COLLOQUIUM MATHEMATICUM

VOL. 80 1999 NO. 1

OPERATORS COMMUTING WITH TRANSLATIONS, AND SYSTEMS OF DIFFERENCE EQUATIONS

BY

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Abstract. Let $\mathcal{B} = \{f : \mathbb{R}\to\mathbb{R} : f \text{ is bounded}\}$, and $\mathcal{M} = \{f : \mathbb{R}\to\mathbb{R} : f \text{ is Lebesgue measurable}\}$. We show that there is a linear operator $\Phi : \mathcal{B} \to \mathcal{M}$ such that $\Phi(f) = f$ a.e. for every $f \in \mathcal{B} \cap \mathcal{M}$, and Φ commutes with all translations. On the other hand, if $\Phi : \mathcal{B} \to \mathcal{M}$ is a linear operator such that $\Phi(f) = f$ for every $f \in \mathcal{B} \cap \mathcal{M}$, then the group

$$G_\Phi = \{a \in \mathbb{R} : \Phi \text{ commutes with the translation by } a\}$$

is of measure zero and, assuming Martin's axiom, is of cardinality less than continuum.

Let Φ be a linear operator from $\mathbb{C}^{\mathbb{R}}$ into the space of complex-valued measurable functions. We show that if $\Phi(f)$ is non-zero for every $f(x) = e^{cx}$, then G_Φ must be discrete. If $\Phi(f)$ is non-zero for a single polynomial-exponential f, then G_Φ is countable, moreover, the elements of G_Φ are commensurable. We construct a projection from $\mathbb{C}^{\mathbb{R}}$ onto the polynomials that commutes with rational translations. All these results are closely connected with the solvability of certain systems of difference equations.

1. Introduction.

Let $\mathcal{B} = \{f : \mathbb{R} \to \mathbb{R} : f \text{ is bounded}\}$, and $\mathcal{M} = \{f : \mathbb{R} \to \mathbb{R} : f \text{ is Lebesgue measurable}\}$. Putting $f \sim g$ if $f = g$ a.e. and factorizing $\mathcal{B} \cap \mathcal{M}$ with respect to the equivalence relation $\sim$ we obtain the space L^∞. Our starting point is the following observation.

THEOREM 1.1. *There is a positive linear operator $\Phi : \mathcal{B} \to L^\infty$ such that $\Phi(f) = f$ a.e. for every $f \in \mathcal{B}\cap\mathcal{M}$ and Φ commutes with every translation.*

Proof. Let μ be a Banach measure on $\mathbb{R}$, that is, a finitely additive translation-invariant extension of the Lebesgue measure to all subsets of $\mathbb{R}$. If $f \in \mathcal{B}$ then we define $\Phi(f)$ as the class containing F', where $F(x) = \int_0^x f(t)\,d\mu(t)$ for every $x \in \mathbb{R}$. Here we integrate a bounded function with respect to a finitely additive measure (see [6], p. 147). If $|f| \le K$ then we have $|F(y) - F(x)| \le K|y - x|$ for every $x, y \in \mathbb{R}$ and thus F is Lipschitz. Therefore F is differentiable a.e., and F' is bounded. It is clear that the operator Φ defined in this way satisfies the requirements. ■

1991 *Mathematics Subject Classification*: 28A20, 47B38, 47B39, 39A10, 39A70.

Supported by the Hungarian National Foundation for Scientific Research, Grant T016094.

The following result is a slight improvement of 1.1.

THEOREM 1.2. *There is a positive linear operator $\Psi : \mathcal{B} \to \mathcal{B} \cap \mathcal{M}$ such that $\Psi(f) = f$ a.e. for every $f \in \mathcal{B} \cap \mathcal{M}$ and Ψ commutes with every translation.*

Proof. Let $L : L^\infty \to \mathcal{B} \cap \mathcal{M}$ be a linear lifting, that is, a positive linear operator satisfying $L(\bar{f}) \in \bar{f}$ for every $\bar{f} \in L^\infty$. It is clear that if L commutes with translations and $\Phi : \mathcal{B} \to L^\infty$ is the operator constructed in Theorem 1.1, then $\Psi = L \circ \Phi$ satisfies the requirements.

A simple way of constructing a linear lifting L is the following. Let ℓ^∞ be the Banach space of bounded sequences, and let Λ be a norm one linear functional on ℓ^∞ such that $\Lambda(c_k) = \lim_{k\to\infty} c_k$ for every convergent sequence (c_k). If $\bar{f} \in L^\infty$, then for every $x \in \mathbb{R}^n$ we define

$$L(\bar{f})(x) = \Lambda(c_k), \quad \text{where} \quad c_k = k \int_x^{x+1/k} f(t)\,dt \qquad (k = 1, 2, \ldots).$$

If x is a Lebesgue point of f then $\lim_{k\to\infty} c_k = f(x)$. Therefore $L(\bar{f})(x) = f(x)$ at every Lebesgue point of f; that is, $L(\bar{f}) = f$ a.e. This implies that L is a linear lifting; moreover, it is easy to check that L commutes with all translations. ∎

Theorem 1.1 is, in fact, a special case of [3, Theorem 2], where a positive linear operator Φ is defined with the following properties: Φ maps the space of bounded functions defined on $\mathbb{R}^n$ into $L^\infty(\mathbb{R}^n)$, $\Phi(f) = f$ a.e. for every bounded measurable f, and Φ commutes with the elements of a prescribed amenable subgroup G of the isometry group of $\mathbb{R}^n$. It is easy to see that Theorem 1.2 has a similar generalization. Moreover, as Prof. Z. Lipecki pointed out, Theorem 1.1 can be further generalized to linear lattices, using Kantorovich' extension theorem. However, we restrict our attention to the case when $n = 1$ and G is the group of translations. As we shall see, already this special case leaves several interesting problems open.

In this note we consider the following questions.

(i) Does Theorem 1.2 remain true if we replace "almost everywhere" by "everywhere"?

(ii) Does Theorem 1.1 remain true if $\mathcal{B}$ is replaced by the class $\mathcal{F}$ of all functions defined on $\mathbb{R}$ and, accordingly, L^∞ is replaced by L^0, the set of equivalence classes of measurable functions? If not, what can we say about the size of the set

$$G_\Phi = \{a \in \mathbb{R} : \Phi \text{ commutes with the translation by } a\},$$

in the cases when

(i) $\varPhi : \mathcal{B} \to \mathcal{M}$ is a linear operator such that $\varPhi(f) = f$ for every $f \in \mathcal{B} \cap \mathcal{M}$, or

(ii) $\varPhi : \mathcal{F} \to L^0$ is a linear operator such that $\varPhi(f) = f$ a.e. for "many" f?

We prove that, in case (i), $G_\varPhi$ is always of measure zero (Theorem 3.2). We also show that, supposing Martin's axiom, $\mathrm{card}(G_\varPhi) < 2^\omega$ is also true; moreover, if G is any subgroup of $\mathbb{R}$ with $\mathrm{card}(G) < 2^\omega$, then there is a linear operator $\varPhi : \mathcal{B} \to \mathcal{M}$ such that $\varPhi(f) = f$ for every $f \in \mathcal{B} \cap \mathcal{M}$, and $G \subset G_\varPhi$ (Theorem 3.5).

In the results concerning question (ii), the trigonometric polynomials will play a special role. Therefore, in order to simplify notation, it will be more convenient to work with complex-valued functions. Let $\mathbb{C}^\mathbb{R}$ denote the set of complex-valued functions defined on $\mathbb{R}$, and let L^0 denote the set of equivalence classes of complex-valued measurable functions with respect to the relation $\sim$. We prove that if $\varPhi : \mathbb{C}^\mathbb{R} \to L^0$ is a linear operator such that $\varPhi(e^{cx}) \neq 0$ for every $c \in \mathbb{C}$, then $G_\varPhi$ is discrete (Theorem 5.1). Even if $\varPhi(f) \neq 0$ holds for a single function f of the form $\sum_{i=1}^n p_i(x)e^{c_i x}$, where each p_i is a polynomial, then $G_\varPhi$ must be countable; moreover, the elements of $G_\varPhi$ must be pairwise commensurable (Theorem 5.2). In the other direction we show that if $G \subset \mathbb{R}$ is a group such that the elements of G are pairwise commensurable, then there is a projection $\varPhi$ from $\mathbb{C}^\mathbb{R}$ onto the set of polynomials such that $G \subset G_\varPhi$ (Theorem 5.3). (By a projection we mean an idempotent linear map onto a subspace.)

The methods applied in the cases (i) and (ii) are different, but they both depend on the solvability of some systems of difference equations. Let $K = \mathbb{R}$ or $\mathbb{C}$. We say that

$$\sum_{k=1}^{n_i} a_k^i f(x + b_k^i) = h_i(x) \qquad (x \in \mathbb{R},\ i \in I)$$

is a system of difference equations if $a_k^i \in K$, $b_k^i \in \mathbb{R}$ $(i \in I,\ k = 1, \dots, n_i)$, $h_i : \mathbb{R} \to K$ is a given function for every $i \in I$, and f is the unknown function. (For a formal definition we refer to the next section.) Theorem 2.2 says that a system S of difference equations is solvable if and only if every finite subsystem of S is solvable. It is possible that each h_i is Lebesgue measurable, the system S is solvable, but S does not have measurable solutions. For example, $f(x + a) - f(x) = 0$, $f(x + b) - f(x) = 1$ is such a system if a/b is irrational (see Lemma 4.3).

It can also happen that every finite subsystem of S has a measurable solution, but S itself does not have measurable solutions (Theorem 4.4). In this example, all finite subsystems of S have solutions which are trigonometric polynomials. On the other hand, if every finite subsystem of S has a polynomial solution, then S itself has a polynomial solution (Theorem 4.5).

We also show that if h_i is a polynomial for every i and the numbers b_k^i are rational, then S has a polynomial solution if and only if S is solvable (Theorem 4.7). We conclude this section by formulating some problems concerning our topic.

PROBLEM 1. *What can we say about G_Φ if Φ is a projection from $\mathcal{B}$ onto $\mathcal{B} \cap \mathcal{M}$?*

We remark that, by a theorem of S. A. Argyros [1], a projection from $\mathcal{B}$ onto $\mathcal{B} \cap \mathcal{M}$ cannot be bounded.

PROBLEM 2. *Is it possible to characterize in ZFC the groups G_Φ where $\Phi : \mathcal{B} \to \mathcal{M}$ are linear operators fixing the elements of $\mathcal{B}\cap\mathcal{M}$? In particular, is it true that these are exactly the groups of cardinality less than* $\mathrm{non}(\mathcal{N})$, *the smallest cardinal of a set of positive Lebesgue outer measure?*

PROBLEM 3. *Let S be a system of difference equations, and suppose that every countable subsystem of S has a measurable solution. Does this imply that S itself has a measurable solution?* ([1])

2. General systems of difference equations. Let G be a commutative group written additively, and let K be a field. The set of functions from G to K is denoted by K^G. We say that an operator $D : K^G \to K^G$ is a *difference operator* if there are elements $a_i \in K$ and $g_i \in G$ $(i = 1, \ldots, n)$ such that

$$(Df)(x) = \sum_{i=1}^{n} a_i f(x + g_i)$$

for every $f \in K^G$ and $x \in G$. The set of all difference operators is denoted by $\mathcal{D}$. If $A \in \mathcal{D}$ and $a \in K$ then we define $aA \in \mathcal{D}$ by $(aA)f = a(Af)$. If $A, B \in \mathcal{D}$ then the sum and product of A, B are defined by $(A + B)f = Af + Bf$ and $(AB)f = A(Bf)$. It is easy to check that under these operations $\mathcal{D}$ becomes a commutative algebra. (Clearly, $\mathcal{D}$ is the same as the group ring $K[G]$; see [4].) Let T_g denote the translation operator $T_g f(x) = f(x+g)$. Clearly, every difference operator is a linear combination of translation operators. Moreover, every $D \in \mathcal{D}$ has a unique representation $D = \sum_{i=1}^{n} a_i T_{g_i}$ in which $g_1, \ldots, g_n$ are different and $a_1, \ldots, a_n$ are non-zero. To see this, it is enough to show that if $\sum_{i=1}^{n} a_i T_{g_i} = 0$ where $g_1, \ldots, g_n$ are different, then $a_1 = \ldots = a_n = 0$. Let f denote the characteristic function of $\{0\}$. Then

$$\Big(\sum_{i=1}^{n} a_i T_{g_i}\Big) f = 0.$$

([1]) Added in proof: recently I gave a negative answer.

Since the value of the left hand side at the point $-g_i$ equals a_i, this proves $a_1 = \ldots = a_n = 0$.

We investigate (possibly infinite) systems of difference equations of the form $D_i f = h_i$ $(i \in I)$, where D_i is a given difference operator and h_i is a given function for every i, and f is the unknown function.

Formally, by a *system of difference equations* we mean a set of pairs $S = \{(D_i, h_i) : i \in I\}$, where $D_i \in \mathcal{D}$ and $h_i \in K^G$ for every $i \in I$. By a *solution* of the system S we mean a function $f \in K^G$ such that $D_i f = h_i$ for every $i \in I$.

We say that the system S is *non-contradictory* if, whenever $i_1, \ldots, i_n \in I$, $E_1, \ldots, E_n \in \mathcal{D}$ and $\sum_{j=1}^n E_j D_{i_j} = 0$, then $\sum_{j=1}^n E_j h_{i_j} = 0$.

THEOREM 2.1. *A system S is solvable if and only if it is non-contradictory.*

Proof. Let f be a solution of S. If $\sum_{i=1}^n E_i D_i = 0$, where $E_i \in \mathcal{D}$ and $(D_i, h_i) \in S$ for every $i = 1, \ldots, n$, then

$$\sum_{i=1}^n E_i h_i = \sum_{i=1}^n E_i (D_i f) = \Big(\sum_{i=1}^n E_i D_i\Big) f = 0.$$

This proves the "only if" part of the theorem. In the other direction, suppose that S is non-contradictory, and let

$$\mathcal{A} = \Big\{ \sum_{i=1}^n E_i D_i : n \in \mathbb{N},\ E_i \in \mathcal{D},\ (D_i, h_i) \in S\ (i = 1, \ldots, n) \Big\}.$$

Then $\mathcal{A}$ is a subalgebra of $\mathcal{D}$. If $A \in \mathcal{A}$ and $A = \sum_{i=1}^n E_i D_i$, where $E_i \in \mathcal{D}$ and $(D_i, h_i) \in S$ for every $i = 1, \ldots, n$, then we define

$$L(A) = \sum_{i=1}^n (E_i h_i)(0).$$

The map $L : \mathcal{A} \to K$ is well-defined. Indeed, if

$$\sum_{i=1}^n E_i D_i = \sum_{j=1}^k E'_j D'_j \qquad (E_i, E'_j \in \mathcal{D},\ (D_i, h_i), (D'_j, h'_j) \in S),$$

then, as S is non-contradictory, we have $\sum_{i=1}^n E_i h_i = \sum_{j=1}^k E'_j h'_j$. Clearly, L is linear on $\mathcal{A}$. Since $\mathcal{A}$ is also a linear subspace of $\mathcal{D}$, L can be extended to $\mathcal{D}$ as a linear map. Let L^* be an extension, and define

$$f(x) = L^*(T_x) \qquad (x \in G). \tag{1}$$

We claim that f is a solution of S. First we show that

$$(Df)(0) = L^*(D) \qquad \text{for every } D \in \mathcal{D}. \tag{2}$$

Since L^* is linear, it is enough to check this for $D = T_g$ $(g \in G)$. Now $(T_g f)(0) = f(g) = L^*(T_g)$ by the definition of f, which proves (2). Let $(D, h) \in S$ and $x \in G$ be given. Then $T_x D \in \mathcal{A}$, and thus (2) and the definition of L imply

$$(Df)(x) = (T_x Df)(0) = L^*(T_x D) = L(T_x D) = (T_x h)(0) = h(x). \ \blacksquare$$

The following theorem is an immediate corollary of Theorem 2.1.

THEOREM 2.2. *A system of difference equations is solvable if and only if each of its finite subsystems is solvable.* $\blacksquare$

We say that $D \in \mathcal{D}$ is *supported* by a set $H \subset G$ if $D = \sum_{i=1}^n a_i T_{g_i}$, where $g_i \in H$ for every $i = 1, \ldots, n$. The family of all difference operators supported by H is denoted by $\mathcal{D}_H$.

LEMMA 2.3. *Let $S = \{(D_i, h_i) : i \in I\}$ be a system of difference equations, and let H be a subgroup of G such that every D_i is supported by H. If S is contradictory, then there are indices $i_1, \ldots, i_n \in I$ and difference operators $A_1, \ldots, A_n \in \mathcal{D}_H$ such that $\sum_{j=1}^n A_j D_{i_j} = 0$ and $\sum_{j=1}^n A_j h_{i_j} \neq 0$.*

Proof. Since S is contradictory, we have $\sum_{i=1}^n E_i D_i = 0$ and $\sum_{j=1}^n E_j h_j \neq 0$, with suitable $(D_i, h_i) \in S$ and $E_i \in \mathcal{D}$ $(i = 1, \ldots, n)$. There are finitely many cosets $U^j = H + u_j$ $(j = 1, \ldots, k)$ of the subgroup H such that each E_i is supported by $\bigcup_{j=1}^k U^j$. Let $E_i = \sum_{j=1}^k A_i^j$, where A_i^j is supported by U^j for every $i = 1, \ldots, n$ and $j = 1, \ldots, k$. If we represent E_i and D_i as linear combinations of translations, then the sum $\sum_{i=1}^n E_i D_i$ must be formally equal to zero. Since the terms belonging to different cosets cannot cancel each other, and the terms of D_i belong to H, this implies that $\sum_{i=1}^n A_i^j D_i = 0$ for every j. On the other hand,

$$0 \neq \sum_{i=1}^n E_i h_i = \sum_{j=1}^k \sum_{i=1}^n A_i^j h_i,$$

and thus $\sum_{i=1}^n A_i^j h_i \neq 0$ for at least one j. Fix such a j, and put $A_i = T_{-u_j} A_i^j$ $(i = 1, \ldots, n)$. Since A_i^j is supported by $U^j = H + u_j$, we have $A_i \in \mathcal{D}_H$ for every i. Also,

$$\sum_{i=1}^n A_i D_i = T_{-u_j} \sum_{i=1}^n A_i^j D_i = 0, \quad \text{and} \quad 0 \neq T_{-u_j} \sum_{i=1}^n A_i^j h_i = \sum_{i=1}^n A_i h_i. \ \blacksquare$$

3. Operators on bounded functions. In this section we shall consider linear operators mapping the class $\mathcal{B} = \{f : \mathbb{R} \to \mathbb{R} : f \text{ is bounded}\}$ into the class $\mathcal{M} = \{f : \mathbb{R} \to \mathbb{R} : f \text{ is measurable}\}$ and satisfying $\Phi(f) = f$ for every bounded measurable f. Recall that G_Φ denotes the set of those numbers

$a \in \mathbb{R}$ for which Φ commutes with the translation T_a. It is easy to see that G_Φ is a subgroup of $\mathbb{R}$.

We use the notation $\Delta_h = T_h - T_0$ for every $h \in \mathbb{R}$; thus

$$\Delta_h f(x) = f(x+h) - f(x) \quad (f : \mathbb{R} \to \mathbb{R},\ x \in \mathbb{R}).$$

LEMMA 3.1. *Let $\Phi : \mathcal{B} \to \mathcal{M}$ be a linear operator such that $\Phi(f) = f$ for every $f \in \mathcal{B} \cap \mathcal{M}$. Then for every $f \in \mathcal{B}$ there is $g : \mathbb{R} \to \mathbb{R}$ such that $f = g$ almost everywhere, and g is periodic mod each element of the set*

$$P_f = \{h \in G_\Phi : \Delta_h f = 0 \textit{ almost everywhere}\}.$$

Proof. It is easy to see that P_f is a subgroup of $\mathbb{R}$. If P_f is discrete then either $P_f = \{0\}$ or $P_f = \{na : n \in \mathbb{Z}\}$ where a is a fixed positive number. In the first case we put $g = f$. In the second case let $g = f$ on $[0, a)$, and let g be periodic mod a.

Suppose that P_f is not discrete. If $h \in P_f$ then $\Delta_h f = 0$ almost everywhere. Then $\Delta_h f \in \mathcal{B} \cap \mathcal{M}$ and, consequently, $\Phi(\Delta_h f) = \Delta_h f$. Let $\phi = \Phi(f)$. If $h \in G_\Phi$, then Φ commutes with T_h and Δ_h, and thus

$$\Delta_h \phi = \Delta_h(\Phi(f)) = \Phi(\Delta_h f) = \Delta_h f = 0 \quad \text{a.e.}$$

for every $h \in P_f$. As P_f is dense and ϕ is measurable, this implies that $\phi = c$ a.e., where c is a constant. We put $g = f - (\phi - c)$. Then $g = f$ a.e., and $\Delta_h g = \Delta_h f - \Delta_h \phi = 0$ for every $h \in P_f$. ■

THEOREM 3.2. *If $\Phi : \mathcal{B} \to \mathcal{M}$ is a linear operator such that $\Phi(f) = f$ for every $f \in \mathcal{B} \cap \mathcal{M}$, then G_Φ is of measure zero.*

Proof. We apply a variant of Sierpiński's argument in [5, Théorème II, p. 24]. The Lebesgue outer measure on $\mathbb{R}$ is denoted by λ. Suppose $\lambda(G_\Phi) > 0$, and let $\kappa = \min\{\mathrm{card}(H) : H \subset G_\Phi,\ \lambda(H) > 0\}$. Choose $H \subset G_\Phi$ such that $\lambda(H) > 0$ and $\mathrm{card}(H) = \kappa$. We may assume that H is a group. Let U be a maximal subset of H such that the elements of U are linearly independent over the rationals. Then

$$\mathrm{card}(U) = \mathrm{card}(H) = \kappa.$$

Let $\{u_\alpha : \alpha < \kappa\}$ be a well-ordering of U. Then every $x \in H \setminus \{0\}$ has a unique representation of the form

$$x = r_1 u_{\alpha_1} + \ldots + r_n u_{\alpha_n}, \tag{3}$$

where $n \geq 1$, $r_i \neq 0$, $r_i \in \mathbb{Q}$ and $\alpha_1 < \ldots < \alpha_n$. Let $r(x)$ denote the coefficient r_n in this representation. Let $A = \{x \in H \setminus \{0\} : r(x) > 0\}$. Since

$$A \cup (-A) = H \setminus \{0\},$$

we have $\lambda(A) > 0$. We claim that

$$\lambda((A + x) \triangle A) = 0 \quad \text{for every } x \in H, \tag{4}$$

where $\triangle$ denotes symmetric difference. Let the representation of x be given by (3) and suppose

$$y = s_1 u_{\beta_1} + \ldots + s_k u_{\beta_k} \in (A+x) \triangle A,$$

where $s_i \neq 0$, $s_i \in \mathbb{Q}$ $(i = 1, \ldots, k)$ and $\beta_1 < \ldots < \beta_k$. We prove that $\beta_k \le \alpha_n$. Indeed, if $\alpha_n < \beta_k$ then $y \in A \Leftrightarrow y - x \in A$, which contradicts $y \in (A+x) \triangle A$. Therefore every non-zero element of $(A+x) \triangle A$ is a linear combination of the numbers u_β $(\beta \le \alpha_n)$ with rational coefficients. Thus

$$\mathrm{card}((A+x) \triangle A) \le \mathrm{card}(\alpha_n) < \kappa,$$

and then $\lambda((A+x) \triangle A) = 0$ by the choice of κ. This proves (4).

Let f be the characteristic function of A; then $H \subset P_f$ by (4). Applying Lemma 3.1, we obtain a function g such that $f = g$ a.e., and g is periodic mod each element of H. Since $\lambda(A) > 0$, there is a point $x \in A$ such that $g(x) = f(x) = 1$. If $h \in (-A) \subset H$ then $g(h) = g(0) = g(x) = 1$, since g is periodic mod h and also mod x. Thus $g = 1$ on $-A$. Now, since $A \cap (-A) = 0$, we have $f = 0$ on $-A$, and hence $g = 0$ at almost every point of $-A$. This, however, contradicts $\lambda(-A) = \lambda(A) > 0$. ∎

In the following theorem we give a "lower estimate" for the possible sizes of the groups G_Φ. Let $\mathcal{N}$ denote the ideal of sets of Lebesgue measure zero. We put

$$\mathrm{add}(\mathcal{N}) = \min\Big\{\mathrm{card}(\mathcal{A}) : \mathcal{A} \subset \mathcal{N},\ \bigcup \mathcal{A} \notin \mathcal{N}\Big\};$$

then $\omega < \mathrm{add}(\mathcal{N}) \le 2^\omega$.

THEOREM 3.3. *Let G be a subgroup of $\mathbb{R}$ with* $\mathrm{card}(G) < \mathrm{add}(\mathcal{N})$. *Then there is a linear operator $\Phi : \mathcal{B} \to \mathcal{M}$ such that $\Phi(f) = f$ for every $f \in \mathcal{B} \cap \mathcal{M}$, and $G \subset G_\Phi$.*

Proof. By Theorem 1.2 there exists a linear operator $\Psi : \mathcal{B} \to \mathcal{B} \cap \mathcal{M}$ such that Ψ commutes with every translation, and $\Psi(f) = f$ almost everywhere for each $f \in \mathcal{B} \cap \mathcal{M}$. Our aim is to construct a linear operator $\Phi : \mathcal{B} \to \mathcal{M}$ such that $\Phi(f) = f$ for every $f \in \mathcal{B} \cap \mathcal{M}$, $G \subset G_\Phi$, and $\Phi(f) = \Psi(f)$ a.e. for every $f \in \mathcal{B}$. Let $\mathcal{W}$ denote the set of pairs (V, Φ) with the following properties:

(i) V is a G-invariant subspace of $\mathcal{B}$ containing $\mathcal{B} \cap \mathcal{M}$;
(ii) $\Phi : V \to \mathcal{M}$ is a linear operator commuting with translations by elements of G;
(iii) $\Phi(f) = f$ for every $f \in \mathcal{B} \cap \mathcal{M}$; and
(iv) $\Phi(f) = \Psi(f)$ a.e. for every $f \in V$.

Then $\mathcal{W}$ is non-empty, as $(\mathcal{B} \cap \mathcal{M}, \text{identity}) \in \mathcal{W}$. We define a partial order on $\mathcal{W}$ by writing $(V_1, \Phi_1) \le (V_2, \Phi_2)$ if $V_1 \subset V_2$ and Φ_2 is an extension of Φ_1. By Zorn's lemma there is a maximal $(V, \Phi) \in \mathcal{W}$. In order to prove the theorem, it is enough to show that $V = \mathcal{B}$. Suppose this is not true, and let $f_0 \in \mathcal{B} \setminus V$. Let $\mathcal{D}_G$ denote the set of difference operators of the form $\sum_{i=1}^{n} a_i T_{g_i}$, where $a_i \in \mathbb{R}$ and $g_i \in G$ for every $i = 1, \ldots, n$, and let

$$S = \{(D, h) : D \in \mathcal{D}_G,\ Df_0 \in V,\ h = \Phi(Df_0)\}. \tag{5}$$

We claim that S is non-contradictory. By Lemma 2.3, it is enough to show that if $A_i \in \mathcal{D}_G$ and $(D_i, h_i) \in S$ are such that $\sum_{i=1}^{n} A_i D_i = 0$, then $\sum_{i=1}^{n} A_i h_i = 0$. Since $G \subset G_\Phi$, it follows that Φ commutes with each element of $\mathcal{D}_G$. Therefore

$$\sum_{i=1}^{n} A_i h_i = \sum_{i=1}^{n} A_i \Phi(D_i f_0) = \sum_{i=1}^{n} \Phi(A_i D_i f_0) = \Phi\Big(\sum_{i=1}^{n} A_i D_i f_0\Big) = \Phi(0) = 0,$$

and thus S is non-contradictory. By Theorem 2.1, this implies that S is solvable.

Let $\mathcal{E}_G = \{D \in \mathcal{D}_G : D(f_0) \in V\}$; then $\mathcal{E}_G$ is a linear subspace of $\mathcal{D}_G$. Since the translations T_g $(g \in G)$ generate $\mathcal{D}_G$, we have $\dim \mathcal{D}_G \le \mathrm{card}(G)$. Therefore $\dim \mathcal{E}_G \le \mathrm{card}(G)$, and we can choose a basis $\mathcal{U}_G$ of $\mathcal{E}_G$ such that $\mathrm{card}(\mathcal{U}_G) \le \mathrm{card}(G)$. If $D \in \mathcal{E}_G$ then $Df_0 \in V$ and thus the set

$$A_D = \{x \in \mathbb{R} : \Phi(Df_0)(x) \neq \Psi(Df_0)(x)\}$$

is of measure zero. We put

$$A = \bigcup \{A_D : D \in \mathcal{U}_G\}.$$

Since $\mathrm{card}(\mathcal{U}_G) \le \mathrm{card}(G) < \mathrm{add}(\mathcal{N})$, we have $A \in \mathcal{N}$. We claim that if $D \in \mathcal{E}_G$ then

$$\Phi(Df_0)(x) = \Psi(Df_0)(x) \quad \text{for every } x \notin A. \tag{6}$$

Indeed, as $\mathcal{U}_G$ is a basis of $\mathcal{E}_G$, we have $D = \sum_{i=1}^{n} a_i D_i$, where $a_i \in \mathbb{R}$ and $D_i \in \mathcal{U}_G$ for every $i = 1, \ldots, n$. If $x \notin A$ then $\Phi(D_i f_0)(x) = \Psi(D_i f_0)(x)$ for every i and thus (6) follows by the linearity of Φ and Ψ.

We define

$$C = A + G = \{a + g : a \in A,\ g \in G\} = \bigcup_{g \in G} (A + g).$$

Then $C \in \mathcal{N}$, as $\mathrm{card}(G) < \mathrm{add}(\mathcal{N})$. Let f_1 be a solution of S defined in (5), and put

$$f^*(x) = \begin{cases} f_1(x) & \text{if } x \in C, \\ \Psi(f_0)(x) & \text{if } x \notin C. \end{cases}$$

We show that f^* is also a solution of the system S. Let $(D, h) \in S$. If $x \in C$ then $x + g \in C$ for every $g \in G$, and thus $(Df^*)(x) = (Df_1)(x) = h(x)$, since f_1 is a solution of S. On the other hand, if $x \notin C$ then $x + g \notin C$ for every $g \in G$, which implies

$$(Df^*)(x) = (D(\Psi(f_0)))(x) = \Psi(Df_0)(x) = \Phi(Df_0)(x) = h(x)$$

by (6). Now we define

$$\Phi^*(v+Df_0)=\Phi(v)+Df^* \qquad (v\in V,\ D\in\mathcal{D}_G).$$

Using the fact that f^* is a solution of S it is easy to check that Φ^* is a well-defined extension of Φ onto the subspace

$$V^*=\{v+Df_0 : v\in V,\ D\in\mathcal{D}_G\}.$$

Also, Φ^* commutes with translations by the elements of G. Indeed, if $g\in G$ then

$$\begin{aligned}\Phi^*(T_g(v+Df_0)) &= \Phi^*(T_gv+T_gDf_0)\\ &= \Phi(T_gv)+T_gDf^* = T_g\Phi(v)+T_gDf^*\\ &= T_g(\Phi(v)+Df^*) = T_g\Phi^*(v+Df_0).\end{aligned}$$

Let $v\in V$ and $D\in\mathcal{D}_G$. Since $\lambda(C)=0$ and $\Psi(v)=\Phi(v)$ a.e., we have $\Phi^*(v+Df_0)=\Psi(v+Df_0)$ a.e. Therefore the pair (V^*,Φ^*) satisfies the conditions (i)–(iv). This, however, contradicts the maximality of (V,Φ), completing the proof. ■

The results of Theorems 3.2 and 3.3 are rather far apart. Next we show that, under some set-theoretical assumptions, this gap can be filled; this will also indicate that the result of Theorem 3.3 is probably closer to the truth than that of Theorem 3.2. We shall use the additional notation

$$\begin{aligned}&\mathrm{non}(\mathcal{N})=\min\{\mathrm{card}(A) : A\subset\mathbb{R},\ A\notin\mathcal{N}\};\ \text{and}\\ &\mathrm{cof}(\mathcal{N})=\min\{\mathrm{card}(\mathcal{A}) : \mathcal{A}\subset\mathcal{N},\ \text{and}\\ &\qquad\qquad \text{for every } X\in\mathcal{N} \text{ there is } Y\in\mathcal{A} \text{ such that } X\subset Y\}.\end{aligned}$$

It is well known that

$$\omega<\mathrm{add}(\mathcal{N})\le\mathrm{non}(\mathcal{N})\le\mathrm{cof}(\mathcal{N})\le 2^\omega.$$

THEOREM 3.4. *Suppose that* $\mathrm{non}(\mathcal{N})=\mathrm{cof}(\mathcal{N})=\kappa$. *If* $\Phi:\mathcal{B}\to\mathcal{M}$ *is a linear operator such that* $\Phi(f)=f$ *for every* $f\in\mathcal{B}\cap\mathcal{M}$, *then* $\mathrm{card}(G_\Phi)<\kappa$.

Proof. Suppose that $\mathrm{card}(G_\Phi)\ge\kappa$. Then there is a subset $U\subset G_\Phi$ such that the elements of U are linearly independent over $\mathbb{Q}$ and $\mathrm{card}(U)=\kappa$. Let $\{g_\alpha : \alpha<\kappa\}$ be a well-ordering of U, and let G_α denote the group generated by $\{g_\beta : \beta<\alpha\}$.

Since $\kappa=\mathrm{cof}(\mathcal{N})$, there is a family $\{N_\alpha : \alpha<\kappa\}\subset\mathcal{N}$ such that for every $N\in\mathcal{N}$ there is an $\alpha<\kappa$ such that $N\subset N_\alpha$.

We define a transfinite sequence of real numbers c_α as follows. We put $c_0=0$. Let $0<\alpha<\kappa$ and suppose that c_β are defined for every $\beta<\alpha$. Then the cardinality of the set

$$H_\alpha=\bigcup_{\beta<\alpha}(G_\beta+c_\beta-g_\alpha)\cup\bigcup_{\beta<\alpha}(G_\alpha+c_\beta)$$

is less than κ, and thus $H_\alpha \in \mathcal{N}$. Therefore we can select a point

$$c_\alpha \notin N_\alpha \cup (N_\alpha - g_\alpha) \cup H_\alpha, \tag{7}$$

since the set on the right hand side is of measure zero. In this way we have selected the points c_α for every $\alpha < \kappa$. Now we define $A = \bigcup_{\alpha<\kappa}(G_\alpha + c_\alpha)$, and prove that $\lambda((A + g_\alpha) \triangle A) = 0$ for every $\alpha < \kappa$. If $\beta > \alpha$ then $G_\beta + c_\beta + g_\alpha = G_\beta + c_\beta$, since G_β is a group containing g_α. Therefore

$$(A + g_\alpha) \triangle A \subset \bigcup_{\beta\le\alpha} [(G_\beta + c_\beta + g_\alpha) \triangle (G_\beta + c_\beta)].$$

This implies $\mathrm{card}((A+g_\alpha)\triangle A) < \kappa = \mathrm{non}(\mathcal{N})$, and thus $\lambda((A+g_\alpha)\triangle A) = 0$, as we stated.

Let f denote the characteristic function of A. Then $\Delta_h f=0$ a.e. for every $h \in \{g_\alpha : \alpha < \kappa\}$ and thus, by Lemma 3.1, there is a function $g : \mathbb{R} \to \mathbb{R}$ such that $f = g$ a.e. and g is periodic mod each g_α. Since $\{x : f(x) \ne g(x)\} \in \mathcal{N}$, there is $\alpha < \kappa$ such that $\{x : f(x) \ne g(x)\} \subset N_\alpha$. Therefore

$$g(x) = \begin{cases} 1 & \text{for every } x \in A \setminus N_\alpha, \\ 0 & \text{for every } x \in \mathbb{R} \setminus (A \cup N_\alpha). \end{cases} \tag{8}$$

Next we prove that

$$c_\alpha \in A \setminus N_\alpha \quad \text{and} \quad c_\alpha + g_\alpha \in \mathbb{R} \setminus (A \cup N_\alpha). \tag{9}$$

Since $c_\alpha \in G_\alpha + c_\alpha$ and $c_\alpha \notin N_\alpha$ by (7), we have $c_\alpha \in A \setminus N_\alpha$. Also, $c_\alpha + g_\alpha \notin N_\alpha$ by (7), so it remains to show that $c_\alpha + g_\alpha \notin A$. If $\beta < \alpha$ then $c_\alpha \notin H_\alpha$ gives $c_\alpha + g_\alpha \notin G_\beta + c_\beta$. If $\beta > \alpha$, then $c_\beta \notin H_\beta$ implies $c_\beta \notin G_\beta + c_\alpha$, and thus $c_\alpha + g_\alpha \notin G_\beta + c_\beta$, taking into account that G_β is a group containing g_α. Finally, $c_\alpha + g_\alpha \notin G_\alpha + c_\alpha$; that is, $g_\alpha \notin G_\alpha$, since the elements g_γ $(\gamma \le \alpha)$ are linearly independent over the rationals. This shows

$$c_\alpha + g_\alpha \notin \bigcup_{\gamma<\kappa} (G_\gamma + c_\gamma) = A,$$

and hence (9) is proved.

Comparing (8) and (9) we obtain $g(c_\alpha) = 1$ and $g(c_\alpha + g_\alpha) = 0$. This, however, contradicts the fact that g is periodic mod g_α. ∎

THEOREM 3.5. *Assume Martin's axiom.*

(i) *If $\Phi : \mathcal{B} \to \mathcal{M}$ is a linear operator such that $\Phi(f) = f$ for every $f \in \mathcal{B} \cap \mathcal{M}$, then* $\mathrm{card}(G_\Phi) < 2^\omega$.

(ii) *If $G \subset \mathbb{R}$ is a group with* $\mathrm{card}(G) < 2^\omega$ *then there is a linear operator $\Phi : \mathcal{B} \to \mathcal{M}$ such that $\Phi(f) = f$ for every $f \in \mathcal{B} \cap \mathcal{M}$ and $G \subset G_\Phi$.*

Proof. Martin's axiom implies that $\mathrm{add}(\mathcal{N}) = \mathrm{non}(\mathcal{N}) = \mathrm{cof}(\mathcal{N}) = 2^\omega$ (see [2, Theorem 2.21, p. 59]). Thus the statement follows from Theorems 3.3 and 3.4. ∎

4. Some special systems of difference equations. Let G and K be as in Section 2. Recall that Δ_g denotes the difference operator $T_g - T_0$. Then we have $\Delta_g f(x) = f(x+g) - f(x)$ for every $f \in K^G$ and $x \in G$.

The elements $g, h \in G$ are said to be *independent* if $ng + kh = 0$ $(n, k \in \mathbb{Z})$ implies $n = k = 0$.

LEMMA 4.1. *Suppose that* $g, h \in G$ *are independent, and let* H *denote the group generated by* g *and* h. *Let* p, q *are positive integers,* $A, B \in \mathcal{D}_H$, *and suppose* $A\Delta_g^p = B\Delta_h^q$. *Then there is a* $C \in \mathcal{D}_H$ *such that* $A = C\Delta_h^q$ *and* $B = C\Delta_g^p$.

Proof. Every element of $\mathcal{D}_H$ is a linear combination of translation operators of the form T_{ng+kh}. If $n, k \ge 0$ then $T_{ng+kh} = T_g^n T_h^k$, and hence every $D \in \mathcal{D}_H$ is of the form

$$D = T_{-g}^N T_{-h}^N p(T_g, T_h),$$

where $p \in K[x, y]$, that is, p is a polynomial in two variables with coefficients from K. Let $\phi(D) = p(x, y)x^{-N}y^{-N}$. It is easy to check that ϕ is a well-defined map from $\mathcal{D}_H$ into $K(x, y)$, and that ϕ is an algebra isomorphism between $\mathcal{D}_H$ and the algebra of rational functions of the form $p(x, y)x^{-N}y^{-N}$, where $p \in K[x, y]$ and N is a non-negative integer.

Suppose that $A, B \in \mathcal{D}_H$ and $A\Delta_g^p = B\Delta_h^q$. Let $\phi(A) = p(x, y)x^{-N}y^{-N}$ and $\phi(B) = q(x, y)x^{-N}y^{-N}$ $(p, q \in K[x, y],\ N \ge 0)$. From $\phi(\Delta_g) = x - 1$, $\phi(\Delta_h) = y - 1$ it follows that $p(x, y)(x-1)^p = q(x, y)(y-1)^q$. Since there is unique factorization in $K[x, y]$, this implies that $p(x, y) = r(x, y)(y-1)^q$ and $q(x, y) = r(x, y)(x-1)^p$ with a suitable $r \in K[x, y]$. If

$$C = r(T_g, T_h)T_{-g}^N T_{-h}^N,$$

then we have $A = C\Delta_h^q$ and $B = C\Delta_g^p$. ∎

LEMMA 4.2. *Suppose that* $g, h \in G$ *are independent,* p, q *are positive integers, and* $u, v \in K^G$. *Then the system* $\Delta_g^p f = u$, $\Delta_h^q f = v$ *is solvable if and only if* $\Delta_h^q u = \Delta_g^p v$.

Proof. Let H denote the group generated by g and h. By Lemma 2.3, the system is non-contradictory if and only if $A, B \in \mathcal{D}_H$ and $A\Delta_g^p = B\Delta_h^q$ imply $Au = Bv$. Thus the statement follows from Theorem 2.1 and Lemma 4.1. ∎

In the sequel we consider the case when $G = \mathbb{R}$ and $K = \mathbb{C}$.

LEMMA 4.3. *Let* $f \in \mathbb{C}^{\mathbb{R}}$ *be measurable, and suppose that*

$$\Delta_a^n f(x) = 0 \quad \textit{and} \quad \Delta_b^m f(x) = c$$

for almost every x, *where* a/b *is irrational,* $0 < n \le m$ *are integers, and* c *is a constant. Then* $c = 0$, *and* f *equals a.e. a polynomial of degree* $< n$.

P r o o f. In the course of the proof, by equality of functions we mean equality almost everywhere. Suppose first $n = m = 1$. Then we have

$$f(x+kb) - f(x) = \sum_{i=0}^{k-1} \Delta_b f(x+ib) = kc.$$

Let $j_k \in \mathbb{Z}$ be such that $h_k = kb - j_k a \in [0, |a|]$. Since $\Delta_a f = 0$, we have $f(x+h_k) - f(x) = f(x+kb) - f(x) = kc$; that is,

$$\frac{1}{k} f(x+h_k) = c + \frac{1}{k} f(x)$$

for every k. However, the sequence of functions $f(x+h_k)/k$ converges in measure to zero on every interval $[u,v]$, since

$$\lambda\left(\left\{x \in [u,v] : \left|\frac{1}{k} f(x+h_k)\right| \geq \varepsilon\right\}\right) \leq \lambda(\{x \in [u, v+|a|] : |f(x)| \geq k\varepsilon\}) \to 0$$

as $k \to \infty$. This gives $c = 0$, and hence $\Delta_a f = \Delta_b f = 0$. This implies that $\Delta_{ia+jb} f = 0$ for every $i, j \in \mathbb{Z}$. Since the set $\{ia+jb : i, j \in \mathbb{Z}\}$ is everywhere dense, it follows that if x, y are points of approximate continuity of f, then $f(x) = f(y)$. But f is approximately continuous almost everywhere, so that f is constant almost everywhere.

Next we consider the case when $n = 1$ and m is arbitrary. We prove the statement by induction on m. The case $m = 1$ was proved above, so we may assume that $m > 1$ and that the statement is true for $m - 1$. Let $g = \Delta_b f$. Then g is measurable and $\Delta_a g = 0$ and $\Delta_b^{m-1} g = c$. By the induction hypothesis this implies that $c = 0$ and g is constant; that is, $\Delta_b f = d$. According to the case $n = m = 1$, this implies that f is constant.

Finally, we prove the general statement by induction on n (for arbitrary $m \geq n$). The case $n = 1$ was proved above, so we may assume that $n > 1$ and that the statement is true for $n - 1$. Let $g = \Delta_a f$. Then g is measurable and $\Delta_a^{n-1} g = 0$ and $\Delta_b^m g = 0$. By the induction hypothesis, this gives $\Delta_a f = g = p$, where p is a polynomial of degree $< n - 1$. Let q be a polynomial of degree $< n$ such that $\Delta_a q = p$, and put $h = f - q$. Then $\Delta_a h = \Delta_a f - \Delta_a q = p - p = 0$ and $\Delta_b^m h = \Delta_b^m f - \Delta_b^m q = c - 0 = c$. This implies, according to the case $n = 1$, that $c = 0$ and h is constant. Then f equals the polynomial $q + h$ of degree $< n$. ■

Let $S = \{(D_i, h_i) : i \in I\}$ be a system of difference equations (we still consider the case $G = \mathbb{R}$ and $K = \mathbb{C}$). The system S can have a *measurable* solution only if h_i is measurable for every i and if S is non-contradictory. However, this condition is not sufficient for the existence of a measurable solution. Consider the system $\Delta_a f = 0$, $\Delta_b f = 1$, where a/b is irrational. This system is non-contradictory by Lemma 4.2, but, according to Lemma 4.3, does not have a measurable solution.

This example, together with Theorem 2.2, motivates the following question. Suppose that every finite subsystem of S has a measurable solution. Does this imply that S itself has a measurable solution? We show next that the answer is negative.

THEOREM 4.4. *There exists a system S such that every finite subsystem of S has a solution which is a trigonometric polynomial, but S itself does not have a measurable solution.*

Proof. Let $e(x) = e^{2\pi i x}$ and $a_n = 2^{-n}$. Then

$$\Delta_{a_n} e(2^j x) = \varepsilon_{j,n} e(2^j x),$$

where $\varepsilon_{j,n} = e(2^{j-n}) - 1$. Note that $\varepsilon_{j,n} = 0$ if and only if $j \geq n$. Let c_j $(j = 0, 1, \ldots)$ be a sequence of complex numbers, and consider the system S of the equations

$$\Delta_{a_n} f = h_n, \quad \text{where} \quad h_n = \sum_{j=0}^{n-1} c_j \varepsilon_{j,n} e(2^j x) \quad (n = 1, 2, \ldots).$$

Then the trigonometric polynomial $\sum_{j=0}^{n-1} c_j e(2^j x)$ is a solution of the first n equations of S. On the other hand, we shall choose the numbers c_j in such a way that S does not have measurable solutions.

If $f : \mathbb{R} \to \mathbb{C}$ is measurable and $a_n \to 0$, then the sequence of functions $\Delta_{a_n} f$ converges in measure to zero on every bounded interval. Therefore, if S has a measurable solution, then h_n should converge in measure to zero on $[0, 1]$. But we can prevent this by a suitable choice of the sequence c_j. We define c_j inductively. If c_j has been defined for every $j < n - 1$, then we choose c_{n-1} so large that $\lambda(\{x \in [0, 1] : |h_n(x)| > 1\}) > 1/2$. This is possible since $\varepsilon_{n-1,n} e(2^{n-1} x) \neq 0$ in $[0, 1]$. Therefore, with this choice, h_n does not converge (in measure) to zero on $[0, 1]$, and thus S cannot have measurable solutions. ∎

Our next aim is to show that a similar example with polynomials instead of trigonometric polynomials does not exist. Moreover, the following is true.

THEOREM 4.5. *A system S has a polynomial solution if and only if every at most two-element subsystem of S has a polynomial solution.*

Let $D = \sum_{i=1}^{n} a_i T_{b_i}$ be a difference operator, where $b_i \neq b_j$ for every $i \neq j$. Let $s_k = \sum_{i=1}^{n} a_i b_i^k$ $(k = 0, 1, \ldots)$. We define the *order* of D to be $\min\{k : s_k \neq 0\}$. Since the determinant of the elements b_i^k $(i = 1, \ldots, n;$ $k = 0, \ldots, n - 1)$ is non-zero, it follows that if $D \neq 0$ then the order of D is not greater than $n - 1$.

LEMMA 4.6. *Let D be a non-zero difference operator of order m. Then*

(i) *A polynomial $f(x) = \sum_{i=0}^{N} c_i x^i$ satisfies $Df = 0$ if and only if $c_i = 0$ for every $i \geq m$; and*

(ii) *if p is a polynomial of degree r then there is a polynomial f of degree $r + m$ such that $Df = p$.*

Proof. It is easy to check that $Dx^i = 0$ if $i < m$, and Dx^i is a polynomial of degree $i - m$ if $i \geq m$. This gives (i). This also implies that choosing the coefficients $c_{r+m}, c_{r+m-1}, \dots, c_m$ appropriately (in this order), $D(c_{m+r}x^{m+r} + c_{m+r-1}x^{m+r-1} + \ldots + c_m x^m)$ can be any prescribed polynomial of degree r. ■

Proof of Theorem 4.5. We only have to prove the "if" statement. Let $S = \{(D_j, p_j) : j \in I\}$. Since $D_j f = p_j$ has a polynomial solution, it follows that p_j itself has to be a polynomial for every $j \in I$. Also, $D_j = 0$ implies $p_j = 0$. Deleting those pairs (D_j, p_j) for which $D_j = 0$, we may assume that $D_j \neq 0$ for every $j \in I$.

Let m_j be the order of D_j and let r_j be the degree of p_j. Lemma 4.6 implies that there are numbers c_i^j $(i = m_j, m_j + 1, \ldots)$ such that $c_i^j = 0$ for every $i > m_j + r_j$, and a polynomial $f(x) = \sum_{i=0}^{N} c_i x^i$ satisfies $D_j f = p_j$ if and only if $N \geq m_j + r_j$, and $c_i = c_i^j$ for every $m_j \leq i \leq N$. Since any two-element subsystem of S has a polynomial solution, the sequences c_i^j must be compatible, that is, if $j_1, j_2 \in I$ and $i \geq \max(m_{j_1}, m_{j_2})$ then $c_i^{j_1} = c_i^{j_2}$. Consequently, there exists an infinite sequence c_i $(i = 0, 1, \ldots)$ such that $c_i = c_i^j$ for every $j \in I$ and $i \geq m_j$. Then $f(x) = \sum_{i=0}^{\infty} c_i x^i$ is a polynomial that satisfies $D_j f = p_j$ for every $j \in I$. ■

In Theorem 4.4 we constructed a system $S = \{(D_n, h_n)\}$ with the following properties: D_n is supported by the rationals for every n, S is non-contradictory, h_n is a trigonometric polynomial for every n, but S does not have measurable solutions. We next show that this phenomenon cannot happen for polynomials.

THEOREM 4.7. *Let $S = \{(D_i, p_i) : i \in I\}$ be a system such that every D_i is supported by the rationals. Then S has a polynomial solution if and only if S is non-contradictory and p_i is a polynomial for every $i \in I$.*

For the proof we shall need the notion of transform. If $D = \sum_{i=1}^{n} a_i T_{b_i}$ is a difference operator, then the *transform* of D is $D^\wedge(z) = \sum_{i=1}^{n} a_i e^{b_i z}$, as a function defined on $\mathbb{C}$. It is easy to check that if $D \neq 0$ then $D^\wedge$ is not identically zero. We also have $(A + B)^\wedge = A^\wedge + B^\wedge$ and $(AB)^\wedge = A^\wedge B^\wedge$ for every $A, B \in \mathcal{D}$.

Proof of Theorem 4.7. The "only if" part is obvious, so we need only prove the "if" part. Suppose that S is non-contradictory, and that p_i is a polynomial for every $i \in I$.

If $D_i = 0$ for an $i \in I$ then $p_i = 0$, since otherwise S would be contradictory. That is, we may assume $D_i \neq 0$ for every $i \in I$, since otherwise we delete from S those pairs (D_i, p_i) in which $D_i = 0$.

By Theorem 4.5, it is enough to show that if $j, k \in I$ then the system $D_j f = p_j$, $D_k f = p_k$ has a polynomial solution. Let $D_j = \sum_{i=1}^{n} a_i T_{b_i}$ and $D_k = \sum_{i=1}^{m} c_i T_{d_i}$ where $b_1 < \ldots < b_n$ and $d_1 < \ldots < d_m$. Then $(D_j)^\wedge = \sum_{i=1}^{n} a_i e^{b_i t}$ and $(D_k)^\wedge = \sum_{i=1}^{m} c_i e^{d_i t}$. Since, by assumption, the numbers b_i, d_i are rational, there is an integer $N > 0$ and there are polynomials $P, Q \in \mathbb{C}[x]$ such that

$$(D_j)^\wedge(t) = e^{b_1 t} P(e^{t/N}), \qquad (D_k)^\wedge(t) = e^{d_1 t} Q(e^{t/N}).$$

Let R be the g.c.d. of P and Q, and choose polynomials S, T such that $PS + QT = R$. Let $A, B \in \mathcal{D}$ be chosen with

$$A^\wedge(t) = e^{-b_1 t} S(e^{t/N}) \quad \text{and} \quad B^\wedge(t) = e^{-d_1 t} T(e^{t/N}).$$

If $E = D_j A + D_k B$, then

$$E^\wedge = (D_j)^\wedge A^\wedge + (D_k)^\wedge B^\wedge = P(y)S(y) + Q(y)T(y) = R(y) \qquad (y = e^{t/N}).$$

Since $R \neq 0$, this gives $E \neq 0$. Thus, by Lemma 4.6(ii), there is a polynomial f such that $Ef = Ap_j + Bp_k$. We shall complete the proof by showing that f satisfies both $D_j f = p_j$ and $D_k f = p_k$.

Since R divides P, there is a polynomial U with $P = RU$. Let $F \in \mathcal{D}$ be chosen with $F^\wedge(t) = e^{b_1 t} U(e^{t/N})$. Then

$$(EF)^\wedge = R(y) e^{b_1 t} U(y) = e^{b_1 t} P(y) = (D_j)^\wedge(y) \qquad (y = e^{t/N}),$$

so that $D_j = EF$. Therefore $D_j = (D_j A + D_k B)F$ and $(T_0 - FA)D_j = FBD_k$. Since the system S is non-contradictory, this implies $(T_0 - FA)p_j = FBp_k$ and

$$p_j = F(Ap_j + Bp_k) = F(Ef) = D_j f.$$

We obtain $p_k = D_k f$ in the same way. ∎

5. Operators on arbitrary functions. First we show that Theorem 1.1 does not have an analogue for operators defined on the set $\mathbb{C}^{\mathbb{R}}$ of all complex-valued functions defined on $\mathbb{R}$. If we factorize the space of complex-valued measurable functions with respect to the equivalence relation $f \sim g \Leftrightarrow f = g$ a.e., then we obtain the space L^0.

THEOREM 5.1. *Let $\Phi : \mathbb{C}^{\mathbb{R}} \to L^0$ be a linear operator such that $\Phi(e^{cx}) \neq 0$ for every $c \in \mathbb{C}$. Then G_Φ is discrete.*

Proof. Suppose that $G = G_\Phi$ is not discrete; then G is dense. We prove first that for every $c \in \mathbb{C}$ there is a non-zero constant $K(c)$ such that $\Phi(e^{cx}) = K(c)e^{cx}$. (Since Φ maps into L^0, by equality of functions we mean equality almost everywhere.) Fix c, and put $\Phi(e^{cx}) = s$. If $g \in G$ then Φ commutes with T_g, and thus

$$T_g s = T_g \Phi(e^{cx}) = \Phi(T_g e^{cx}) = \Phi(e^{cx+cg}) = e^{cg} s. \tag{10}$$

Let $y \in \mathbb{R}$ be arbitrary, and let $g_n \in G$ be a sequence converging to y. Since s is measurable, the sequence of functions $T_{g_n} s$ converges in measure to $T_y s$ on every compact interval. On the other hand, $e^{cg_n} s \to e^{cy} s$ pointwise, thus (10) gives $T_y s = e^{cy} s$ for every $y \in \mathbb{R}$. Let $A = \{(u, y) \in \mathbb{R}^2 : s(u + y) \neq e^{cy} s(u)\}$. Then A is measurable, and each horizontal section of A is of measure zero. Then, by Fubini's theorem, almost every vertical section of A is also of measure zero. Let u be such that the section $A_u = \{y : (u, y) \in A\}$ is of measure zero. Then $s(u+y) = e^{cy} s(u)$ for a.e. y, and thus $s(x) = K(c)e^{cx}$ a.e., where $K(c) = e^{-cu} s(u)$. Since $s \neq 0$ by assumption, we have $K(c) \neq 0$.

Next we show that the elements of G are pairwise commensurable. Let $\mathbf{0}$ and $\mathbf{1}$ denote the identically 0 and identically 1 function, respectively, and put $s = \Phi(\mathbf{1})$. Then, as we proved above, $s = K(0)\cdot\mathbf{1}$ is a non-zero constant. Suppose that $a, b \in G$ are not commensurable. Then, by Lemma 4.2, the system of difference equations $\Delta_a f = \mathbf{0}$, $\Delta_b f = \mathbf{1}$ is solvable. Let f be a solution, and put $h = \Phi(f)$. Then

$$\Delta_a h = \Delta_a \Phi(f) = \Phi(\Delta_a f) = \Phi(\mathbf{0}) = \mathbf{0},$$
$$\Delta_b h = \Delta_b \Phi(f) = \Phi(\Delta_b f) = \Phi(\mathbf{1}) = s.$$

Since s is a non-zero constant and $h \in L^0$, this contradicts Lemma 4.3. This proves that the elements of G are pairwise commensurable.

By rescaling $\mathbb{R}$ if necessary, we may assume that $1 \in G$. Then $G \subset \mathbb{Q}$. We prove that there is a sequence of positive integers $n_0 < n_1 < \ldots$ such that n_k divides n_{k+1} and $1/n_k \in G$ for every k. Let $n_0 = 1$, and suppose that n_k has been chosen. Since G is dense, there is $g \in G$ such that $0 < g < 1/n_k$. Let H denote the (additive) group generated by g and $1/n_k$. Since $g \in \mathbb{Q}$, H is discrete, and thus $H = \{ka : k \in \mathbb{Z}\}$ with a suitable positive number a. We put $n_{k+1} = 1/a$. Then $1/n_k \in H$ implies that n_{k+1} is an integer multiple of n_k, and $g \in H$ shows that $n_{k+1} > n_k$.

We write $a_k = 1/n_k$ for every $k = 0, 1, \ldots$ Our aim is to construct a sequence of trigonometric polynomials h_k such that the system

$$\Delta_{a_k} f = h_k \quad (k = 0, 1, \ldots) \tag{11}$$

is solvable, but the system

$$\Delta_{a_k} f = \Phi(h_k) \quad (k = 0, 1, \ldots) \tag{12}$$

does not have any measurable solution. This will provide the contradiction we are looking for, since if f is a solution of (11), then $\Phi(f)$ is a measurable solution of (12), as Φ commutes with each a_k.

In the following construction we repeat the argument of the proof of Theorem 4.4. Let $e(x) = e^{2\pi i x}$. Then

$$\Delta_{a_k} e(n_j x) = \eta_{j,k} e(n_j x),$$

where $\eta_{j,k} = e(n_j/n_k) - 1$. Note that $\varepsilon_{j,k} = 0$ if and only if $j \geq k$. Let c_j $(j = 0, 1, \ldots)$ be a sequence of complex numbers, and define

$$h_k = \sum_{j=0}^{k-1} c_j \eta_{j,k} e(n_j x) \qquad (k = 1, 2, \ldots).$$

Then the trigonometric polynomial $\sum_{j=0}^{k-1} c_j e(n_j x)$ is a solution of the first k equations of the system (11). Thus every finite subsystem of (11) is solvable. By Theorem 2.2, this implies that (11) itself is solvable.

On the other hand, we shall choose the numbers c_j in such a way that (12) does not have measurable solutions. If $f : \mathbb{R} \to \mathbb{C}$ is measurable and $a_k \to 0$, then the sequence of functions $\Delta_{a_k} f$ converges in measure to $\mathbf{0}$ on every bounded interval. Therefore, if (12) has a measurable solution, then $\Phi(h_k)$ should converge in measure to zero on $[0,1]$. But we can prevent this by a suitable choice of the sequence c_j. We define c_j inductively. If c_j has been defined for every $j < k-1$, then we choose c_{k-1} so large that

$$\lambda(\{x \in [0,1] : |\Phi(h_k)(x)| > 1\}) > 1/2.$$

This is possible since

$$\eta_{k-1,k} \Phi(e(n_{k-1} x)) = \eta_{k-1,k} K(2\pi i n_{k-1}) e(n_{k-1} x) \neq 0$$

in $[0,1]$. Therefore, with this choice, $\Phi(h_k)$ does not converge in measure to zero on $[0,1]$, and thus (12) cannot have measurable solutions. ∎

In the first part of the proof of Theorem 5.1 we showed that if $\Phi(e^{cx}) \neq 0$ for any $c \in \mathbb{C}$, then the elements of G_Φ must be commensurable. The next theorem generalizes this result. By a *polynomial-exponential function* we mean a function of the form $\sum_{i=1}^n p_i(x) e^{c_i x}$, where p_i is a polynomial and $c_i \in \mathbb{C}$ for every i.

THEOREM 5.2. *Let $\Phi : \mathbb{C}^{\mathbb{R}} \to L^0$ be a linear operator, and suppose that there is a polynomial-exponential function h such that $\Phi(h) \neq 0$. Then the elements of the group G_Φ are pairwise commensurable.*

Proof. Let $h(x) = \sum_{i=1}^n p_i(x) e^{c_i x}$, where p_i is a polynomial and $c_i \in \mathbb{C}$ for every i. If $\Phi(h) \neq 0$, then there is an i such that $\Phi(p_i(x) e^{c_i x}) \neq 0$. Let

$$\Phi_1(g) = e^{-c_i x} \Phi(g e^{c_i x}) \qquad (g \in \mathbb{C}^{\mathbb{R}}).$$

Then Φ_1 is also a linear operator from $\mathbb{C}^{\mathbb{R}}$ into L^0, $\Phi_1(p_i) \neq 0$, and, as an easy computation shows, $G_{\Phi_1} = G_{\Phi}$.

Therefore, we may assume that h itself is a polynomial. We may also suppose that $G = G_{\Phi}$ is everywhere dense, since otherwise G is discrete, and the statement of the theorem is obviously satisfied.

Let n be the smallest integer for which there exists a polynomial p of degree n such that $\Phi(p) \neq 0$. If $g \in G$ then $\Delta_g p$ is a polynomial of degree $n-1$ and thus $\Phi(\Delta_g p) = 0$. Since Φ and Δ_g commute, this implies that $\Delta_g(\Phi(p)) = \mathbf{0}$ for every $g \in G$. As $\Phi(p)$ is measurable and G is dense, this implies that $\Phi(p)$ is constant. We may assume that p is a monic polynomial and that $\Phi(p) = \mathbf{1}$. Therefore, if q is a polynomial of degree n with leading coefficient a_n, then $\Phi(q) = a_n \cdot \mathbf{1}$.

The function $\Phi(\mathbf{1})$ is also constant. Indeed, if $n = 0$ then $\mathbf{1} = p$ and thus $\Phi(\mathbf{1}) = \mathbf{1}$. If $n > 0$, then $\Phi(\mathbf{1}) = \mathbf{0}$ by the choice of n.

Suppose that the elements of G are not commensurable, and let $a, b \in G$ be such that a/b is irrational. Then, by Lemma 4.2, there is a function s such that $\Delta_a s = \mathbf{0}$ and $\Delta_b^n s = \mathbf{1}$. Then $\Delta_a(\Phi(s)) = \mathbf{0}$ and $\Delta_b^n(\Phi(s)) = c \cdot \mathbf{1}$. Since $\Phi(s)$ is measurable, it follows from Lemma 4.3 that $\Phi(s)$ is constant. We shall distinguish between two cases.

Case I: $\Phi(s) = d \cdot \mathbf{1}$ where $d \neq 0$. Let q be a polynomial of degree n such that $\Delta_b p = \Delta_a q$. Then the system

$$\Delta_a f = p, \qquad \Delta_b f = q + es$$

is solvable for every $e \in \mathbb{C}$, since $\Delta_b p = \Delta_a(q + es) = \Delta_a q$. If f is a solution then $\Delta_a(\Phi(f)) = \mathbf{1}$ and $\Delta_b(\Phi(f)) = \Phi(q) + ed \cdot \mathbf{1}$. Since q is a polynomial of degree n, $\Phi(q)$ is constant, and we can choose e such that $\Delta_b(\Phi(f)) = \mathbf{0}$. In this case, however, $\Delta_a(\Phi(f)) = \mathbf{1}$ contradicts Lemma 4.3.

Case II: $\Phi(s) = \mathbf{0}$. Let r be a polynomial of degree n such that $\Delta_b^n r = \mathbf{1}$, and let $h = s - r$. Then

$$\Delta_b^n h = \Delta_b^n s - \Delta_b^n r = \mathbf{1} - \mathbf{1} = \mathbf{0},$$

and $\Phi(h) = \Phi(s) - \Phi(r)$ is a non-zero constant. Let t be a polynomial of degree n such that $\Delta_b^n p = \Delta_a^n t$. Then the system

$$\Delta_a^n f = p + eh, \qquad \Delta_b^n f = t$$

is solvable, since $\Delta_b^n(p + eh) = \Delta_b^n p = \Delta_a^n t$. If f is a solution, then

$$\Delta_a^n(\Phi(f)) = \mathbf{1} + e\Phi(h) \quad \text{and} \quad \Delta_b^n(\Phi(f)) = \Phi(t).$$

Since $\Phi(h)$ and $\Phi(t)$ are both non-zero constants, we can choose e such that $\Delta_a^n(\Phi(f)) = \mathbf{0}$, and $\Delta_b^n(\Phi(f))$ is a non-zero constant. This, again, contradicts Lemma 4.3, completing the proof. ∎

We conclude with three constructions of linear operators that commute with the elements of a prescribed group.

First we note that if V is a translation-invariant subspace of $\mathbb{C}^{\mathbb{R}}$, then for every discrete group G there is a projection $\Phi : \mathbb{C}^{\mathbb{R}} \to V$ such that $G \subset G_\Phi$. Indeed, let $G = \{ka : k \in \mathbb{Z}\}$, where $a > 0$. Let $V_0 = \{f|_{[0,a)} : f \in V\}$; then V_0 is a linear subspace of $\mathbb{C}^{[0,a)}$, the space of all complex-valued functions defined on $[0, a)$. Let $\Psi : \mathbb{C}^{[0,a)} \to V_0$ be a projection, that is, a linear map such that $\Psi(g) = g$ for every $g \in V_0$. Then we define

$$\Phi(f)(x) = \Psi((T_{ka}f)|_{[0,a)})(x - ka)$$

for every $f \in \mathbb{C}^{\mathbb{R}}$ and $x \in \mathbb{R}$, where k is the unique integer such that $ka \le x < (k + 1)a$. It is easy to check that Φ satisfies the requirements.

Our next construction is a complement to Theorem 5.2. We denote by $\mathcal{P}$ the set of polynomials.

THEOREM 5.3. *Let G be a subgroup of $\mathbb{R}$ such that the elements of G are pairwise commensurable. Then there is a projection $\Phi : \mathbb{C}^{\mathbb{R}} \to \mathcal{P}$ such that $G \subset G_\Phi$.*

Proof. We may assume that $G = \mathbb{Q}$. Let $\mathcal{W}$ denote the set of those pairs (V, Φ) in which V is a $\mathbb{Q}$-invariant subspace of $\mathbb{C}^{\mathbb{R}}$ containing $\mathcal{P}$, and $\Phi : V \to \mathcal{P}$ is a projection commuting with rational translations. Then $\mathcal{W}$ is non-empty, as $(\mathcal{P}, \text{identity}) \in \mathcal{W}$. We define a partial order on $\mathcal{W}$ by writing $(V_1, \Phi_1) \le (V_2, \Phi_2)$ if $V_1 \subset V_2$ and Φ_2 is an extension of Φ_1. By Zorn's lemma there is a maximal $(V, \Phi) \in \mathcal{W}$. In order to prove the theorem, it is enough to show that $V = \mathbb{C}^{\mathbb{R}}$. Suppose this is not true, and let $f_0 \in \mathbb{C}^{\mathbb{R}} \setminus V$. Then $V^* = \{v + Df_0 : v \in V,\ D \in \mathcal{D}_{\mathbb{Q}}\}$ is the smallest linear subspace of $\mathbb{C}^{\mathbb{R}}$ that contains $V \cup \{f_0\}$ and which is invariant under rational translations. We prove that Φ can be extended to V^* as a linear operator commuting with rational translations. Since (V, Φ) is maximal, this will be a contradiction, proving the theorem. Let

$$S = \{(D, p) : D \in \mathcal{D}_{\mathbb{Q}},\ Df_0 \in V,\ p = \Phi(Df_0)\}.$$

We claim that the system S is non-contradictory. By Lemma 2.3, it is enough to show that if $A_i \in \mathcal{D}_{\mathbb{Q}}$ and $(D_i, p_i) \in S$ $(i = 1, \dots, n)$, then $\sum_{i=1}^n A_i D_i = 0$ implies $\sum_{i=1}^n A_i p_i = 0$. Since Φ commutes with rational translations, it also commutes with the elements of $\mathcal{D}_{\mathbb{Q}}$, and thus we have

$$\sum_{i=1}^n A_i p_i = \sum_{i=1}^n A_i \Phi(D_i f_0) = \sum_{i=1}^n \Phi(A_i D_i f_0)$$
$$= \Phi\Big(\Big(\sum_{i=1}^n A_i D_i\Big) f_0\Big) = \Phi(\mathbf{0}) = \mathbf{0}.$$

Then S is non-contradictory, as we stated.

By Theorem 4.7, S has a polynomial solution. Let q be such a solution; then $Dq = p$ for every $(D,p) \in S$. Let $\Phi^*(v + Df_0) = \Phi(v) + Dq$ for every $v \in V$ and $D \in \mathcal{D}_{\mathbb{Q}}$. It is easy to check that Φ^* is a well-defined extension of Φ and that it commutes with rational translations. ∎

Our third construction is also related to Theorem 5.2. It shows that if f_0 has linearly independent translates (that is, if $Df_0 \neq 0$ for every $D \in \mathcal{D}$, $D \neq 0$), then there is a linear operator $\Phi : \mathbb{C}^{\mathbb{R}} \to \mathcal{P}$ such that $\Phi(f_0) \neq 0$, and Φ commutes with *every* translation.

THEOREM 5.4. *Let $f_0 \in \mathbb{C}^{\mathbb{R}}$ be a function with linearly independent translates. Then there exists a linear operator Φ mapping $\mathbb{C}^{\mathbb{R}}$ into the set of polynomials such that $\Phi(f_0) = \mathbf{1}$ and $G_\Phi = \mathbb{R}$.*

Proof. Let V be a maximal translation-invariant subspace of $\mathbb{C}^{\mathbb{R}}$ that does not contain any function of the form Af_0 ($A \in \mathcal{D}$, $A \neq 0$). Zorn's lemma easily implies that such a maximal subspace exists. Then for every $f \in \mathbb{C}^{\mathbb{R}}$ there are $v \in V$ and $A, D \in \mathcal{D}$ such that $D \neq 0$ and $v + Df = Af_0$. Indeed, if $f \in V$ then we may take $v = -f$, $D = T_0$ (the identity operator) and $A = 0$. On the other hand, if $f \notin V$, then the set $\{v + Df : v \in V,\ D \in \mathcal{D}\}$ is a translation-invariant subspace of $\mathbb{C}^{\mathbb{R}}$ strictly larger than V. By the maximality of V, there are difference operators A, D such that $A \neq 0$ and $v + Df = Af_0$. Clearly, this implies $D \neq 0$.

If $v + Df = Af_0$ ($v \in V$; $D, A \in \mathcal{D}$, $D \neq 0$) then we define $\phi_f(z) = A^\wedge(z)/D^\wedge(z)$. This definition makes sense, that is, ϕ_f is independent of v, A and D. Indeed, suppose that $v' + D'f = A'f_0$, where $v' \in V$ and $D', A' \in \mathcal{D}$, $D' \neq 0$. Then $Dv' - D'v = (DA' - D'A)f_0$. Since $Dv' - D'v \in V$, this implies that $DA' = D'A$, $D^\wedge A'^\wedge = D'^\wedge A^\wedge$ and $A^\wedge/D^\wedge = A'^\wedge/D'^\wedge$.

We show that the map $f \mapsto \phi_f$ is linear. Let $f \in \mathbb{C}^{\mathbb{R}}$ and $a \in \mathbb{R}$. If $v + Df = Af_0$ then $av + D(af) = aAf_0$, from which $\phi_{af} = (aA)^\wedge/D^\wedge = aA^\wedge/D^\wedge = a\phi_f$. If $v + Df = Af_0$ and $w + Eg = Bf_0$ then $(Ev + Dw) + DE(f + g) = (EA + DB)f_0$. If $D, E \neq 0$ then $DE \neq 0$ and thus

$$\phi_{f+g} = (EA + DB)^\wedge/(DE)^\wedge = (A^\wedge/D^\wedge) + (B^\wedge/E^\wedge) = \phi_f + \phi_g.$$

If $\phi(z)$ is a meromorphic function on the complex plane then we denote by $L\phi$ the constant term of the Laurent expansion of ϕ around zero. Then L is a linear functional, and for every ϕ, the function $x \mapsto L(\phi(z)e^{xz})$ is a polynomial. Indeed, if $\phi(z) = \sum_{n=-k}^{\infty} a_n z^n$ where $k \geq 0$, then

$$L(\phi(z)e^{xz}) = L\biggl(\sum_{n=-k}^{\infty} a_n z^n \cdot \sum_{m=0}^{\infty} \frac{x^m}{m!} z^m\biggr) = \sum_{i=0}^{k} a_{-i} \frac{x^i}{i!}.$$

Now we define

$$\Phi(f)(x) = L(\phi_f(z)e^{xz}) \qquad (x \in \mathbb{R})$$

for every $f \in \mathbb{C}^{\mathbb{R}}$. Then $\Phi(f)$ is a polynomial for every f, and Φ is a linear operator. We have $\phi_{f_0} \equiv 1$, and $\Phi(f_0)(x) = L(e^{xz}) = 1$ for every x.

In order to prove that Φ commutes with translations, let $f \in \mathbb{C}^{\mathbb{R}}$ and $g = T_b f$ where $b \in \mathbb{R}$. If $v + Df = Af_0$, where $v \in V$ and $D \neq 0$, then $T_b v + Dg = T_b v + D(T_b f) = T_b A f_0$. This gives $\phi_g = (T_b A)^\wedge / D^\wedge$; that is,

$$\phi_g(z) = (T_b A)^\wedge(z)/D^\wedge(z) = T_b^\wedge(z) A^\wedge(z)/D^\wedge(z) = e^{bz}\phi_f(z).$$

Then

$$\begin{aligned}\Phi(T_b f)(x) &= \Phi(g)(x) = L(\phi_g e^{xz}) \\ &= L(\phi_f e^{bz} e^{xz}) = L(\phi_f e^{(b+x)z}) = \Phi(f)(x+b),\end{aligned}$$

and $\Phi T_b = T_b \Phi$. ∎

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Received 26 August 1996;
revised 15 February 1997