on smooth curves.

Both authors were partially supported by MURST and GNSAGA of CNR (Italy).

1. Notations. Let C be a smooth projective curve, $P \in C$ and E, F vector bundles on C.

Call $\mu(E) := \deg(E)/\mathrm{rank}(E)$ the *slope* of E. The bundle E is called *stable* (resp. *semistable*) if for all proper subbundles G of E we have $\mu(G) < \mu(E)$ (resp. $\mu(G) \le \mu(E)$). The bundle E is called *polystable* if it is a direct sum of stable vector bundles with the same slope. Hence a polystable bundle is semistable. A polystable vector bundle E is called *superpolystable* if no two among the indecomposable factors of E are isomorphic.

We will say that F is *obtained from E by making a positive elementary transformation supported by $P \in C$* if E and F fit in an exact sequence $0 \to E \to F \to \mathcal{O}_P \to 0$, where $\mathcal{O}_P$ is the skyscraper sheaf on C supported by P. Note that in this case we have $\mathrm{rank}(F) = \mathrm{rank}(E)$ and $\deg(F) = \deg(E)+1$.

Dualizing the above exact sequence, we obtain the exact sequence $0 \to F^\vee \to E^\vee \to \mathcal{O}_P \to 0$. Then F is uniquely determined by E and a point $v \in \mathbb{P}(E^\vee)_P$.

More generally, we can say that F is *obtained from E by making a positive elementary transformation supported by a* 0*-dimensional subscheme S of C* if E and F fit in an exact sequence $0 \to E \to F \to \mathcal{O}_S \to 0$.

If the 0-dimensional scheme $S \subset C$ is of length s, then F is obtained from E by making s positive elementary transformations.

Remark 1.1. We will use the following parameter spaces for finite sequences of positive elementary transformations of a fixed vector bundle E on C:

(i) There is an integral quasi-projective variety parametrizing sequences of s positive elementary transformations supported by s different points varying in C.

(ii) Fix s distinct points $P_1, \ldots, P_s$ of C. The space of bundles obtained from E by making s positive elementary transformations supported respectively by $P_1, \ldots, P_s$ is a closed irreducible subset of the space considered in (i).

(iii) We fix a bundle F obtained from E by making s positive elementary transformations. We take a local deformation space of F as parameter space, having an open and dense subset which parametrizes bundles in (i).

A reduced curve X is called a *nodal curve* if the only singularities of X are ordinary nodes. We will use only nodal curves with smooth irreducible components.

Let X be a nodal curve in $\mathbb{P}^n$. Then its normal sheaf $N_X := (\mathcal{I}/\mathcal{I}^2)^\vee$ is locally free of rank $n-1$ and degree $\deg(N_X) = (n+1)\deg(X) + 2p_a(X) - 2$.

Positive elementary transformations are involved in the description of the normal bundle N_X. In fact, if $X = Y_1 \cup Y_2$ is a nodal curve, then the normal bundle N_X is a glueing of $N_{X|Y_1}$ and $N_{X|Y_2}$. Moreover, for $i = 1, 2$,

$N_{X|Y_i}$ is obtained from N_{Y_i} by making $s = \mathrm{card}(Y_1 \cap Y_2)$ positive elementary transformations supported by the points of $Y_1 \cap Y_2$; at every $P \in Y_1 \cap Y_2$ the positive elementary transformation needed to obtain $N_{X|Y_i}$ from N_{Y_i} is given by the plane K determined by the tangent lines of Y_1 and Y_2 at P (see [12], Cor. 3.2, Prop. 3.3 and their proofs).

The definition of stability and semistability of a vector bundle on a smooth curve C is extended in a similar way to a vector bundle on a reducible nodal curve X (see e.g. [16]).

In the following we denote by rF the direct sum of r copies of the bundle F and by $[x]$ the integer part of a real number x.

2. Preliminary remarks on rational and elliptic curves. We want to prove a result due to Sacchiero (see [15]), i.e. Proposition 2.2 below.

We need the following trivial extension of the terminology and proof of [14], Prop. 1.3.5 and Prop. 2.1.4.

LEMMA 2.1. *Let C be a smooth curve in $\mathbb{P}^n$, $n \geq 3$, $P \in C$ and let D be a line passing through P different from the tangent line T_PC to C at P. Set $X := C \cup D$. Denote by K the plane defined by the lines D and T_PC. Let M be the maximal line subbundle of N_D passing through K and L a line subbundle of N_C. Let G be the rank 1 saturated subsheaf of N_X with $L \subset G_{|C}$ and $M \subset G_{|D}$.*

(a) *If L does not pass through K, then* $\deg(G) = \deg(L)+1$. *If L passes through K and does not glue together with M at P in N_X, then* $\deg(G) = \deg(L)+2$; *if L and M glue together at P in N_X, then* $\deg(G) = \deg(L)+3$.

(b) *If P is a general point of C and D is a general line passing through P, then L and M do not glue together at P in N_X.*

Recall that the normal bundle of a line D in $\mathbb{P}^n$ is $N_D \cong (n-1)\mathcal{O}_{\mathbb{P}^1}(1)$.

PROPOSITION 2.2 (Sacchiero [15]). *Fix integers n, d with $d \geq n \geq 3$. Let $C \subset \mathbb{P}^n$ be a general rational curve of degree d. Then the normal bundle N_C is rigid. More precisely, we have $N_C \cong r\mathcal{O}_{\mathbb{P}^1}(a+1) \oplus (n-1-r)\mathcal{O}_{\mathbb{P}^1}(a)$, where the integers r and a are such that $(n+1)d-2 = a(n-1)+r$, $0 \leq r \leq n-2$.*

Proof. STEP 1. First we prove the proposition for $d=n$. We use induction on n. The case $n = 3$ is classical (see e.g. [7], [8] or [10]). Now assume $n \geq 4$ and that the assertion is true in $\mathbb{P}^{n-1}$. Let H be a hyperplane of $\mathbb{P}^n$ and $Y \subset H$ a rational normal curve of degree $n-1$ contained in H. By the inductive assumption, $N_{Y/H} \cong (n-2)\mathcal{O}_{\mathbb{P}^1}(n+1)$.

Let P be a general point of Y and D a general line of $\mathbb{P}^n$ passing through P. Then $X := Y \cup D$ is smoothable to a degree n rational normal curve C in $\mathbb{P}^n$. By the openness of semistability (see e.g. [13], Thm. 2.4), it suffices to prove that N_X is semistable.

We have $N_Y \cong N_{Y/H} \oplus \mathcal{O}_Y(1) \cong (n-2)\mathcal{O}_{\mathbb{P}^1}(n+1) \oplus \mathcal{O}_{\mathbb{P}^1}(n-1)$. Let K be the plane defined by D and the tangent line of Y at P. Since P and D are general, by Lemma 2.1 the maximal line subbundle M of N_D passing through the plane K does not glue together with a maximal line subbundle of N_Y. Hence N_X is semistable.

Step 2. We use induction on d. For $d = n$ the assertion is proved in Step 1. Assume $d > n$ and the result true for the general rational curve Y in $\mathbb{P}^n$ with degree $d-1$. Take a general point P of Y and a general line D passing through P. The nodal curve $X := Y \cup D$ is smoothable to a degree d smooth rational curve in $\mathbb{P}^n$. We have to prove that the Harder–Narasimhan polygon of N_X (see e.g. [4] and also [13] for its definition) is general.

By the inductive assumption, $N_Y \cong r'\mathcal{O}_{\mathbb{P}^1}(a'+1) \oplus (n-1-r')\mathcal{O}_{\mathbb{P}^1}(a')$, where the integers r' and a' are such that $(n+1)(d-1)-2 = a'(n-1)+r'$ and $0 \le r' \le n-2$.

Let K be the plane defined by D and the tangent line of Y at P. Since P and D are general, $N_{X|Y}$ is a general positive elementary transformation of N_Y and hence $N_{X|Y}$ is rigid. Also $N_{X|D}$ is rigid. With the terminology of [14], N_X is a glueing of $N_{X|Y}$ and $N_{X|D}$.

If $r' = n-2$, we are done. Let $r' \le n-3$. Since D is general, the (maximal) line subbundles of N_Y with degree $a'+1$ do not pass through K and the maximal line subbundle M of N_D passing through K does not glue together with a degree a' line subbundle of N_Y passing through K. Thus the Harder–Narasimhan polygon of N_X is general.

Remark 2.3. Let C be a general rational curve in $\mathbb{P}^n$ of degree $d \ge n \ge 3$. Let r be the integer defined in Proposition 2.2. Take $t := n-1-r$ for $0 \le r \le n-2$. Then the bundle obtained from N_C by making t general positive elementary transformations is semistable.

Write $d = n+\beta+(n-1)\gamma$ with $\beta, \gamma \in \mathbb{N}$ and $0 \le \beta \le n-2$. The above integer t depends only on β and n, in fact it is equal to

$$t_\beta := \begin{cases} -2\beta+n-1 & \text{if } 0 \le \beta \le [n/2]-1, \\ -2\beta+2(n-1) & \text{if } [n/2] \le \beta \le n-2. \end{cases}$$

For elliptic curves we have the following result:

Proposition 2.4 (Ein–Lazarsfeld [5]). *A linearly normal elliptic curve C in $\mathbb{P}^n$ has a semistable normal bundle.*

Remark 2.5. The above result is the case $i=1$ of the Corollary in [5]. The authors of [5] wrote in the introduction of that paper that this particular case of their Corollary was due to Ellingsrud.

For an elliptic curve C, the vector bundles on C were classified by Atiyah [1]. For all integers r, s with $r > 0$, there are polystable bundles of rank r

and degree s. A semistable bundle E on C is stable if and only if $\deg(E)$ and $\operatorname{rank}(E)$ are coprime.

Let C be a linearly normal elliptic curve in $\mathbb{P}^n$. Then C is of degree $n+1$, its normal bundle N_C is a semistable bundle of rank $n-1$ and degree $(n+1)^2$. Therefore the normal bundle N_C is stable if and only if n is even.

LEMMA 2.6. *Let C be an elliptic curve and E a superpolystable bundle on C. Then the bundle F obtained from E by making s general positive elementary transformations is superpolystable.*

Proof. Note that for all integers r, t with $r > 0$, there is a superpolystable bundle on C of rank r and degree t. In fact, let $G = \sum_{i=1}^{m} G_i$ be a polystable bundle of rank r and degree t on C with each G_i stable. Take m general line bundles $L_1, \ldots, L_m$ in $\operatorname{Pic}^0(C) \cong C$. Then $G' = \sum_{i=1}^{m} G_i \otimes L_i$ is superpolystable.

Put $r = \operatorname{rank}(E)$ and $d = \deg(E)$. Let F_0 be a superpolystable bundle on C of rank r and degree $d+s$. By the Riemann–Roch Theorem, $\operatorname{Hom}(E, F_0) \cong H^0(C, E^\vee \otimes F_0) \neq 0$ and, from Thm. 1 of [2], a general $f \in \operatorname{Hom}(E, F_0)$ is injective.

Then we have an exact sequence $0 \to E \to F_0 \to \mathcal{O}_{S_0} \to 0$, i.e. F_0 is a positive elementary transformation of E supported by a 0-dimensional subscheme S_0 of C of length s. By the openness of superpolystability, we have the assertion (see Remark 1.1).

3. Proof of Theorem 1. We use the following result contained in [3], Lemma 1.1:

LEMMA 3.1. *Let X be a nodal curve whose irreducible components $Y_1, \ldots, Y_m$ are smooth. Let E be a bundle on X such that $E_{|Y_i}$ is semistable for every $i = 1, \ldots, m$ and moreover $E_{|Y_1}$ is stable. Then the bundle E is stable.*

We recall the following result of Eisenbud and Harris on the rational normal curve:

LEMMA 3.2 ([6], Thm. 1(b)). *Let Γ be a 0-dimensional subscheme of $\mathbb{P}^n$ in linearly general position (i.e. for every proper linear subspace $\Lambda \subset \mathbb{P}^n$ the length of $\Lambda \cap \Gamma$ is $\leq 1 + \dim(\Lambda)$). If Γ is of length $n+3$, then Γ is contained in a unique rational normal curve of degree n.*

LEMMA 3.3. *Let n be even and $n \geq 4$. Consider integers $\alpha, \beta, \gamma \in \mathbb{N}$ with $\alpha \geq 3$ and $0 \leq \beta \leq n-2$. Put*

$$d = (n+1)\alpha + n + \beta + (n-1)\gamma$$

and let t_β be the integer defined in Remark 2.3. *Then*, *for every integer g such that*

$$n + t_\beta + \alpha - 4 \leq g \leq n - 1 + t_\beta + (\alpha - 3)(n/2 + 1),$$

there exists a smooth curve C in $\mathbb{P}^n$ of degree d and genus g having a stable normal bundle N_C.

Proof. It is sufficient to exhibit a smoothable nodal curve X of degree d and arithmetic genus g with a stable normal bundle.

We consider the following types of "polygonal" curves X. The irreducible components of X are α linearly normal elliptic curves $Y_1, \ldots, Y_\alpha$ and a general rational curve D of degree $n + \beta + (n-1)\gamma$.

The curve D intersects Y_1 in ν_1 points, Y_{i-1} intersects Y_i in ν_i points for $2 \leq i \leq \alpha$, Y_α intersects D in $\nu_{\alpha+1}$ points, and there are no further intersections.

We put the following conditions on the intersections: $\nu_i \geq 1$ for every $1 \leq i \leq \alpha$, $\nu_{\alpha+1} \geq 0$, $\nu_i \leq n/2 + 1$ for every $1 \leq i \leq \alpha + 1$, and moreover $t_\beta = \nu_1 + \nu_{\alpha+1}$ and $n - 1 = \nu_2 + \nu_3$.

Note that $1 \leq t_\beta \leq n - 1$ (Remark 2.3).

Now we show that the above curve X exists. It is sufficient to consider nodal reducible curves Y_i of arithmetic genus 1 and degree $n+1$ that are the union of a normal rational curve D_i of degree n and a bisecant line ℓ_i.

Consider the rational curve D and fix ν_1 general points of D. Since $\nu_1 < n + 3$, the scheme Σ of degree n rational curves passing through the above ν_1 points is of dimension $(n^2 + 2n - 3) - (n-1)\nu_1 > 0$. Moreover the curves of Σ meeting the curve D give a scheme Σ' of dimension $(n^2 + 2n - 3) - (n-1)(\nu_1 + 1) + 1$. Then the general curve D_1 of Σ intersects D exactly in ν_1 points. Now consider a general bisecant ℓ_1 of D_1. The line ℓ_1 is not a tangent line of D_1 and does not intersect the curve D.

By proceeding in this way, we can construct a "polygonal" configuration $D \cup (D_1 \cup \ell_1) \cup \ldots \cup (D_{\alpha-1} \cup \ell_{\alpha-1})$ satisfying the above conditions on the intersections.

Now take ν_α general points of $D_{\alpha-1}$ and $\nu_{\alpha+1}$ general points of D. Since $\nu_\alpha + \nu_{\alpha+1} < n+3$, a general degree n rational curve D_α passing through the above $\nu_\alpha + \nu_{\alpha+1}$ points does not intersect the curves of the configuration in further points. We conclude by taking a general bisecant ℓ_α of D_α.

Since $\mathrm{card}((\bigcup_{j=1}^{i-1} Y_j) \cap Y_i) \leq n + 1$ for $2 \leq i \leq \alpha$, and $\mathrm{card}((\bigcup_{j=1}^{\alpha} Y_j) \cap D) \leq n + 1$, we see that X is smoothable ([12]).

Note that X is of degree d and genus $g = n - 1 + t_\beta + \sum_{i=4}^{\alpha} \nu_i$.

From Remark 2.3 we know that the bundle on the rational curve D obtained from the normal bundle N_D by making t_β general positive elementary transformations is semistable.

Since $\nu_1, \nu_{\alpha+1} \leq n/2+1$, given $t_\beta = \nu_1 + \nu_{\alpha+1}$ general points $P_1, \ldots, P_{t_\beta}$ of D and for each P_j a general line ℓ_j passing through it, with $j = 1, \ldots, t_\beta$, there exists a "polygonal" curve X of the above type such that the tangent lines of Y_1 and Y_α at $P_1, \ldots, P_\alpha$ are $\ell_1, \ldots, \ell_\alpha$ (that is a consequence of Lemma 3.2).

So there exists a nodal curve X of the above type such that $N_{X|D}$ is semistable.

The normal bundle N_{Y_i} of the linearly normal elliptic curve Y_i is stable (see Remark 2.5). Thus for every positive integer s the bundle obtained from N_{Y_i} by making s general positive elementary transformations is semistable (Lemma 2.6). Given $\nu_i + \nu_{i+1}$ general points of Y_i (with $\nu_i, \nu_{i+1} \leq n/2+1$) and for each of them a general line passing through it, there exists a nodal curve X of the above type such that Y_{i-1} and Y_{i+1} (put $Y_0 = D = Y_{i+1}$) intersect Y_i in the given points and have the given lines as tangent lines at those points (Lemma 3.2).

Thus, for every $1 \leq i \leq \alpha$, there exists a nodal curve X of the above type such that $N_{X|Y_i}$ is semistable.

Moreover $\nu_2 + \nu_3 = n-1$ and so $\deg(N_{X|Y_2}) = \deg(N_{Y_2}) + n - 1 = (n+1)^2 + n - 1$. Hence $\deg(N_{X|Y_2})$ and $\operatorname{rank}(N_{X|Y_2})$ are coprime, and thus $N_{X|Y_2}$ is stable.

As our "polygonal" curve X varies in an irreducible scheme, from the openness of semistability and stability we deduce that the general nodal curve X of the above type has a normal bundle N_X whose restriction to each irreducible component is semistable and to one irreducible component is stable.

Then, by Lemma 3.1, for such a nodal curve X the normal bundle N_X is stable. By the openness of stability (see e.g. [13]), we have the assertion.

Proof of Theorem 1 for n even. We use the notations of Lemma 3.3. Note that

$$\alpha \leq \alpha_d := \left[\frac{d-n}{n+1}\right].$$

Since $1 \leq t_\beta \leq n-1$, for $5 \leq \alpha \leq \alpha_d$ and for every integer g such that $2n + \alpha - 5 \leq g \leq 2n + (\alpha - 5)n/2 + \alpha - 3$, by Lemma 3.3 the pair (d, g) satisfies the assertion of the theorem, i.e. there exists a smooth curve in $\mathbb{P}^n$ of degree d and genus g having a stable normal bundle.

Thus for $d \geq 6n+5$ and

$$2n \leq g \leq 2n + (\alpha_d - 5)n/2 + \alpha_d - 3$$

the pair (d, g) satisfies the assertion of the theorem. We have $d - n = (n+1)\alpha_d + r_d$ with $0 \leq r_d \leq n$. The last displayed inequality is equivalent to

$$d \geq 2g + 2n + r_d + 5 - \frac{2g+4}{n+2} := d(g, n).$$

Since $g \geq 2n$ and $r_d \leq n$, we have $2g + 3n + 1 \geq d(g, n)$ and Theorem 1 for n even is proved.

LEMMA 3.4. *Fix an odd integer $n \geq 5$ and let H be a hyperplane of $\mathbb{P}^n$. Let $C \subset H$ be a linearly normal elliptic curve contained in H. Let $s(5) := 4$ and $s(n) := 3$ for every $n \geq 7$. Then, for every integer $s \geq s(n)$, the bundle obtained from the normal bundle N_C of C in $\mathbb{P}^n$ by making s general positive elementary transformations is semistable.*

Proof. Denote by $N_{C/H}$ the normal bundle of C in H. We have $N_C \cong N_{C/H} \oplus \mathcal{O}_C(1)$ and by Remark 2.5 the bundle $N_{C/H}$ is stable.

Let t be an integer such that

$$\frac{t}{n-1} > \frac{n^2}{n-2} = \mu(N_{C/H}).$$

Let F be a superpolystable bundle on C of degree t and rank $n-1$ (see the proof of Lemma 2.6).

By the Riemann–Roch Theorem, $\mathrm{Hom}(N_{C/H}, F) \cong H^0(C, N_{C/H}^{\vee} \otimes F) \neq 0$ and, by [2], Thm. 1, a general $f \in \mathrm{Hom}(N_{C/H}, F)$ is injective and such that $\mathrm{coker}(f)$ is locally free.

Since $F(-1)$ and $N_{C/H}(-1)$ are semistable with degree > 0, we also have $h^1(C, F(-1)) = h^1(C, N_{C/H}(-1)) = 0$ (see [1]).

Hence by the Riemann–Roch Theorem and the assumptions on t, we have $h^0(C, \mathrm{Hom}(\mathcal{O}_C(1), F)) > h^0(C, \mathrm{Hom}(\mathcal{O}_C(1), N_{C/H}))$ and there exists a map $g : \mathcal{O}_C(1) \to F$ which does not factor through $f(N_{C/H})$, where f is the map described above.

Thus the map $(f, g) : N_C \cong N_{C/H} \oplus \mathcal{O}_C(-1) \to F$ has generic rank $n - i$ and it gives an exact sequence $0 \to N_C \xrightarrow{(f,g)} F \to \mathcal{O}_S \to 0$, where S is a 0-dimensional subscheme of C of length $s = \deg(F) - \deg(N_C) = t - n(n+1)$.

On the other hand, the superpolystable bundle F is obtained from N_C by making a positive elementary transformation supported by a 0-dimensional subscheme of C of length $s = t - n(n+1)$.

By the openness of superpolystability, the bundle obtained from N_C by making $s = t - n(n+1)$ general positive elementary transformations is superpolystable, and hence semistable. Note that $\frac{t}{n-1} > \frac{n^2}{n-2}$ if and only if $s = t - n(n+1) > 2 + \frac{4}{n-2}$, i.e. $s \geq s(n)$.

LEMMA 3.5. *Let $n \geq 5$ be an odd integer. Consider integers $\alpha, \beta, \gamma \in \mathbb{N}$ with $\alpha \geq 3$ and $0 \leq \beta \leq n-2$. Let*

$$d = (n+1)\alpha - 1 + n + \beta + (n-1)\gamma.$$

Put $s_n := n - 2$ for $n \geq 7$ and $s_5 := 5$. Consider the integer t_β defined in

Remark 2.3. *Then for every integer g with*

$$s_n + t_\beta + \alpha - 3 \leq g \leq s_n + t_\beta + (\alpha - 3)\left(\frac{n+1}{2} + 1\right),$$

there exists a smooth curve C in $\mathbb{P}^n$ of degree d and genus g having a stable normal bundle N_C.

Proof. As in the proof of Lemma 3.3, we consider smoothable nodal curves X that are "polygonal". In this case the irreducible components of X are $(\alpha - 1)$ linearly normal elliptic curves of degree $n + 1$, $Y_1, Y_3, \ldots, Y_\alpha$, a linearly normal elliptic curve Y_2 of degree n contained in a hyperplane, and a general rational curve D of degree $n + \beta + (n - 1)\gamma$.

The curve D intersects Y_1 in ν_1 points, Y_{i-1} intersects Y_i in ν_i points for $2 \leq i \leq \alpha$, Y_α intersects D in $\nu_{\alpha+1}$ points and there are no further intersections.

We put the following conditions on the intersections: $\nu_i \geq 1$ for every $1 \leq i \leq \alpha$, $\nu_{\alpha+1} \geq 0$, $\nu_i \leq (n+1)/2+1$ for every $1 \leq i \leq \alpha+1$, $\nu_2, \nu_3 \leq (n+1)/2$, and moreover $t_\beta = \nu_1 + \nu_{\alpha+1}$ and $s_n = \nu_2 + \nu_3$. Note that $1 \leq t_\beta \leq n - 1$.

The smoothable curve X is of degree d and genus $g = s_n + t_\beta + \sum_{i=4}^{\alpha} \nu_i$.

By Lemma 3.4, the bundle G_{s_n} obtained from the normal bundle N_{Y_2} of Y_2 in $\mathbb{P}^n$ by making s_n general positive elementary transformations is semistable. Since $\deg(G_{s_n}) = n(n + 1) + s_n$ and $\operatorname{rank}(G_{s_n}) = n - 1$ are coprime, we infer that G_{s_n} is stable.

Thus we can proceed as in the proof of Lemma 3.3 to conclude.

Proof of Theorem 1 for n odd and $n \geq 7$. We use the notations of Lemma 3.5. Note that

$$\alpha \leq \alpha_d := \left[\frac{d - n + 1}{n + 1}\right].$$

By Lemma 3.5, for $5 \leq \alpha \leq \alpha_d$ and for every integer g such that $2n+\alpha-6 \leq g \leq \frac{1}{2}((\alpha - 1)n + \alpha - 5) + \alpha - 3$, the pair (d, g) satisfies the assertion of the theorem. We obtain the result for $d \geq 6n + 4$ and

$$2n - 1 \leq g \leq \frac{1}{2}((\alpha_d - 1)n + \alpha_d - 5) + \alpha_d - 3.$$

We have $d - n + 1 = (n + 1)\alpha_d + r_d$ with $0 \leq r_d \leq n$. The last displayed inequality is equivalent to

$$d \geq 2g + 2n + r_d + 8 - \frac{4g + 16}{n + 3}.$$

So we obtain the range $d \geq 2g + 3n + 1$.

Proof of Theorem 1 for $n = 5$. For $n = 5$ we have $s_n = n = 5$. We proceed as above to obtain the range $g \geq 10$ and $d \geq \frac{3}{2}g + 18$.

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Reçu par la Rédaction le 25.5.1998
Révisé le 18.2.1999