COLLOQUIUM MATHEMATICUM

VOL. 80 1999 NO. 2

ON QUASI-p-BOUNDED SUBSETS

BY

M. SANCHIS (CASTELLÓ) AND
A. TAMARIZ-MASCARÚA (MÉXICO)

Abstract. The notion of quasi-p-boundedness for $p \in \omega^*$ is introduced and investigated. We characterize quasi-p-pseudocompact subsets of $\beta(\omega)$ containing ω, and we show that the concepts of RK-compatible ultrafilter and P-point in ω^* can be defined in terms of quasi-p-pseudocompactness. For $p \in \omega^*$, we prove that a subset B of a space X is quasi-p-bounded in X if and only if $B \times P_{\mathrm{RK}}(p)$ is bounded in $X \times P_{\mathrm{RK}}(p)$, if and only if $\mathrm{cl}_{\beta(X\times P_{\mathrm{RK}}(p))}(B \times P_{\mathrm{RK}}(p)) = \mathrm{cl}_{\beta X} B \times \beta(\omega)$, where $P_{\mathrm{RK}}(p)$ is the set of Rudin–Keisler predecessors of p.

1. Introduction. All the spaces considered in this paper are Tikhonov spaces. The *Rudin–Keisler pre-order* $\le_{\mathrm{RK}}$ on $\beta(\omega)$ is defined by $p \le_{\mathrm{RK}} q$ if there exists a function $g : \omega \to \omega$ such that $g^\beta(q) = p$ where g^β is the continuous extension of g to $\beta(\omega)$. If $p \le_{\mathrm{RK}} q$ and $q \le_{\mathrm{RK}} p$, for $p, q \in \omega^*$, then we say that p and q are RK-*equivalent* and we write $p \approx_{\mathrm{RK}} q$. It is not difficult to verify that $p \approx_{\mathrm{RK}} q$ if and only if there is a permutation σ of ω such that $\sigma^\beta(p) = q$. For $p \in \omega^*$, we set $P_{\mathrm{RK}}(p) = \{r \in \beta(\omega) : r \le_{\mathrm{RK}} p\}$. The *type* of $p \in \omega^*$ is the set $T(p) = \{r \in \omega^* : p \approx_{\mathrm{RK}} r\}$. We denote by $\Sigma(p)$ the set $T(p) \cup \omega$.

For $p, q \in \beta(\omega)$ we write $p <_{\mathrm{R}} q$ if there is a surjection $f : \omega \to \omega$ such that $f^\beta(q) = p$ and for every $A \in q$ there is $n < \omega$ for which $|A \cap f^{-1}(n)| = \omega$. If $p <_{\mathrm{R}} q$, $r \approx_{\mathrm{RK}} p$ and $s \approx_{\mathrm{RK}} q$, then $r <_{\mathrm{R}} s$. The *Rudin pre-order* $\le_{\mathrm{R}}$ on $\beta(\omega)$, introduced in [17], is defined by $p \le_{\mathrm{R}} q$ if either $p \approx_{\mathrm{RK}} q$ or $p <_{\mathrm{R}} q$. It is obvious that $p \le_{\mathrm{R}} q$ implies $p \le_{\mathrm{RK}} q$. For $p \in \omega^*$ let $P_{\mathrm{R}}(p)$ be the set $\{r \in \beta(\omega) : r \le_{\mathrm{R}} p\}$.

An ω-*partition* of ω is a cover of ω consisting of infinite pairwise disjoint subsets. For each $A \subset \omega$ the symbol $\widehat{A}$ indicates the set $\{p \in \beta(\omega) : A \in p\}$.

1991 *Mathematics Subject Classification*: 54A20, 54A25, 54C50, 54G20.

Key words and phrases: free ultrafilter, p-limit point, bounded subset, (quasi)-p-bounded subset, (quasi)-p-pseudocompact space, Rudin–Keisler pre-order, P-point.

The first author acknowledges the support of the DEGS, under grant PB95-0737. The second-listed author was supported by Proyecto de Cooperación Intercampus. This author is also grateful to the Department of Mathematics of Jaume I University for its generous hospitality during February–March, 1997.

Two ultrafilters $p, q \in \omega^*$ are RK-*compatible* if there is $s \in \omega^*$ such that $s \leq_{\mathrm{RK}} p$ and $s \leq_{\mathrm{RK}} q$.

1.1. Definition. For $p \in \omega^*$, a point $x \in X$ is said to be a *p-limit point* of a sequence $(U_n)_{n<\omega}$ of nonempty subsets of X (in symbols: $x = p\text{-}\lim(U_n)$) if, for each neighborhood V of x, the set $\{n < \omega : U_n \cap V \neq \emptyset\}$ belongs to p.

This notion was introduced by Ginsburg and Saks [10] by generalizing the notion of p-limit point discovered and investigated by Bernstein [1]. It should be mentioned that Bernstein's p-limit concept was also introduced, in a different form, by Frolík [5] and Katětov [13], [14]. A subset B of a space X is said to be *bounded* (*in* X) if every real-valued continuous function on X is bounded on B. In [15] N. Noble proved that B is bounded in X if (and only if) every sequence of (pairwise disjoint) open sets of X meeting B has a cluster point. Starting from this fact and the above concept of p-limit point, S. García-Ferreira [7] introduced the notion of p-bounded subset for $p \in \omega^*$: a subset B is *p-bounded* (*in* X) if every sequence of open subsets meeting B has a p-limit point. Obviously, for each $p \in \omega^*$, every p-bounded subset (in X) is bounded but the converse does not hold in general (see e.g. [7, Theorem 1.10]). Later, p-boundedness was widely studied by the authors in [18]. Here we are concerned with quasi-p-boundedness, a notion weaker than p-boundedness:

1.2. Definition. Let $p \in \omega^*$. A subset B of a space X is called *quasi-p-bounded in* X if every sequence of pairwise disjoint open subsets of X meeting B has a subsequence which admits a p-limit point.

Recall that a space is said to be *pseudocompact* if it is bounded in itself. Analogously, for $p \in \omega^*$, a space X is *quasi-p-pseudocompact* (respectively, *p-pseudocompact*) if it is quasi-p-bounded (resp., p-bounded) in itself. If either $q <_{\mathrm{RK}} p$ or q and p are $\leq_{\mathrm{RK}}$-incomparable, then $\Sigma(p)$ is a pseudocompact space which is not quasi-q-pseudocompact (see Corollary 3.4 and Example 3.5). So, p-boundedness implies quasi-p-boundedness and quasi-p-boundedness implies boundedness but none of these implications can be reversed.

The paper is organized as follows: Section 2 is devoted to proving several basic results on quasi-p-boundedness. In Section 3, we characterize the subsets of $\beta(\omega)$ which are quasi-p-bounded for some $p \in \omega^*$ and we apply these results to determine when $\Sigma(q)$, $P_{\mathrm{RK}}(q)$ and $T(q)$ are quasi-p-pseudocompact. Finally, in Section 4, we show that, for $p \in \omega^*$, a bounded subset B of X is quasi-p-bounded if and only if its product with $P_{\mathrm{RK}}(p)$ is bounded in $X \times P_{\mathrm{RK}}(p)$, if and only if $\mathrm{cl}_{\beta X} B \times \beta(\omega) = \mathrm{cl}_{\beta(X \times P_{\mathrm{RK}}(p))}(B \times P_{\mathrm{RK}}(p))$.

Our notation is standard: $\mathrm{cl}_X A$ and $\mathrm{int}_X A$ denote the closure and the interior, respectively, of a subset A of X. A subset A of X is called *regular-*

closed if $A = \mathrm{cl}_X(\mathrm{int}_X A)$. The symbol $\mathbb{R}$ stands for the real numbers endowed with the usual topology. For terminology and notation not defined here and for general background see [4] and [8].

2. Basic results on quasi-p-bounded subsets. We begin by showing several useful lemmas.

2.1. Lemma. *Let $p \in \omega^*$, $(U_n)_{n<\omega}$ be a sequence of subsets of a space X, and $x \in X$. Then*:

(1) *If $g : \omega \to \omega$ is a function satisfying $g^\beta(p) = r$, then $x = r\text{-}\lim(U_n)$ if and only if $x = p\text{-}\lim(U_{g(n)})$*;

(2) *If there are $r' \in \omega^*$ with $r' \le_{\mathrm{RK}} p$, and a subsequence $(V_n)_{n<\omega}$ of $(U_n)_{n<\omega}$ such that $x = r'\text{-}\lim(V_n)$, then there is an r-limit point in X of $(U_n)_{n<\omega}$ with $r \le_{\mathrm{RK}} p$.*

Proof. We obtain (1) because $W \subset \omega$ belongs to r if and only if $g^{-1}(W) \in p$, and $\{n < \omega : U_{g(n)} \cap A \neq \emptyset\} = g^{-1}(\{n < \omega : U_n \cap A \neq \emptyset\})$ for every $A \subset X$.

Now we prove (2). For each $n<\omega$ there is $k(n)<\omega$ such that $V_n = U_{k(n)}$. Let $g : \omega \to \omega$ be defined by $g(n) = k(n)$. By (1), $x = r\text{-}\lim(U_n)$ where $r = g^\beta(r')$. Moreover, $r \le r' \le p$. ■

The following lemma is already known and we omit the proof.

2.2. Lemma. *Let X be a Hausdorff space and let $(A_n)_{n<\omega}$ be a sequence of nonempty open subsets of X. Then either there exists $n_0 < \omega$ such that $A_n = A_{n_0}$ for every $n \ge n_0$ and $|A_{n_0}| < \aleph_0$, or there is a sequence $(k_n)_{n<\omega}$ of natural numbers and a sequence $(B_n)_{n<\omega}$ of nonempty disjoint open subsets of X such that $B_n \subset A_{k_n}$ for every $n < \omega$.*

2.3. Theorem. *Let X be a topological space and let $p \in \omega^*$. For each subset B of X, the following conditions are equivalent*:

(1) *B is quasi-p-bounded in X*;

(2) *Every sequence of open nonempty subsets of X meeting B has a subsequence which has a p-limit point in X*;

(3) *For every sequence $(U_n)_{n<\omega}$ of nonempty open subsets of X meeting B there are $r \in \omega^*$, with $r \le_{\mathrm{RK}} p$, and $x \in X$ such that $x = r\text{-}\lim(U_n)$*;

(4) *For every sequence $(U_n)_{n<\omega}$ of open nonempty subsets of X meeting B, there are a subsequence $(V_n)_{n<\omega}$ of $(U_n)_{n<\omega}$, an $r \in \omega^*$ with $r \le_{\mathrm{RK}} p$, and $x \in X$ such that $x = r\text{-}\lim(V_n)$.*

Proof. The implications (2)⇒(1) and (3)⇒(4) are trivial. Moreover, the implications (1)⇒(2), (2)⇒(3), (4)⇒(3) and (3)⇒(2) are consequences of Lemmas 2.2, 2.1(1), 2.1(2) and 2.1(1), respectively. ■

In view of this last theorem, the concept of quasi-p-pseudocompactness is equivalent to the concept of M-pseudocompactness, with $M = P_{\mathrm{RK}}(p)$, introduced in [7], which coincides with condition (3) of Theorem 2.3.

The proof of the following lemma is left to the reader.

2.4. Lemma. *For each $p \in \omega^*$, the following conditions hold*:

(1) *Quasi-p-boundedness is preserved under continuous functions*;
(2) *Quasi-p-pseudocompactness is inherited by regular closed subsets.*

A *Frolík sequence* in a space X is a sequence $(U_n)_{n<\omega}$ of open subsets of X such that for each filter $\mathcal{G}$ of infinite subsets of ω,

$$\bigcap_{F\in\mathcal{G}} \mathrm{cl}_X \Big(\bigcup_{n\in F} U_n\Big) \neq \emptyset.$$

A subset B of a space X is *strongly bounded in X* (see [19]) if each infinite family of mutually disjoint open subsets of X meeting B contains a disjoint subfamily $(U_n)_{n<\omega}$ which is a Frolík sequence. The *Frolík class $\mathcal{P}$* is the class of pseudocompact spaces whose product with each pseudocompact space is also pseudocompact. So, Theorem 3.6 of [6] says:

2.5. Theorem. *A pseudocompact space belongs to the Frolík class $\mathcal{P}$ if, and only if, it is strongly bounded in itself.*

2.6. Theorem. *If a subset B is strongly bounded in X, then B is quasi-p-bounded in X for each $p \in \omega^*$.*

Proof. Let $p \in \omega^*$ and let $(U_n)_{n<\omega}$ be a sequence of pairwise disjoint open sets whose elements meet B. Since B is strongly bounded in X, there exist a subsequence $(U_{n(k)})_{k<\omega}$ of $(U_n)_{n<\omega}$ and $x \in X$ such that

$$x \in \bigcap_{F\in p} \mathrm{cl}_X \Big(\bigcup_{k\in F} U_{n(k)}\Big).$$

It is apparent that x is a p-limit point of $(U_{n(k)})_{k<\omega}$. ∎

As an immediate consequence of the previous result, pseudocompact spaces in the Frolík class $\mathcal{P}$ are quasi-p-pseudocompact for every $p \in \omega^*$. We shall explore this fact in the following. Consider the (proper) subclass $\mathcal{P}^*$ of $\mathcal{P}$ defined as the class of spaces X with the property that each sequence of disjoint open sets in X has a subsequence such that each of its elements meets some fixed compact set. This class was introduced and studied by N. Noble in [16]. In particular, Noble showed that $X \in \mathcal{P}^*$ whenever $k_{\mathrm{R}}X$, the k_{R}-space associated with X (that is, the set X endowed with the weak topology induced by the real-valued functions on X which are continuous on all compact subsets of X) is pseudocompact. Thus, pseudocompact spaces which are locally compact or sequential are quasi-p-pseudocompact for every $p \in \omega^*$ (for an example of a space in $\mathcal{P}^*$ such that $k_{\mathrm{R}}X$ is not pseudocompact, see [2] and [12]). As every completely regular space can be embedded as a closed subspace of a pseudocompact k_{R}-space [16, 2.3], we have the following result.

2.7. THEOREM. *Every pseudocompact space can be embedded as a closed subspace of a space which is quasi-p-pseudocompact for each $p \in \omega^*$. So, quasi-p-pseudocompactness is not inherited by closed pseudocompact subsets.*

In the context of this result the question of characterizing quasi-p-pseudocompact spaces whose closed sets are also quasi-p-pseudocompact arises. We are concerned with this question in the following theorem.

2.8. THEOREM. *Let $p \in \omega^*$. Every closed subset of a space X is quasi-p-pseudocompact if and only if every sequence in X contains a subsequence which admits a p-limit.*

Proof. Suppose that every closed subset of X is a quasi-p-pseudocompact space and let $(x_n)_{n<\omega}$ be a sequence in X. We can assume, without loss of generality, that $\{x_n : n < \omega\}$ contains no p-limit points of $(x_n)_{n<\omega}$. We prove, by induction on n, that there is a subsequence $(y_n)_{n<\omega}$ of $(x_n)_{n<\omega}$ which is a copy of ω. In fact, put $y_0 = x_0$ and suppose that, for $k < \omega$, there exist a subset $\{y_0, \ldots, y_k\}$ where $y_s = x_{g(s)}$ and $g(s) < g(s+1)$, $s = 0, 1, \ldots, k-1$, and a family $(U_n)_{n\le k}$ of pairwise disjoint open subsets such that

$$y_n \in U_n, \quad n = 0, 1, \ldots, k, \tag{1}$$

$$M_n = \{t < \omega : x_t \in \mathrm{cl}_X\, U_n\} \notin p. \tag{2}$$

By inductive hypothesis, $M = \bigcap_{n\le k}(\omega \setminus M_n)$ belongs to p. Let $m \in M$ be such that $m > g(k)$. We define $y_{k+1} = x_m$. The induction step is finished by taking an open neighborhood V of y_{k+1} which does not meet U_n for every $n \le k$ and such that $\{n < \omega : x_n \in V\} \notin p$, and by taking an open set U_{k+1} containing y_{k+1} and such that its closure is a subset of V (so, U_{k+1} is an open neighborhood of y_{k+1} which does not meet U_n for every $n \le k$ and such that $\{t < \omega : x_t \in \mathrm{cl}_X\, U_{k+1}\} \notin p$).

Now, consider $H = \mathrm{cl}_X\{y_n\}_{n<\omega}$. Since $\{y_n\}_{n<\omega}$ is a copy of ω, it is a sequence of open sets in H. By assumption, $(y_n)_{n<\omega}$ admits a subsequence having a p-limit point, as was to be proved. The converse is clear. ∎

Relating to the previous theorems, we construct a space in the class $\mathcal{P}$ which is not p-pseudocompact for any $p \in \omega^*$.

2.9. EXAMPLE. For each $p \in \omega^*$, let $X(p) = \beta(\omega) \setminus \{p\}$. Since $P_{\mathrm{RK}}(p)$ is not contained in $X(p)$, $X(p)$ is not p-pseudocompact [7, Lemma 1.9]. Let $Y = \prod_{p\in\omega^*} X(p)$. For every $p \in \omega^*$, the space Y is not p-pseudocompact because the image of Y under the p-projection is $X(p)$. But $X(p) \in \mathcal{P}$ for each $p \in \omega^*$ [6, Example 4.4] and, since the class $\mathcal{P}$ is closed under arbitrary products [16, Theorem 3.1], Y is also in $\mathcal{P}$. In particular, by Theorem 2.6, Y is quasi-q-pseudocompact for every $q \in \omega^*$.

Later (in Example 3.2) we will see an example of a quasi-p-pseudocompact space for every $p \in \omega^*$ which does not belong to $\mathcal{P}$.

Let α be a cover of a space X. A function g from X into a space Y is *α-continuous* if the restriction of g to each member of α is continuous. A space X for which every real-valued α-continuous function is continuous is called an *α_{R}-space*. We say that a point $x \in X$ is an *α_{R}-point* if there exists a neighborhood of x which is an α_{R}-space. For instance, k_{R}-spaces are α_{R}-spaces when α is the cover of compact sets. In the following, if $p \in \omega^*$, we denote by $\alpha(p)$ the cover of all quasi-p-pseudocompact subsets of X.

2.10. THEOREM. *Let $p \in \omega^*$ and let B be a bounded subset of a space X. If every point of X is either an $\alpha(q)_{\mathrm{R}}$-point for some $q \leq_{\mathrm{RK}} p$ or a P-point, then B is quasi-p-bounded in X.*

Proof. If B is not quasi-p-bounded in X, by Lemma 2.2 and Theorem 2.3(4), there exists a sequence $(U_n)_{n<\omega}$ of pairwise disjoint open sets in X meeting B such that for each quasi-q-pseudocompact subset Y of X, with $q \leq_{\mathrm{RK}} p$, only a finite subcolection of $\{U_n : n < \omega\}$ meet Y. We shall see that this fact leads us to a contradiction. Consider a sequence $(V_n)_{n<\omega}$ of regular-closed sets meeting B and that $V_n \subset U_n$ for every $n < \omega$. For all $n < \omega$, let $x_n \in \operatorname{int}_X V_n$ and define a real-valued continuous function f_n such that $f_n(x_n) = n$ and $f_n(X \setminus V_n) = 0$.

We prove that the function $f(x) = \sum_{n<\omega} f_n$ is continuous. Let $x \in X$. Since $V_n \cap V_m = \emptyset$ when $n \neq m$, f is continuous in $\bigcup_{m<\omega} \operatorname{int} V_m$. If x is a P-point of X belonging to $X \setminus \bigcup_{n<\omega} V_n$, then f is zero on the neighborhood $\bigcap_{n<\omega}(X \setminus V_n)$ of x. So, f is continuous at x.

Suppose now that $x \in X \setminus \bigcup_{m<\omega} \operatorname{int} V_m$ is not a P-point. By assumption x is an $\alpha(q)_{\mathrm{R}}$-point for some $q \leq_{\mathrm{RK}} p$. So there exists a neighborhood V of x which is an $\alpha(q)_{\mathrm{R}}$-space. Let $Q \subset V$ be a quasi-q-pseudocompact space. Then Q only meets a finite subcollection of $\{V_n : n < \omega\}$ and, consequently, f agrees on Q with a finite sum of continuous functions. Hence, f is continuous at Q. Thus, since V is an $\alpha(q)_{\mathrm{R}}$-space, $f|_V$ is continuous; but V is a neighborhood of x, so f is continuous at x. As f is continuous on all of X and unbounded on B, we have just obtained a contradiction. ∎

2.11. COROLLARY. *Let $p \in \omega^*$. Each open pseudocompact subset of a quasi-p-pseudocompact space is quasi-p-pseudocompact.*

Proof. Let X be a quasi-p-pseudocompact space and consider an open pseudocompact subset P of X. Since each point of P belongs to a regular-closed subset contained in P, each point of P is an $\alpha(p)_{\mathrm{R}}$-point. Thus, the result is a consequence of Theorem 2.10. ∎

2.12. Corollary. *Let $p \in \omega^*$. A free topological sum $X = \bigoplus_{\alpha\in A} X_\alpha$, where $X_\alpha \neq \emptyset$, is quasi-p-pseudocompact if and only if each X_α is quasi-p-pseudocompact and $|A| < \aleph_0$.*

3. Quasi-p-pseudocompactness in $\beta(\omega)$. This section is devoted to studying the notion of quasi-p-pseudocompactness in $\beta(\omega)$. In [7, Lemma 1.9] it was proven that $P_{\mathrm{RK}}(p)$ is p-pseudocompact for every $p \in \omega^*$. Our first result in this section relates quasi-p-pseudocompactness to $P_{\mathrm{RK}}(p)$.

3.1. Theorem. *Let $\omega \subset X \subset \beta(\omega)$ and $p \in \omega^*$. Then the following assertions are equivalent*:

(1) *X is quasi-p-pseudocompact*;
(2) *$X \cap P_{\mathrm{RK}}(p)$ is quasi-p-pseudocompact*;
(3) *$(X \cap P_{\mathrm{RK}}(p)) \setminus \omega$ is dense in ω^*.*

Proof. (1)⇒(2). Assume that X is quasi-p-pseudocompact, and let $(U_n)_{n<\omega}$ be a sequence of pairwise disjoint open sets in $X \cap P_{\mathrm{RK}}(p)$. For each $n < \omega$, choose $k_n \in U_n \cap \omega$. The sequence $(\{k_n\})_{n<\omega}$ has an r-limit point $x \in X$ where $r \in \omega^*$ and $r \leq_{\mathrm{RK}} p$. Define $g : \omega \to \omega$ by $g(n) = k_n$. If $B \in x$, then $\{n < \omega : k_n \in B\} = \{n < \omega : g(n) \in B\} = g^{-1}(B) \in r$. So $B \in g^\beta(r)$. Thus, $g^\beta(r) = x$; that is, $x \leq_{\mathrm{RK}} r \leq_{\mathrm{RK}} p$. We have just proved that $x \in X \cap P_{\mathrm{RK}}(p)$ and $x = r\text{-}\lim(U_{k_n})$.

(2)⇒(3). Let A be an infinite subset of ω. We are going to prove that there exists a free ultrafilter on ω that belongs to $P_{\mathrm{RK}}(p) \cap X \cap \widehat{A}$. Let $g : \omega \to \omega$ be an injective function which enumerates A: $A = \{g(n) : n < \omega\}$. By assumption, there is a subsequence of $(\{g(n)\})_{n<\omega}$ which has a p-limit point in $P_{\mathrm{RK}}(p) \cap X$. By Lemma 2.1, the sequence $(\{g(n)\})_{n<\omega}$ has an r-limit point $x \in X$ where $r \in \omega^*$ and $r \leq_{\mathrm{RK}} p$. Thus, $g^\beta(r) = x$; that is, for every $B \in x$, we have $g^{-1}(b) = \{n < \omega : g(n) \in B\} \in r$. So, $B \cap A \neq \emptyset$. Then $A \in x$; and this means that $x \in \widehat{A} \cap X$. Moreover, $x \leq_{\mathrm{RK}} r \leq_{\mathrm{RK}} p$, and x is free because otherwise we contradict the injectivity of g.

(3)⇒(1). Let $(A_n)_{n<\omega}$ be a sequence of nonempty subsets of ω. We are going to prove that the sequence $(\widehat{A}_n \cap X)_{n<\omega}$ of nonempty open subsets of X has an r-limit point in X, where $r \in \omega^*$ and $r \leq_{\mathrm{RK}} p$. For each $n < \omega$, let $g(n)$ be an element of A_n. Take the set $A = \{g(n) : n < \omega\}$. Using our hypothesis, we obtain an $x_g \in X \cap P_{\mathrm{RK}}(p) \cap \widehat{A} \cap \omega^*$. Hence, $A \in x_g$, $x_g \leq_{\mathrm{RK}} p$, $x_g \in X$ and x_g is a free ultrafilter. The collection $\{g^{-1}(g(n)) : n < \omega\}$ defines a partition on ω, so it defines an equivalence relation R in ω. Let ω/R be the collection of equivalence classes, and let $c : \omega \to \omega/R$ be the function which assigns to each $n < \omega$ its equivalence class. We choose a function ξ

on $\{c(n) : n < \omega\}$ with values in ω such that $\xi(c(n)) \in g^{-1}(g(n))$. Also, we take a function $h : \omega \to \omega$ which satisfies $h^\beta(p) = x_g$. Finally, we define $\phi : \omega \to \omega$ in the following way: $\phi(n) = \xi(c(m))$ if $h(n) = g(m)$, and $\phi(n) = 0$ if $h(n) \notin A$. The relation ϕ is a function from ω to ω. Let r_g be the image of p under ϕ^β. In particular, we have $r_g \leq_{\mathrm{RK}} p$.

We are going to prove that $x_g = r_g\text{-}\lim\{g(n)\}$, that is, for every $B \in x_g$, $g^{-1}(B) \in r_g$. In order to do this, it is enough to prove that for every $B \in x_g$, $\phi^{-1}g^{-1}(B) \in p$. But $g^{-1}(B) \supset g^{-1}(B \cap A)$ (recall that $B \cap A \in x_g$). Then $\phi^{-1}(g^{-1}(B)) \supset \phi^{-1}(g^{-1}(B \cap A))$, and this last set contains $h^{-1}(B \cap A)$. In fact, let $x \in h^{-1}(B \cap A)$, so $h(x) = g(m)$ for some $m < \omega$. This means that $\phi(x) = \xi(c(m)) \in g^{-1}(g(m))$. Hence, $g(\phi(x)) = g(m) \in B \cap A$. Therefore, $\phi(x) \in g^{-1}(B \cap A)$. Since $h^{-1}(B \cap A) \in p$, $\phi^{-1}(g^{-1}(B)) \in p$. This implies that $g^{-1}(B) \in r_g$, so $x_g = r_g\text{-}\lim\{g(n)\}$. ∎

Now, we obtain some results that are consequences of the previous theorem.

3.2. Example. Let p be a free non-RK-minimal ultrafilter on ω. The space $X = \beta(\omega) \setminus T(p)$ is quasi-q-pseudocompact for all $q \in \omega^*$ and does not belong to $\mathcal{P}$.

Proof. In fact, let $q \in \omega^*$. If $p \not\approx_{\mathrm{RK}} q$ then $T(q) \subset X \cap P_{\mathrm{RK}}(q)$, and if $p \approx_{\mathrm{RK}} q$ then $X \cap P_{\mathrm{RK}}(q) \supset T(r)$ where $r \in \omega^*$ is strictly less that p in the Rudin–Keisler pre-order. So, in both cases, $X \cap P_{\mathrm{RK}}(q)$ is dense in ω^*. By Theorem 3.1 we conclude that X is quasi-q-pseudocompact for every $q \in \omega^*$.

Now we are going to prove that X does not belong to $\mathcal{P}$. Let $U_n = \{n\}$ for each $n \in \omega$, and let $\{V_n : n < \omega\}$ be a subsequence of $\{U_n : n < \omega\}$ such that $V_n \neq V_m$ if $n \neq m$; that is, for each $n < \omega$ there is $k_n < \omega$ such that $V_n = U_{k_n}$. The function $f : \omega \to \omega$ defined by $f(n) = k_n$ is one-to-one. Moreover,

$$\bigcap_{N\in p} \mathrm{cl}_X \Big(\bigcup_{n\in N} V_n\Big) = \bigcap_{N\in p} \mathrm{cl}_X(f(N)) = \Big(\bigcap_{N\in p} \mathrm{cl}_{\beta(\omega)} f(N)\Big) \cap X.$$

But $\bigcap_{N\in p} \mathrm{cl}_{\beta(\omega)} f(N) = \{f^\beta(p)\}$ and $f^\beta(p) \in T(p)$; therefore,

$$\bigcap_{N\in p} \mathrm{cl}_X \Big(\bigcup_{n\in N} V_n\Big) = \emptyset.$$

We conclude, using Theorem 2.5, that X is not in $\mathcal{P}$. ∎

Another consequence of Theorem 3.1 is the following.

3.3. Corollary. *For $p, q \in \omega^*$, $P_{\mathrm{RK}}(q)$ is quasi-p-pseudocompact if and only if p and q are* RK-*compatible.*

Blass and Shelah [3] have defined a model $\mathfrak{M}$ of ZFC in which

$$\mathfrak{M} \models \forall p, q \in \omega^* \ \exists r \in \omega^* \ (r \leq_{\mathrm{RK}} p \wedge r \leq_{\mathrm{RK}} q),$$

so, by Corollary 3.3,

$$\mathfrak{M} \models \forall p \in \omega^* \ (P_{\mathrm{RK}}(p) \text{ is quasi-}q\text{-pseudocompact for every } q \in \omega^*).$$

(Observe that $P_{\mathrm{RK}}(p)$ does not belong to $\mathcal{P}$ because if $p <_{\mathrm{RK}} q$, then $P_{\mathrm{RK}}(p) \times \Sigma(q)$ is not pseudocompact.)

By definition, if $q \le_{\mathrm{RK}} p$, then every quasi-q-pseudocompact space is quasi-p-pseudocompact. Moreover, Theorem 3.1 shows that $\Sigma(q)$ is quasi-q-pseudocompact, and if $\Sigma(q)$ is quasi-p-pseudocompact, then we must have $q \le_{\mathrm{RK}} p$. So, we obtain:

3.4. Corollary. *Let $p, q \in \omega^*$. The following are equivalent*:

(1) $q \le_{\mathrm{RK}} p$;
(2) *Every quasi-q-pseudocompact space is quasi-p-pseudocompact*;
(3) *$\Sigma(q)$ is quasi-p-pseudocompact.*

Now we are able to give an example of a pseudocompact space which is not quasi-p-pseudocompact for any $p \in \omega^*$.

3.5. Example. Let K be the one-point compactification of the space $\bigoplus_{p\in\omega^*}(\beta(\omega) \times \{p\})$. The subspace $X = \bigoplus_{p\in\omega^*}(\Sigma(p) \times \{p\}) \cup \{x_0\}$ of K, where x_0 is the distinguished point in K, is a pseudocompact space. Also, X contains a clopen copy of $\Sigma(p)$ for each $p \in \omega^*$. Since ω^* does not have $\le_{\mathrm{RK}}$-maximal elements, and because of Lemma 2.4 and Corollary 3.4, X is not quasi-p-pseudocompact for any $p \in \omega^*$.

We finish this section by studying the space $T(p)$ related to the properties that we are analyzing. We begin by determining when $T(q)$ is quasi-p-pseudocompact and we characterize P-points in ω^* in terms of quasi-p-pseudocompactness of $T(p)$. The following result, proved in [7], will help us.

3.6. Theorem. *For $p, q \in \omega$, $p <_{\mathrm{R}} q$ if and only if $T(q)$ is p-pseudocompact.*

3.7. Theorem. *Let $p, q \in \omega^*$. The space $T(q)$ is quasi-p-pseudocompact if and only if $(P_{\mathrm{RK}}(p) \cap P_{\mathrm{R}}(q)) \setminus \Sigma(q) \neq \emptyset$.*

Proof. Assume that $T(q)$ is quasi-p-pseudocompact and let $(A_n)_{n<\omega}$ be an ω-partition of ω. There are $r \le_{\mathrm{RK}} p$ and $s \in T(q)$ such that $s = r\text{-}\lim \widehat{A}_n$. Thus, for each $A \in s$, $\{n<\omega : \widehat{A} \cap \widehat{A}_n \neq \emptyset\} \in r$. Since $\{n<\omega : |A \cap A_n| = \aleph_0\} \supset \{n < \omega : \widehat{A} \cap \widehat{A}_n \neq \emptyset\}$, it follows that $\{n < \omega : |A \cap A_n| = \aleph_0\} \in r$. Let $f : \omega \to \omega$ be defined by $f(m) = n$ if $m \in A_n$. The function f is surjective and $\{n < \omega : |A \cap f^{-1}(n)| = \aleph_0\} \in r$ for each $A \in s$. Then $r <_{\mathrm{R}} s$. Since $s \approx_{\mathrm{RK}} q$, we have $r <_{\mathrm{R}} q$. Therefore, $r \in (P_{\mathrm{RK}}(p) \cap P_{\mathrm{R}}(q)) \setminus \Sigma(q)$.

Now, if $r \in (P_{\mathrm{RK}}(p) \cap P_{\mathrm{R}}(q)) \setminus \Sigma(q)$, then $r <_{\mathrm{R}} q$, so $T(q)$ is r-pseudocompact (Theorem 3.6). In particular, $T(q)$ is quasi-p-pseudocompact. ∎

The result that follows generalizes Theorem 5.3 in [10].

3.8. Corollary. *Let $q \in \omega^*$. The following are equivalent*:

(1) *q is a P-point in ω^**;
(2) *$T(q)$ is not pseudocompact*;
(3) *$T(q)$ is not quasi-q-pseudocompact*;
(4) *$T(q)$ is not quasi-p-pseudocompact for any $p \in \omega^*$*;
(5) *$T(q)$ is not p-pseudocompact for any $p \in \omega^*$.*

Proof. The implications (2)⇒(3)⇒(4)⇒(5) are trivial, and (5)⇒(1) is a consequence of Theorem 3.6 (it is also a result due to Ginsburg and Saks in [10]). Finally, (1)⇒(2) holds because if q is a P-point in ω^*, then $T(q)$ is a P-space, and so it cannot be pseudocompact because pseudocompact P-spaces are finite. ■

Also, as a consequence of Theorems 3.6 and 3.7, the space $T(q)$ is quasi-p-pseudocompact if and only if $T(q)$ is r-pseudocompact for some $r \leq_{\mathrm{RK}} p$.

4. Products of quasi-p-bounded subsets. Let $p \in \omega^*$. In [7] it was proved that, if X and Y are p-pseudocompact spaces, then so is $X \times Y$. However, in Example 2.9 a space Y in the Frolík class $\mathcal{P}$ has been constructed which is not p-pseudocompact for any $p \in \omega^*$. Since $Y \in \mathcal{P}$, the product space $X \times Y$ is pseudocompact for each pseudocompact space X. These facts suggest the question of characterizing the spaces whose product with every p-pseudocompact space is pseudocompact. The following theorem answers this question.

4.1. Theorem. *Let $p \in \omega^*$. For a subset A of a space X the following conditions are equivalent*:

(1) *A is quasi-p-bounded in X*;
(2) *For each p-bounded subset B of a space Y, $A \times B$ is quasi-p-bounded in $X \times Y$*;
(3) *For each p-bounded subset B of a space Y, $A \times B$ is bounded in $X \times Y$*;
(4) *$A \times P_{\mathrm{RK}}(p)$ is bounded in $X \times P_{\mathrm{RK}}(p)$.*

Proof. (1)⇒(2). Let $(U_n \times V_n)_{n<\omega}$ be a sequence of open sets in $X \times Y$ meeting $A \times B$. We prove that there is a subsequence of $(U_n \times V_n)_{n<\omega}$ which admits a p-limit point. By assumption, $(U_n)_{n<\omega}$ has a q-limit point for some $q \leq_{\mathrm{RK}} p$. So, by Lemma 2.1(1), there exists a subsequence $(U_{g(n)})_{n<\omega}$ of $(U_n)_{n<\omega}$ and a point $x \in X$ such that $x = p\text{-}\lim(U_{g(n})$. Now, since B is p-bounded in Y, we can find $y \in Y$ such that $y = p\text{-}\lim(V_{g(n)})$. Thus, $(x, y) = p\text{-}\lim(U_{g(n)} \times V_{g(n)})_{n<\omega}$.

(2)⇒(3) and (3)⇒(4) are clear.

(4)⇒(1). Let $(U_n)_{n<\omega}$ be a sequence of open sets in X meeting A. Since $X \times P_{\mathrm{RK}}(p)$ is bounded, $(U_n \times \{n\})_{n<\omega}$ has a cluster point (x, r). We claim that $x = r\text{-}\lim(U_n)$. In fact, suppose to the contrary that there exists a neighborhood U of x such that the set $M = \{n < \omega : U_n \cap U \neq \emptyset\} \notin r$. Since r is an ultrafilter, $\omega \setminus M \in r$. So, $U \times \widehat{\omega \setminus M}$ is a neighborhood of (x, r) missing $U_n \times \{n\}$ for all $n < \omega$, which leads us to a contradiction. ∎

Consequently, a space X is quasi-p-pseudocompact for a $p \in \omega^*$ if and only if $X \times P_{\mathrm{RK}}(p)$ is pseudocompact.

We remind the reader that a *compactification* K of a space X is a compact space containing X as a dense subset. Two compactifications K_1 and K_2 of X are said to be *equivalent* if the identity map on X admits a continuous extension to a homeomorphism from K_1 onto K_2. In this case we write $K_1 = K_2$.

For bounded subsets A and B of two topological spaces X and Y, respectively, the equality $\mathrm{cl}_{\beta(X\times Y)}(A \times B) = \mathrm{cl}_{\beta X} A \times \mathrm{cl}_{\beta Y} B$ has been widely studied (see e.g. [9], [11], [18]). In what follows we analyze this equality in the field of quasi-p-bounded subsets. The following lemma is necessary for our purposes. A proof is available in [9, Lemma 2.5].

4.2. Lemma. *Let A and B be bounded subsets of X and Y, respectively. If $\mathrm{cl}_{\beta(X)} A \times \mathrm{cl}_{\beta(Y)} B = \mathrm{cl}_{\beta(X\times Y)}(A \times B)$, then $A \times B$ is bounded in $X \times Y$.*

We remind the reader that a family $\{f_\delta\}_{\delta\in D}$ of real-valued functions on a space X is said to be *equicontinuous at* $x_0 \in X$ if for every $\varepsilon > 0$ there exists a neighborhood V of x_0 such that, for each $\delta \in D$, $|f_\delta(x) - f_\delta(x_0)| < \varepsilon$ whenever $x \in V$. For each real-valued bounded continuous function on a product space $X \times Y$ we denote by $\beta(f)$ its continuous extension to $\beta(X \times Y)$. Given $x \in X$, $\beta(f)(a, -)$ stands for the continuous extension to βY of the bounded function g on Y defined by the requirement $g(y) = f(x, y)$ whenever $y \in Y$. For each $y \in \beta Y$, $\beta(f)(a, y)$ stands for $\beta(f)(a, -)(y)$ and, if $y \in \beta Y$, $\beta(f)(-, y)$ for the function from X into $\mathbb{R}$ defined by

$$\beta(f)(-, y)(x) = \beta(f)(x, y)$$

whenever $x \in X$. As usual, for each subset U of X, we define the *oscillation* of f in U, $\mathrm{osc}(f, U)$, as $\sup\{|f(x) - f(y)| : (x, y) \in U \times U\}$.

4.3. Theorem. *Let $p \in \omega^*$. For a bounded subset A of X, the following conditions are equivalent*:

(1) *A is quasi-p-bounded*;

(2) *For each p-bounded subset B of a space Y, $\mathrm{cl}_{\beta(X\times Y)}(A \times B) = \mathrm{cl}_{\beta X} A \times \mathrm{cl}_{\beta Y} B$*;

(3) *For each p-pseudocompact space Y, $\mathrm{cl}_{\beta(X\times Y)}(A \times Y) = \mathrm{cl}_{\beta X} A \times \beta Y$*;

(4) $\mathrm{cl}_{\beta(X\times P_{\mathrm{RK}}(p))}(A \times P_{\mathrm{RK}}(p)) = \mathrm{cl}_{\beta X} A \times \beta(\omega)$.

Proof. (1)⇒(2). Let $\beta(i)$ be the continuous extension to $\beta(X \times Y)$ of the identity mapping $i : X \times Y \longrightarrow X \times Y \subset \beta X \times \beta Y$. It is clear that $\beta(i)|_{\mathrm{cl}_{\beta(X\times Y)}(A\times B)}$ maps $\mathrm{cl}_{\beta(X\times Y)}(A\times B)$ onto $\mathrm{cl}_{\beta X} A\times\mathrm{cl}_{\beta Y} B$. We prove that $\beta(i)|_{\mathrm{cl}_{\beta(X\times Y)}(A\times B)}$ is injective. For this, suppose to the contrary that there exist two different points a and b in $\mathrm{cl}_{\beta(X\times Y)}(A\times B)\setminus(A\times B)$ such that $\beta(i)(a) = \beta(i)(b) = (a_0, b_0)$. Choose a real-valued continuous function f on $\beta(X \times Y)$ such that $f(a) = 0$ and $f(b) = 1$.

We begin by checking that the family $\{\beta(f)(a,-) : a \in A\}$ is not equicontinuous at b_0. Indeed, let $(b_\delta)_{\delta\in D}$ be a net in B converging to b_0. Then, if $\{\beta(f)(a,-) : a \in A\}$ were equicontinuous at b_0, the function $\beta(f)(-, a_0)$ is the uniform limit (on A) of the net $(\beta(f)(-, b_\delta))_{\delta\in D}$ and, consequently, it admits a continuous extension g to $\mathrm{cl}_{\beta X} A$. Consider now a net $(a_\delta, b_\delta)_{\delta\in D}$ in $A\times B$ converging to a. Then $(a_\delta, b_\delta)_{\delta\in D}$ converges to (a_0, b_0). Let $\varepsilon > 0$. Since $\{\beta(f)(a,-) : a \in A\}$ is equicontinuous at b_0 and g is continuous on $\mathrm{cl}_{\beta X} A$, there exists $\delta_0 \in D$ such that

$$|f(a_\delta, b_\delta) - \beta(f)(a_\delta, b_0)| < \varepsilon/2, \qquad |\beta(f)(a_\delta, b_0) - g(a_0)| < \varepsilon/2$$

whenever $\delta > \delta_0$. So, by the triangle inequality,

$$|f(a_\delta, b_\delta) - g(a_0)| \le \varepsilon.$$

Thus, $g(a_0) = 0$. In the same way, we obtain $g(a_0) = 1$, a contradiction.

We have just proved that $\{\beta(f)(a,-) : a \in A\}$ is not equicontinuous at b_0. Hence the following condition is satisfied:

(E) there exists $\varepsilon > 0$ such that, for each neighborhood V of b_0 in βY, there are $a \in A$ and $b \in V \cap B$ such that

$$|f(a, b) - \beta(f)(a, b_0)| > \varepsilon.$$

Next, we define by induction a sequence $(a_n, b_n)_{n<\omega} \subset A \times B$ and two sequences $(W_n)_{n<\omega}$, $(U_n\times V_n)_{n<\omega}$ of regular-closed subsets of βY and $X\times Y$, respectively, such that:

(1) $|f(a_n, b_n) - \beta(f)(a_n, b_0)| > \varepsilon$ for each $n < \omega$;

(2) For each $n < \omega$, $b_0 \in \mathrm{int}_{\beta Y} W_n$ and $\mathrm{osc}(\beta(f)(a_n,-), W_n) < \varepsilon/4$;

(3) For each $n < \omega$, $(a_n, b_n) \in \mathrm{int}_{X\times Y}(U_n \times V_n)$ and $\mathrm{osc}(f, U_n \times V_n) < \varepsilon/4$;

(4) For each $n < \omega$, $\mathrm{int}_Y V_n \subset \mathrm{int}_{\beta Y} W_{n-1}$ and $\mathrm{int}_{\beta Y} W_n \subset \mathrm{int}_{\beta Y} W_{n-1}$.

In fact, by condition (E), we can find a point $(a_1, b_1) \in A \times B$ such that

$$|f(a_1, b_2) - \beta(f)(a, b_0)| > \varepsilon.$$

As f and $\beta(f)(a_1,-)$ are both continuous functions on $X \times Y$ and on βY, respectively, there exists a regular-closed neighborhood (in $X \times Y$) $U_1 \times V_1$ of (a_1, b_1) and a regular-closed neighborhood (in βY) W_1 of b_0 such that

$$\mathrm{osc}(f, U_1 \times V_1) < \varepsilon/4, \qquad \mathrm{osc}(\beta(f)(a_1,-), W_1) < \varepsilon/4.$$

This completes step $n = 1$.

For $n > 1$, by condition (E) again, there exist $b_n \in \mathrm{int}_{\beta Y} W_{n-1} \cap B$ and $a_n \in A$ such that

$$|f(a_n, b_n) - \beta(f)(a_n, b_0)| > \varepsilon.$$

From an argument similar to that given in step $n = 1$, we can find a regular-closed neighborhood (in $X \times Y$) $U_n \times V_n$ of (a_n, b_n) with $\mathrm{int}_Y V_n \subset \mathrm{int}_{\beta Y} W_{n-1}$ and a regular-closed neighborhood (in βY) W_n of b_0 with $\mathrm{int}_{\beta Y} W_n \subset \mathrm{int}_{\beta Y} W_{n-1}$ such that

$$\mathrm{osc}(f, U_n \times V_n) < \varepsilon/4, \qquad \mathrm{osc}(\beta(f)(a_n, -), W_n) < \varepsilon/4.$$

This completes the induction.

Now, since B is quasi-p-bounded, there exists a subsequence $(V_{n(k)})_{k<\omega}$ which admits a p-limit y in Y. By (4) it is an easy matter to check that y is a cluster point of $(W_n)_{n<\omega}$ and, consequently, y belongs to W_n for each $n < \omega$. On the other hand, as $\beta(f)(-, y)$ is continuous, we can find a sequence $(M_n)_{n<\omega}$ of regular-closed sets in X with $a_n \in \mathrm{int}_X M_n \subset U_n$ such that $\mathrm{osc}(\beta(f)(-, y), M_n) < \varepsilon/4$ for each $n < \omega$. The subset A being p-bounded, we can choose a p-limit x of the sequence $(M_{n(k)})_{k<\omega}$. It is clear that (x, y) is a cluster point of both $(M_{n(k)}, V_{n(k)})_{k<\omega}$ and $(M_{n(k)}, W_{n(k)})_{k<\omega}$.

Next, let $U \times V$ be a regular-closed neighborhood on $X \times Y$ such that $|f(a, b) - f(x, y)| < \varepsilon/4$ whenever $(a, b) \in U \times V$ and consider the set $J = \{k < \omega : (U \times V) \cap (M_{n(k)} \times V_{n(k)}) \neq \emptyset\}$. According to (4), $J \subset \{k < \omega : (U \times V) \cap (M_{n(k)} \times V_{n(k)}) \neq \emptyset\}$. So, by (3),

$$|f(x, y) - f(a_{n(k)}, b_{n(k)})| < \varepsilon/4$$

whenever $k \in J$.

On the other hand, because $y \in W_{n(k)}$ and $\mathrm{osc}(\beta(f)(-, y), M_{n(k)}) < \varepsilon/4$ for each $k < \omega$, we have

$$|f(a_{n(k)}, y) - \beta(f)(a_{n(k)}, b_0)| < \varepsilon/4, \qquad |f(a, y) - f(a_{n(k)}, y)| < \varepsilon/4$$

whenever $a \in M_{n(k)}$. Therefore, $|\beta(f)(a, y) - \beta(f)(a_{n(k)}, b_0)| < \varepsilon/2$ whenever $k < \omega$. This contradicts the fact that

$$|f(a_{n(k)}, b_{n(k)}) - \beta(f)(a_{n(k)}, b_0)| > \varepsilon.$$

Thus, the function $\beta(i)$ is injective, as was to be proved.

(2)⇒(3) and (3)⇒(4) are clear.

(4)⇒(1). Since $\omega \subset P_{\mathrm{RK}}(p) \subset \beta(\omega)$, we have $\beta P_{\mathrm{RK}}(p) = \beta(\omega)$. So, condition (4) and Lemma 4.2 imply that $A \times P_{\mathrm{RK}}(p)$ is bounded in $X \times P_{\mathrm{RK}}(p)$. The result follows from Theorem 4.1. ∎

4.4. Corollary. *Let $p \in \omega^*$. A bounded subset A of a space X is quasi-p-bounded in X if and only if for each p-bounded subset B of a space Y, the restriction to $A \times B$ of every real-valued continuous function on $X \times Y$ admits a continuous extension to $\mathrm{cl}_{\beta X} A \times \mathrm{cl}_{\beta Y} B$.*

4.5. Corollary. *Let $p \in \omega^*$. A pseudocompact space X is quasi-p-pseudocompact if and only if $\beta(X \times P_{\mathrm{RK}}(p)) = \beta X \times \beta(\omega)$.*

We give an example which points out that quasi-p-boundedness is not preserved under finite products.

4.6. Example. Let $p \in \omega^*$ be a non-RK-minimal free ultrafilter and choose $r <_{\mathrm{RK}} p$. By Corollary 3.4 both $\Sigma(p)$ and $\Sigma(r)$ are quasi-p-pseudocompact subsets. Since the sequence $((n,n))_{n<\omega}$ of open sets in $\Sigma(p) \times \Sigma(r)$ does not have cluster points, the space $\Sigma(p) \times \Sigma(r)$ is not pseudocompact. Now, consider $Z = \Sigma(p) \oplus \Sigma(r)$. By Corollary 2.12, Z is quasi-p-pseudocompact. However, $Z \times Z$ has a clopen copy of $\Sigma(p) \times \Sigma(r)$ which is not pseudocompact and, consequently, $Z \times Z$ is not pseudocompact.

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Departament de Matemàtiques
Universitat Jaume I
Campus de Penyeta Roja s/n
12071, Castelló, Spain
E-mail: sanchis@mat.uji.es

Departamento de Matemáticas
Facultad de Ciencias
U.N.A.M.
Ciudad Universitaria
México 04510, México
E-mail: atamariz@servidor.unam.mx

Received 6 February 1998;
revised 28 September 1998