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ANALYTIC SOLUTIONS OF A SECOND-ORDER FUNCTIONAL DIFFERENTIAL EQUATION WITH A STATE DERIVATIVE DEPENDENT DELAY

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Abstract. This paper is concerned with a second-order functional differential equation of the form $x''(z)=x(az+bx'(z))$ with the distinctive feature that the argument of the unknown function depends on the state derivative. An existence theorem is established for analytic solutions and systematic methods for deriving explicit solutions are also given.

1. Introduction. Functional differential equations of the form

$$x'(t) = f(x(\sigma(t)))$$

have been studied by many authors. However, when the delay function $\sigma(t)$ is state dependent, $\sigma(t) = x(t)$, relatively little is known. In [1], [3], [4], analytic solutions of the state dependent functional differential equations

$$x'(z) = x^{[m]}(z)$$

and

$$x'(z) = x(az + bx(z))$$

are found. In this paper, we will be concerned with analytic solutions of the second-order functional differential equation

$$x''(z) = x(az + bx'(z)), \tag{1}$$

where a and $b \neq 0$ are complex numbers. A distinctive feature of (1) is that the argument of the unknown function depends on the state derivative. In order to construct analytic solutions of (1) in a systematic manner, we first let

$$y(z) = az + bx'(z). \tag{2}$$

Then for any number z_0, we have

$$x(z) = x(z_0) + \frac{1}{b}\int_{z_0}^{z} (y(s) - as)ds \tag{3}$$

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and so

$$x(y(z)) = x(z_0) + \frac{1}{b} \int_{z_0}^{y(z)} (y(s) - as) ds.$$

Therefore, in view of (1) and $x''(z) = \frac{1}{b}(y'(z) - a)$, we have

$$\frac{1}{b}(y'(z) - a) = x(z_0) + \frac{1}{b} \int_{z_0}^{y(z)} (y(s) - as) ds. \tag{4}$$

If z_0 is a fixed point of $y(z)$, i.e., $y(z_0) = z_0$, we see that

$$\frac{1}{b}(y'(z_0) - a) = x(z_0) + \frac{1}{b} \int_{z_0}^{y(z_0)} (y(s) - as) ds,$$

or

$$x(z_0) = \frac{1}{b}(y'(z_0) - a). \tag{5}$$

Furthermore, differentiating both sides of (4) with respect to z, we obtain

$$y''(z) = [y(y(z)) - ay(z)]y'(z). \tag{6}$$

2. Analytic solutions of (6). To find analytic solutions of (6), we first seek an analytic solution $g(z)$ of the auxiliary equation

$$\alpha g''(\alpha z)g'(z) = g'(\alpha z)g''(z) + (g'(z))^2 g'(\alpha z)[g(\alpha^2 z) - ag(\alpha z)] \tag{7}$$

satisfying the initial value conditions

$$g(0) = \mu, \qquad g'(0) = \eta \neq 0, \tag{8}$$

where μ, η are complex numbers, and α satisfies either

(H1) $0 < |\alpha| < 1$; or

(H2) $|\alpha| = 1, \alpha$ is not a root of unity, and

$$\log \frac{1}{|\alpha^n - 1|} \leq T \log n, \qquad n = 2, 3, \ldots,$$

for some positive constant T. Then we show that (6) has an analytic solution of the form

$$y(z) = g(\alpha g^{-1}(z)) \tag{9}$$

in a neighborhood of μ. We begin with the following preparatory lemma the proof of which can be found in [2, Chapter 6].

LEMMA 1. *Assume that* (H2) *holds. Then there is a positive number* δ *such that* $|\alpha^n - 1|^{-1} < (2n)^\delta$ *for* $n = 1, 2, \ldots$ *Furthermore, the sequence*

$\{d_n\}_{n=1}^{\infty}$ *defined by* $d_1 = 1$ *and*

$$d_n = \frac{1}{|\alpha^{n-1} - 1|} \max_{\substack{n=n_1+\ldots+n_t \\ 0<n_1\le\ldots\le n_t,\, t\ge 2}} \{d_{n_1} \ldots d_{n_t}\}, \qquad n = 2, 3, \ldots,$$

satisfies

$$d_n \le \left(2^{5\delta+1}\right)^{n-1} n^{-2\delta}, \qquad n = 1, 2, \ldots$$

LEMMA 2. *Suppose* (H1) *holds. Then for the initial value conditions* (8), *equation* (7) *has an analytic solution of the form*

$$g(z) = \mu + \eta z + \sum_{n=2}^{\infty} b_n z^n \tag{10}$$

in a neighborhood of the origin.

Proof. Rewrite (7) in the form

$$\frac{\alpha g''(\alpha z) g'(z) - g'(\alpha z) g''(z)}{(g'(z))^2} = g'(\alpha z)[g(\alpha^2 z) - a g(\alpha z)],$$

or

$$\left(\frac{g'(\alpha z)}{g'(z)}\right)' = g'(\alpha z)[g(\alpha^2 z) - a g(\alpha z)].$$

Therefore, in view of $g'(0) = \eta \ne 0$, we have

$$g'(\alpha z) = g'(z)\Big[1 + \int_0^z g'(\alpha s)(g(\alpha^2 s) - a g(\alpha s))\, ds\Big]. \tag{11}$$

We now seek a solution of (7) in the form of a power series (10). By defining $b_0 = \mu$ and $b_1 = \eta$ and then substituting (10) into (11), we see that the sequence $\{b_n\}_{n=2}^{\infty}$ is successively determined by the condition

$$\begin{aligned} &(\alpha^{n+1} - 1)(n+2) b_{n+2} \\ &= \sum_{k=0}^{n} \sum_{j=0}^{n-k} \frac{(k+1)(j+1)(\alpha^{2(n-k)-j} - a\alpha^{n-k})}{n-k+1} b_{k+1} b_{j+1} b_{n-k-j}, \quad n = 0, 1, \ldots, \end{aligned} \tag{12}$$

in a unique manner. We need to show that the resulting power series (10) converges in a neighborhood of the origin. First of all, note that

$$\left|\frac{(k+1)(j+1)(\alpha^{2(n-k)-j} - a\alpha^{n-k})}{(n+2)(n-k+1)(\alpha^{n+1} - 1)}\right| \le \frac{1 + |a|}{|\alpha^{n+1} - 1|} \le M$$

for some positive number M, thus if we define a sequence $\{B_n\}_{n=0}^{\infty}$ by $B_0 = |\mu|, B_1 = |\eta|$ and

$$B_{n+2} = M \sum_{k=0}^{n} \sum_{j=0}^{n-k} B_{k+1} B_{j+1} B_{n-k-j}, \qquad n = 0, 1, \ldots,$$

then in view of (12),

$$|b_n| \le B_n, \quad n = 0, 1, \ldots$$

Now if we define

$$G(z) = \sum_{n=0}^{\infty} B_n z^n, \tag{13}$$

then

$$\begin{aligned} G^2(z) &= \Big(|\mu| + \sum_{n=0}^{\infty} B_{n+1} z^{n+1}\Big)\Big(\sum_{n=0}^{\infty} B_n z^n\Big) \\ &= |\mu| \sum_{n=0}^{\infty} B_n z^n + \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} B_{k+1} B_{n-k}\Big) z^{n+1}, \end{aligned}$$

and

$$\begin{aligned} G^3(z) &= \Big(|\mu| + \sum_{n=0}^{\infty} B_{n+1} z^{n+1}\Big)\Big(|\mu| \sum_{n=0}^{\infty} B_n z^n + \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} B_{k+1} B_{n-k}\Big) z^{n+1}\Big) \\ &= |\mu|^2 \sum_{n=0}^{\infty} B_n z^n + 2|\mu| \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} B_{k+1} B_{n-k}\Big) z^{n+1} \\ &\quad + \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}\sum_{j=0}^{n-k} B_{k+1} B_{j+1} B_{n-k-j}\Big) z^{n+2} \\ &= |\mu|^2 G(z) + 2|\mu|(G^2(z) - |\mu| G(z)) + \frac{1}{M}\sum_{n=0}^{\infty} B_{n+2} z^{n+2} \\ &= |\mu|^2 G(z) + 2|\mu|(G^2(z) - |\mu| G(z)) + \frac{1}{M}(G(z) - |\mu| - |\eta| z) \\ &= 2|\mu| G^2(z) + \left(\frac{1}{M} - |\mu|^2\right) G(z) - \frac{1}{M}(|\eta| z + |\mu|), \end{aligned}$$

that is,

$$G^3(z) - 2|\mu| G^2(z) - \left(\frac{1}{M} - |\mu|^2\right) G(z) + \frac{1}{M}(|\eta| z + |\mu|) = 0. \tag{14}$$

Let

$$R(z, \omega) = \omega^3 - 2|\mu|\omega^2 - \left(\frac{1}{M} - |\mu|^2\right)\omega + \frac{1}{M}(|\eta| z + |\mu|)$$

for (z, ω) from a neighborhood of $(0, |\mu|)$. Since $R(0, |\mu|) = 0$ and $R'_\omega(0, |\mu|) = -1/M \neq 0$, there exists a unique function $\omega(z)$, analytic in a neighborhood of zero, such that $\omega(0) = |\mu|$, $\omega'(0) = |\eta|$ and $R(z, \omega(z)) = 0$. By (13) and (14), we have $G(z) = \omega(z)$. It follows that the power series (13), and hence also (10), converges in a neighborhood of the origin. The proof is complete.

LEMMA 3. *Suppose* (H2) *holds. Then if* $0 < |\eta| \le 1$, *equation* (7) *has an analytic solution of the form* (10) *in a neighborhood of the origin.*

Proof. As in the proof of Lemma 2, we seek a power series solution of the form (10). Set $b_0 = \mu$ and $b_1 = \eta$. Then (12) again holds so that

$$(15)\qquad |b_{n+2}| \le \frac{1+|a|}{|\alpha^{n+1}-1|}\sum_{k=0}^{n}\sum_{j=0}^{n-k}|b_{k+1}|\cdot|b_{j+1}|\cdot|b_{n-k-j}|,\qquad n=0,1,\ldots$$

Let us now consider the equation

$$(16)\qquad Q(z,\omega) = \omega^3 - 2|\mu|\omega^2 - \Big(\frac{1}{1+|a|} - |\mu|^2\Big)\omega + \frac{1}{1+|a|}(z+|\mu|) = 0$$

for (z,ω) from a neighborhood of $(0,|\mu|)$. Since $Q(0,|\mu|)=0$ and $Q'_\omega(0,|\mu|) = -1/(1+|a|) \ne 0$, there is a unique function $\omega(z)$, analytic in a neighborhood of zero, such that $\omega(0)=|\mu|$, $\omega'(0)=1$ and $Q(z,\omega(z))=0$. Now if

$$(17)\qquad \omega(z) = |\mu| + z + \sum_{n=2}^{\infty} C_n z^n,$$

where the coefficient sequence $\{C_n\}_{n=0}^{\infty}$ satisfies $C_0 = |\mu|, C_1 = 1$ and

$$(18)\qquad C_{n+2} = (1+|a|)\sum_{k=0}^{n}\sum_{j=0}^{n-k} C_{k+1}C_{j+1}C_{n-k-j},\qquad n=0,1,\ldots,$$

then

$$\begin{aligned}\omega^2(z) &= \Big(|\mu| + \sum_{n=0}^{\infty} C_{n+1}z^{n+1}\Big)\Big(\sum_{n=0}^{\infty} C_n z^n\Big)\\ &= |\mu|\sum_{n=0}^{\infty} C_n z^n + \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} C_{k+1}C_{n-k}\Big)z^{n+1},\end{aligned}$$

and

$$\begin{aligned}\omega^3(z) &= \Big(|\mu| + \sum_{n=0}^{\infty} C_{n+1}z^{n+1}\Big)\Big(|\mu|\sum_{n=0}^{\infty} C_n z^n + \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} C_{k+1}C_{n-k}\Big)z^{n+1}\Big)\\ &= |\mu|^2\sum_{n=0}^{\infty} C_n z^n + 2|\mu|\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} C_{k+1}C_{n-k}\Big)z^{n+1}\\ &\quad + \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}\sum_{j=0}^{n-k} C_{k+1}C_{j+1}C_{n-k-j}\Big)z^{n+2}\\ &= |\mu|^2\omega(z) + 2|\mu|(\omega^2(z) - |\mu|\omega(z)) + \frac{1}{1+|a|}\sum_{n=0}^{\infty} C_{n+2}z^{n+2}\\ &= |\mu|^2\omega(z) + 2|\mu|(\omega^2(z) - |\mu|\omega(z)) + \frac{1}{1+|a|}(\omega(z) - |\mu| - z)\end{aligned}$$

$$= 2|\mu|\omega^2(z) + \left(\frac{1}{1+|a|} - |\mu|^2\right)\omega(z) - \frac{1}{1+|a|}(z+|\mu|),$$

that is, $\omega(z)$ satisfies the equation (16). It follows that the power series (17) converges in a neighborhood of zero, and there is a positive constant T such that

$$C_n < T^n, \qquad n = 1, 2, \ldots \tag{19}$$

Now by induction, we prove that

$$|b_n| \le C_n d_n, \qquad n = 1, 2, \ldots,$$

where the sequence $\{d_n\}_{n=1}^{\infty}$ is defined in Lemma 1. In fact,

$$\begin{aligned}
|b_1| &= |\eta| \le 1 = C_1 d_1,\\
|b_2| &= (1+|a|)|\alpha-1|^{-1}|b_1|\cdot|b_1|\cdot|b_0|\\
&\le (1+|a|)|\alpha-1|^{-1}C_1d_1\cdot C_1d_1\cdot C_0\\
&= C_2|\alpha-1|^{-1}\max_{\substack{n_1+n_2=2\\0<n_1\le n_2}}\{d_{n_1}d_{n_2}\}\\
&= C_2 d_2.
\end{aligned}$$

Assume that the above inequality holds for $n = 1, \ldots, m$. Then

$$\begin{aligned}
|b_{m+1}| &\le (1+|a|)|\alpha^m-1|^{-1}\sum_{k=0}^{m-1}\sum_{j=0}^{m-1-k}|b_{k+1}|\cdot|b_{j+1}|\cdot|b_{m-1-k-j}|\\
&= (1+|a|)|\alpha^m-1|^{-1}\Big(\sum_{k=0}^{m-1}|b_{k+1}|\cdot|b_{m-k}|\cdot|b_0|\\
&\quad+\sum_{k=0}^{m-2}\sum_{j=0}^{m-2-k}|b_{k+1}|\cdot|b_{j+1}|\cdot|b_{m-1-k-j}|\Big)\\
&\le (1+|a|)|\alpha^m-1|^{-1}\Big(\sum_{k=0}^{m-1}C_{k+1}d_{k+1}C_{m-k}d_{m-k}C_0\\
&\quad+\sum_{k=0}^{m-2}\sum_{j=0}^{m-2-k}C_{k+1}d_{k+1}C_{j+1}d_{j+1}C_{m-1-k-j}d_{m-1-k-j}\Big)\\
&\le (1+|a|)|\alpha^m-1|^{-1}\max_{\substack{n_1+\ldots+n_t=m+1\\0<n_1\le\ldots\le n_t,\,t\ge2}}\{d_{n_1}\ldots d_{n_t}\}\\
&\quad\times\Big(\sum_{k=0}^{m-1}C_{k+1}C_{m-k}C_0+\sum_{k=0}^{m-2}\sum_{j=0}^{m-2-k}C_{k+1}C_{j+1}C_{m-1-k-j}\Big)\\
&= C_{m+1}d_{m+1}.
\end{aligned}$$

as desired. In view of (19) and Lemma 1, we finally see that

$$|b_n| \le T^n (2^{5\delta+1})^{n-1} n^{-2\delta}, \quad n = 1, 2, \dots,$$

which shows that the power series (10) converges for

$$|z| < \frac{1}{T 2^{5\delta+1}}.$$

The proof is complete.

THEOREM. *Suppose the conditions of Lemma* 2 *or Lemma* 3 *are satisfied. Then equation* (6) *has an analytic solution* $g(z)$ *of the form* (9) *in a neighborhood of the number* μ, *where* $g(z)$ *is an analytic solution of* (7).

Proof. In view of Lemmas 2 and 3, we may find a sequence $\{b_n\}_{n=2}^{\infty}$ such that the function $g(z)$ of the form (10) is an analytic solution of (7) in a neighborhood of the origin. Since $g'(0) = \eta \ne 0$, the function $g^{-1}(z)$ is analytic in a neighborhood of $g(0) = \mu$. If we now define $y(z)$ by means of (9), then

$$\begin{aligned}
y'(z) &= \alpha g'(\alpha g^{-1}(z))(g^{-1}(z))' = \frac{\alpha g'(\alpha g^{-1}(z))}{g'(g^{-1}(z))}, \\
y''(z) &= \frac{\alpha^2 g''(\alpha g^{-1}(z)) - \alpha g'(\alpha g^{-1}(z)) g''(g^{-1}(z)) \cdot \frac{1}{g'(g^{-1}(z))}}{(g'(g^{-1}(z)))^2} \\
&= \frac{\alpha[\alpha g''(\alpha g^{-1}(z)) g'(g^{-1}(z)) - g'(\alpha g^{-1}(z)) g''(g^{-1}(z))]}{[g'(g^{-1}(z))]^3} \\
&= \frac{\alpha\{(g'(g^{-1}(z)))^2 g'(\alpha g^{-1}(z))[g(\alpha^2 g^{-1}(z)) - ag(\alpha g^{-1}(z))]\}}{[g'(g^{-1}(z))]^3} \\
&= \frac{\alpha g'(\alpha g^{-1}(z))[g(\alpha^2 g^{-1}(z)) - ag(\alpha g^{-1}(z))]}{g'(g^{-1}(z))},
\end{aligned}$$

and

$$\begin{aligned}
[y(y(z)) - ay(z)]y'(z) &= [g(\alpha^2 g^{-1}(z)) - ag(\alpha g^{-1}(z))] \frac{\alpha g'(\alpha g^{-1}(z))}{g'(g^{-1}(z))} \\
&= \frac{\alpha g'(\alpha g^{-1}(z))[g(\alpha^2 g^{-1}(z)) - ag(\alpha g^{-1}(z))]}{g'(g^{-1}(z))}
\end{aligned}$$

as required. The proof is complete.

3. Analytic solutions of (1) via (6). In the last section, we have shown that under the conditions of Lemma 2 or Lemma 3, equation (6) has an analytic solution $y(z) = g(\alpha g^{-1}(z))$ in a neighborhood of the number μ, where g is an analytic solution of (7). Since the function $g(z)$ in (10) can be determined by (12), it is possible to calculate, at least in theory, the explicit form of $y(z)$, an analytic solution of (1), in a neighborhood of the fixed point μ of $y(z)$ by means of (3) and (5). However, knowing that an analytic solution of (1) exists, we can take an alternative route as follows. Assume that $x(z)$ is of the form

$$x(z) = x(\mu) + x'(\mu)(z-\mu) + \frac{x''(\mu)}{2!}(z-\mu) + \dots;$$

we need to determine the derivatives $x^{(n)}(\mu)$, $n = 0, 1, \dots$ First of all, in view of (5) and (2), we have

$$x(\mu) = \frac{1}{b}(y'(\mu) - a) = \frac{1}{b}\left(\frac{\alpha g'(\alpha g^{-1}(\mu))}{g'(g^{-1}(\mu))} - a\right) = \frac{\alpha - a}{b}$$

and

$$x'(\mu) = \frac{1}{b}(y(\mu) - a\mu) = \frac{(1-a)\mu}{b},$$

respectively. Furthermore,

$$x''(\mu) = x(a\mu + bx'(\mu)) = x\left(a\mu + b\frac{(1-a)\mu}{b}\right) = x(\mu) = \frac{\alpha - a}{b}.$$

Next by calculating the derivatives of both sides of (1), we obtain successively

$$\begin{aligned} x'''(z) &= x'(az + bx'(z))(a + bx''(z)), \\ x^{(4)}(z) &= x''(az + bx'(z))(a + bx''(z))^2 + x'(az + bx'(z))(bx''(z)), \end{aligned}$$

so that

$$\begin{aligned} x'''(\mu) &= x'(a\mu + bx'(\mu))(a + bx''(\mu)) = \alpha x'(\mu) = \frac{\alpha\mu(1-a)}{b}, \\ x^{(4)}(\mu) &= x''(\mu)\alpha^2 + x'(\mu)[\alpha\mu(1-a)] \\ &= \frac{\alpha - a}{b}\cdot\alpha^2 + \frac{(1-a)\mu}{b}(\alpha\mu(1-a)) \\ &= \frac{\alpha}{b}[(\alpha - a)\alpha + ((1-a)\mu)^2]. \end{aligned}$$

In general, we can show by induction that

$$\begin{aligned} &(x(az + bx'(z)))^{(m)} \\ &\quad = \sum_{i=1}^{m} P_{im}(a + bx''(z), bx'''(z), \dots, bx^{(m+1)}(z))x^{(i)}(az + bx'(z)), \end{aligned}$$

where $m = 1, 2, \dots$ and P_{im} is a polynomial with nonnegative coefficients. Hence

$$x^{(m+2)}(\mu) = \sum_{i=1}^{m} P_{im}(a + bx''(\mu), bx'''(\mu), \dots, bx^{(m+1)}(\mu))x^{(i)}(\mu) =: \Gamma_m$$

for $m = 1, 2, \ldots$ It is then easy to write out the explicit form of our solution $x(z)$:

$$\begin{aligned} x(z) = {} & \frac{\alpha - a}{b} + \frac{(1-a)\mu}{b}(z-\mu) + \frac{\alpha - a}{2!b}(z-\mu)^2 \\ & + \frac{\alpha\mu(1-a)}{3!b}(z-\mu)^3 + \frac{\alpha}{4!b}[(\alpha - a)\alpha + ((1-a)\mu)^2](z-\mu)^4 \\ & + \sum_{m=3}^{\infty} \frac{\Gamma_m}{(m+2)!}(z-\mu)^{m+2}. \end{aligned}$$

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