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A GENERAL THEOREM COVERING MANY ABSOLUTE SUMMABILITY METHODS

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Abstract. A general theorem concerning many absolute summability methods is proved.

1. Introduction. Let $\sum a_n$ be a given infinite series with the sequence of partial sums (s_n). By σ_n^δ we denote the nth Cesàro mean of order $\delta > -1$ of the sequence (s_n),

$$\sigma_n^\delta = \frac{1}{A_n^\delta} \sum_{v=1}^{n} A_{n-v}^{\delta-1} s_v.$$

Here $A_k^\delta = \binom{k+\delta}{k} = (\delta+1)\dots(\delta+k)/k!$. It may be easily verified that $A_k^\delta \sim k^\delta$. The series $\sum a_n$ is said to be $|C,\delta|_k$ *summable*, $k \ge 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} |\sigma_n^\delta - \sigma_{n-1}^\delta|^k < \infty.$$

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as } n \to \infty.$$

The transformation

$$t_n = \frac{1}{P_n} \sum_{v=0}^{\infty} p_v s_v$$

defines the sequence (t_n) of the Riesz means of the sequence (s_n) generated by the sequence of coefficients (p_n) (see [4]). The series $\sum a_n$ is said to be $|R, p_n|_k$ *summable*, $k \ge 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} |t_n - t_{n-1}|^k < \infty.$$

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The series $\sum a_n$ is said to be $|\overline{N}, p_n|_k$ *summable*, $k \ge 1$ (Bor [1]), if

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |t_n - t_{n-1}|^k < \infty.$$

In the special case when $p_n = 1$ for all values of n, both $|R, p_n|_k$ and $|\overline{N}, p_n|_k$ summability are the same as $|C, 1|_k$ summability.

The series $\sum a_n$ is said to be $|N, p_n|$ *summable* if

$$\sum_{n=1}^{\infty} |T_n - T_{n-1}| < \infty, \tag{1}$$

where

$$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_{n-v} s_v.$$

The sequence class M is defined by

$$M = \left\{ p = \{p_n\} : p_n > 0 \;\&\; \frac{p_{n+1}}{p_n} \le \frac{p_{n+2}}{p_{n+1}} \le 1,\ n = 0, 1, \ldots,\ P_n \to \infty \right\}.$$

It is known (Das [3]) that for $p \in M$, (1) holds iff

$$\sum_{n=1}^{\infty} \frac{1}{nP_n} \Big| \sum_{v=1}^{n} p_{n-v} v a_v \Big| < \infty.$$

For $p \in M$, the series $\sum a_n$ is said to be $|N, p_n|_k$ *summable*, $k \ge 1$ (Sulaiman [5]), if

$$\sum_{n=1}^{\infty} \frac{1}{nP_n^k} \Big| \sum_{v=1}^{n} p_{n-v} v a_v \Big|^k < \infty.$$

In the special case in which $p_n = A_n^{r-1}$, $r > -1$, $|N, p_n|_k$ summability is equivalent to $|C, r|_k$ summability. The series $\sum a_n$ is said to be $|R, \log n, 1|_k$ *summable* if it is $|\overline{N}, p_n|_k$ summable with $p_n = 1/(n+1)$ and $P_n \sim \log(n+1)$. For any sequence $\{f_n\}$, we define $\Delta f_n = f_n - f_{n+1}$.

2. Main result. We prove the following:

THEOREM 1. *Let $\{f_n\}$, $\{g_n\}$, $\{G_n\}$, and $\{H_n\}$ be sequences of positive constants such that $\{f_n\} \in M$ and $F_n = \sum_{v=1}^{n} f_v \to \infty$ as $n \to \infty$. Let $\{\varepsilon_n\}$ be a sequence of constants. Given a sequence $\{x_n\}$ define*

$$X_n = \frac{1}{G_n} \sum_{v=1}^{n} g_v x_v, \qquad Y_n = \frac{1}{F_{n-1} H_n} \sum_{v=1}^{n} v f_{n-v} x_v \varepsilon_n$$

and assume

$$g_{n+1} = O(g_n), \tag{2}$$

$$\frac{H_{n+1} F_{n+1}}{G_{n+1}} = O\left(\frac{H_n F_n}{G_n}\right), \tag{3}$$

$$\Delta g_n = O\left(\frac{g_{n+1}}{n}\right), \tag{4}$$

$$\Delta\left(\frac{g_n H_n F_n}{nG_n}\right) = O\left(\frac{g_n H_n}{nG_n}\right), \tag{5}$$

$$\Delta\left(\frac{nG_n}{g_n H_n F_n}\varepsilon_n\right) = O\left(\frac{1}{F_n}\right), \tag{6}$$

$$\sum_{n=v+1}^{\infty} \frac{f_{n-v}}{F_{n-1}H_n^k} = O\left(\frac{1}{H_v^k}\right). \tag{7}$$

Let $k \geq 1$. Then a necessary and sufficient condition for the implication:

$$\textit{if } \sum |X_n|^k < \infty \textit{ then } \sum |Y_n|^k < \infty$$

to hold (*for any sequence* $\{x_n\}$) *is*

(i) $\varepsilon_n = O(g_n H_n F_n/(nG_n))$, *and*
(ii) $\Delta\varepsilon_n = O(g_{n+1}H_n/(nG_n))$.

3. Lemmas

Lemma 1 (Bor [2]). *Let $k \geq 1$, and let $A = (a_{nv})$ be an infinite matrix that maps ℓ^k into ℓ^k. Then $a_{nv} = O(1)$ for all n and v.*

Proof. By the Closed Graph Theorem, A defines a bounded linear mapping in ℓ^k. Then the bound $|a_{nv}| \leq C$ follows, where C is the norm of A.

Lemma 2 (Sulaiman [6]). *Let $p \in M$. Then for $0 < r \leq 1$,*

$$\sum_{n=v+1}^{\infty} \frac{p_{n-v-1}}{n^r P_{n-1}} = O(v^{-r}).$$

Lemma 3. *Suppose that $\varepsilon_n = O(\alpha_n\beta_n)$, $\alpha_n, \beta_n > 0$, $\alpha_{n+1}\beta_{n+1} = O(\alpha_n\beta_n)$, $\Delta(\alpha_n\beta_n) = O(\alpha_n)$ and $\Delta(\varepsilon_n/(\alpha_n\beta_n)) = O(1/\beta_n)$. Then $\Delta\varepsilon_n = O(\alpha_n)$.*

Proof. We have $\varepsilon_n = k_n\alpha_n\beta_n$ where $k_n = \varepsilon_n/(\alpha_n\beta_n) = O(1)$. Therefore

$$\begin{aligned}\Delta\varepsilon_n &= k_n\Delta(\alpha_n\beta_n) + \Delta k_n(\alpha_{n+1}\beta_{n+1})\\ &= O(1)O(\alpha_n) + O(1/\beta_n)O(\alpha_n\beta_n) = O(\alpha_n).\end{aligned}$$

4. Proof of Theorem 1. *Sufficiency.* We have via Abel's transformation:

$$\begin{aligned}Y_n &= \frac{1}{F_{n-1}H_n}\sum_{v=1}^{n} g_v x_v\left(v\frac{f_{n-v}}{g_v}\varepsilon_v\right)\\ &= \frac{1}{F_{n-1}H_n}\left[\sum_{v=1}^{n-1}\Big(\sum_{r=1}^{v} g_r x_r\Big)\Delta_v\left(v\frac{f_{n-v}}{g_v}\varepsilon_v\right) + \Big(\sum_{r=1}^{n} g_r x_r\Big)n\frac{f_0}{g_n}\varepsilon_n\right]\end{aligned}$$

$$= \frac{1}{F_{n-1}H_n}\sum_{v=1}^{n-1} G_v X_v \biggl\{ -\frac{f_{n-v}}{g_v}\varepsilon_v + (v+1)\Delta g_v^{-1} f_{n-v}\varepsilon_v$$

$$+ (v+1)g_{v+1}^{-1}\Delta_v f_{n-v}\varepsilon_v + (v+1)g_{v+1}^{-1} f_{n-v-1}\Delta\varepsilon_v \biggr\} + \frac{nG_nX_nf_0}{F_{n-1}H_ng_n}\varepsilon_n$$

$$= Y_{n,1} + Y_{n,2} + Y_{n,3} + Y_{n,4} + Y_{n,5}, \quad \text{say.}$$

By Minkowski's inequality,

$$\sum_{n=1}^{m} |Y_{n,1}|^k = O(1)\sum_{n=1}^{m}\sum_{r=1}^{5} |Y_{n,r}|^k.$$

Applying Hölder's inequality gives

$$\begin{aligned}
\sum_{n=2}^{m+1} |Y_{n,1}|^k &= \sum_{n=2}^{m+1} \frac{1}{F_{n-1}^k H_n^k}\left|\sum_{v=1}^{n-1} f_{n-v}\frac{G_v}{g_v}X_v\varepsilon_v\right|^k \\
&\le \sum_{n=2}^{m+1} \frac{1}{F_{n-1}H_n^k} \\
&\quad \times \left\{\sum_{v=1}^{n-1} f_{n-v}\left(\frac{G_v}{g_v}\right)^k |X_v|^k|\varepsilon_v|^k\right\}\left\{\sum_{v=1}^{n-1}\frac{f_{n-v}}{F_{n-1}}\right\}^{k-1} \\
&\le O(1)\sum_{v=1}^{m}\left(\frac{G_v}{g_v}\right)^k |X_v|^k|\varepsilon_v|^k \sum_{n=v+1}^{m+1}\frac{f_{n-v}}{F_{n-1}H_n^k} \\
&\le O(1)\sum_{v=1}^{m}\frac{1}{H_v^k}\left(\frac{v}{F_v}\right)^k\left(\frac{G_v}{g_v}\right)^k |X_v|^k|\varepsilon_v|^k,
\end{aligned}$$

$$\begin{aligned}
\sum_{n=2}^{m+1} |Y_{n,2}|^k &= \sum_{n=2}^{m+1} \frac{1}{F_{n-1}^k H_n^k}\left|\sum_{v=1}^{n-1}(v+1)G_v\Delta g_v^{-1} f_{n-v}X_v\varepsilon_v\right|^k \\
&\le O(1)\sum_{n=2}^{m+1}\frac{1}{F_{n-1}H_n^k} \\
&\quad \times \left\{\sum_{v=1}^{n-1} v^k G_v^k|\Delta g_v^{-1}|^k f_{n-v}|X_v|^k|\varepsilon_v|^k\right\}\left\{\sum_{v=1}^{n-1}\frac{f_{n-v}}{F_{n-1}}\right\}^{k-1} \\
&\le O(1)\sum_{v=1}^{m} v^k G_v^k|\Delta g_v^{-1}||X_v|^k|\varepsilon_v|^k \sum_{n=v+1}^{m+1}\frac{f_{n-v}}{F_{n-1}H_n^k} \\
&= O(1)\sum_{v=1}^{m}\left(\frac{v}{H_v}\right)^k G_v^k\frac{|\Delta g_v|^k}{g_v^k g_{v+1}^k}|X_v|^k|\varepsilon_v|^k
\end{aligned}$$

$$\leq O(1)\sum_{v=1}^{m}\frac{1}{H_v^k}\left(\frac{v}{F_v}\right)^k\left(\frac{G_v}{g_v}\right)^k|X_v|^k|\varepsilon_v|^k,$$

$$\sum_{n=2}^{m+1}|Y_{n,3}|^k = \sum_{n=2}^{m+1}\frac{1}{F_{n-1}^k H_n^k}\Big|\sum_{v=1}^{n-1}(v+1)g_{v+1}^{-1}G_v\Delta_v f_{n-v}X_v\varepsilon_v\Big|^k$$

$$\leq \sum_{n=2}^{m+1}\frac{1}{F_{n-1}^k H_n^k}$$

$$\times\left\{\sum_{v=1}^{n-1}v^k\left(\frac{G_v}{g_{v+1}}\right)^k|\Delta_v f_{n-v}||X_v|^k|\varepsilon_v|^k\right\}\left\{\sum_{v=1}^{n-1}|\Delta f_{n-v}|\right\}^{k-1}$$

$$\leq O(1)\sum_{v=1}^{m}v^k\left(\frac{G_v}{g_v}\right)^k|X_v|^k|\varepsilon_v|^k\sum_{n=v+1}^{m+1}\frac{|\Delta_v f_{n-v}|}{F_{n-1}^k H_n^k}$$

$$\leq O(1)\sum_{v=1}^{m}\frac{1}{H_v^k}\left(\frac{v}{F_v}\right)^k\left(\frac{G_v}{g_v}\right)^k|X_v|^k|\varepsilon_v|^k,$$

$$\sum_{n=2}^{m+1}|Y_{n,4}|^k = \sum_{n=2}^{m+1}\frac{1}{F_{n-1}^k H_n^k}\Big|\sum_{v=1}^{n-1}vg_{v+1}^{-1}f_{n-v-1}G_vX_v\Delta\varepsilon_v\Big|^k$$

$$\leq \sum_{n=2}^{m+1}\frac{1}{F_{n-1}H_n^k}$$

$$\times\left\{\sum_{v=1}^{n-1}v^k\left(\frac{G_v}{g_{v+1}}\right)^k f_{n-v-1}|X_v|^k|\Delta\varepsilon_v|^k\right\}\left\{\sum_{v=1}^{n-1}\frac{f_{n-v-1}}{F_{n-1}}\right\}^{k-1}$$

$$\leq O(1)\sum_{v=1}^{m}v^k\left(\frac{G_v}{g_{v+1}}\right)^k|X_v|^k|\Delta\varepsilon_v|^k\sum_{n=v+1}^{m+1}\frac{f_{n-v-1}}{F_{n-1}H_n^k}$$

$$\leq O(1)\sum_{v=1}^{m}\left(\frac{v}{H_v}\right)^k\left(\frac{G_v}{g_{v+1}}\right)^k|X_v|^k|\Delta\varepsilon_v|^k,$$

$$\sum_{n=1}^{m}|Y_{n,5}|^k = \sum_{n=1}^{m}\left|\frac{nG_nX_nf_0\varepsilon_n}{F_{n-1}H_ng_n}\right|^k$$

$$\leq O(1)\sum_{n=1}^{m}\left(\frac{n}{F_n}\right)^k\left(\frac{G_n}{g_n}\right)^k\frac{1}{H_n^k}|X_n|^k|\varepsilon_n|^k.$$

Necessity of (i). By the result of Bor [1], the transformation from (X_n) into (Y_n) maps ℓ^k into ℓ^k and hence the diagonal elements of this transformation are bounded (by Lemma 1), so (i) is necessary.

Necessity of (ii). This follows from Lemma 3 and necessity of (i) by taking $\alpha_n \equiv g_n H_n/(nG_n)$ and $\beta_n \equiv F_n$, using (2).

This completes the proof of the theorem.

REMARK. (1) If we put $x_n = a_n$, $f_n = p_n$ and $H_n = n^{1/k}$ in the formula defining Y_n, $p \in M$, then the condition $\sum |Y_n|^k < \infty$ is equivalent to $|N, p_n|_k$ summability of $\sum a_n \varepsilon_n$ (note that P_n/P_{n-1} is a bounded sequence).

(2) If we put $x_n = a_n$, $Q_n = q_0 + \ldots + q_n$, $g_n = Q_{n-1}$ and $G_n = Q_{n-1}(Q_n/q_n)^{1/k}$ in the formula defining X_n, then the condition $\sum |X_n|^k < \infty$ simply means $|\overline{N}, q_n|_k$ summability of $\sum a_n$.

(3) If we put $x_n = a_n$, $Q_n = q_0 + \ldots + q_n$, $g_n = Q_{n-1}$ and $G_n = n^{1/k-1} Q_n Q_{n-1}/q_n$ in the formula defining X_n, then the condition $\sum |X_n|^k < \infty$ means $|R, q_n|_k$ summability of $\sum a_n$.

5. Applications. Throughout the rest of the paper we assume that $P_n \to \infty$ and $Q_n \to \infty$ as $n \to \infty$.

THEOREM 2. *Let $p \in M$ and let $nq_n = O(Q_n)$, $Q_n = O(Q_{n-1})$, and*

$$\frac{P_{n+1}}{P_n} = O\biggl(\biggl(\frac{Q_n}{Q_{n-1}}\biggr)\biggl(\frac{q_n Q_{n+1}}{q_{n+1} Q_n}\biggr)^{1/k}\biggr),$$

$$\Delta\biggl(\frac{P_n}{n}\biggl(\frac{nq_n}{Q_n}\biggr)^{1/k}\biggr) = O\biggl(\frac{1}{n}\biggl(\frac{nq_n}{Q_n}\biggr)^{1/k}\biggr),$$

$$\Delta\biggl(\frac{n}{P_n}\biggl(\frac{Q_n}{nq_n}\biggr)^{1/k}\varepsilon_n\biggr) = O\biggl(\frac{1}{P_n}\biggr).$$

Then a necessary and sufficient condition that $\sum a_n \varepsilon_n$ be $|N, p_n|_k$ summable whenever $\sum a_n$ is $|\overline{N}, q_n|_k$ summable, $k \ge 1$, is

$$\varepsilon_n = O\biggl(\frac{P_n}{n}\biggl(\frac{nq_n}{Q_n}\biggr)^{1/k}\biggr), \qquad \Delta\varepsilon_n = O\biggl(\frac{1}{n}\biggl(\frac{nq_n}{Q_n}\biggr)^{1/k}\biggr).$$

THEOREM 3. *Let $p \in M$ and let $nq_n = O(Q_n)$, $Q_n = O(Q_{n-1})$, and*

$$\frac{P_{n+1}}{P_n} = O\biggl(\frac{q_n Q_{n+1}}{q_{n+1} Q_n}\biggr), \qquad \Delta\biggl(\frac{P_n q_n}{Q_n}\biggr) = O\biggl(\frac{q_n}{Q_n}\biggr),$$

$$\Delta\biggl(\frac{Q_n}{P_n q_n}\varepsilon_n\biggr) = O\biggl(\frac{1}{P_n}\biggr).$$

Then a necessary and sufficient condition that $\sum a_n \varepsilon_n$ be $|N, p_n|_k$ summable whenever $\sum a_n$ is $|R, q_n|_k$ summable, $k \ge 1$, is

$$\varepsilon_n = O(P_n q_n/Q_n), \qquad \Delta\varepsilon_n = O(q_n/Q_n).$$

The following results are consequences of Theorem 2.

COROLLARY 4. *A necessary and sufficient condition that* $\sum a_n\varepsilon_n$ *be* $|C,\alpha|_k$ *summable*, $0<\alpha<1$, *whenever* $\sum a_n$ *is* $|C,1|_k$ *summable*, $k\ge 1$, *is*

$$\varepsilon_n = O(n^{\alpha-1}), \qquad \Delta\varepsilon_n = O(n^{-1}),$$

provided that $\Delta(n^{1-\alpha}\varepsilon_n) = O(n^{-\alpha})$.

COROLLARY 5. *A necessary and sufficient condition that* $\sum a_n\varepsilon_n$ *be* $|N,1/(n+1)|_k$ *summable whenever* $\sum a_n$ *is* $|C,1|_k$ *summable*, $k\ge 1$, *is*

$$\varepsilon_n = O(\log n/n), \qquad \Delta\varepsilon_n = O(n^{-1}),$$

provided that

$$\Delta\left(\frac{n}{\log n}\varepsilon_n\right) = O\left(\frac{1}{\log n}\right).$$

COROLLARY 6. *A necessary and sufficient condition that* $\sum a_n\varepsilon_n$ *be* $|N,1/(n+1)|_k$ *summable whenever* $\sum a_n$ *is* $|R,\log n,1|_k$ *summable*, $k\ge 1$, *is*

$$\varepsilon_n = O\{(\log n)^{1-1/k}/n\}, \qquad \Delta\varepsilon_n = O\{1/n(\log n)^{1/k}\},$$

provided that

$$\Delta\left(\frac{n}{(\log n)^{1-1/k}}\varepsilon_n\right) = O\left(\frac{1}{\log n}\right).$$

COROLLARY 7. *A necessary and sufficient condition that* $\sum a_n\varepsilon_n$ *be* $|C,\alpha|_k$ *summable*, $0<\alpha\le 1$, *whenever* $\sum a_n$ *is* $|R,\log n,1|_k$ *summable*, $k\ge 1$, *is*

$$\varepsilon_n = O\{n^{\alpha-1}/(\log n)^{1/k}\}, \qquad \Delta\varepsilon_n = O\{1/(n(\log n)^{1/k})\},$$

provided that $\Delta(n^{1-\alpha}(\log n)^{1/k}\varepsilon_n) = O(n^{-\alpha})$.

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