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ON SYSTEMS OF NULL SETS

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Abstract. The collection of all sets of measure zero for a finitely additive, group-valued measure is studied and characterised from a combinatorial viewpoint.

Let X be a non-empty set and let $\mathbf{A}$ be a class of subsets of X. Then $\mathbf{A}$ is a *field* if $X \in \mathbf{A}$ and $\mathbf{A}$ is closed under the operations of (finite) union and complementation, i.e. $\mathbf{A}$ is a Boolean algebra of subsets of X. If $\mathbf{A}$ is any class of subsets of X, then $a(\mathbf{A})$ denotes the smallest field containing $\mathbf{A}$. A collection $\mathbf{U}$ of subsets of X is a *u-system* if $\emptyset \in \mathbf{U}$ and $\mathbf{U}$ is closed under the operation of proper difference: $U_1 \setminus U_2 \in \mathbf{U}$ whenever $U_1 \supseteq U_2$ for $U_1, U_2 \in \mathbf{U}$. It is easy to show that if $\mathbf{U}$ is a u-system such that $X \in \mathbf{U}$, and $U_1, U_2 \in \mathbf{U}$ with $U_1 \cap U_2 = \emptyset$, then $U_1 \cup U_2 \in \mathbf{U}$: a u-system containing X is closed under formation of disjoint unions (and also complements).

Let $A_1, \ldots, A_m$ and $B_1, \ldots, B_n$ be finite sequences of not necessarily distinct subsets of a set X. For any $k \geq 1$, we define

$$A(k) = \bigcup A_{i_1} \cap \ldots \cap A_{i_k}, \quad B(k) = \bigcup B_{i_1} \cap \ldots \cap B_{i_k},$$

in each case intending the union of all k-fold intersections: the $(i_1, \ldots, i_k)$ are k-tuples of distinct indices i_j. Then we have

$$A(1) = A_1 \cup \ldots \cup A_m, \quad A(m) = A_1 \cap \ldots \cap A_m,$$
$$B(1) = B_1 \cup \ldots \cup B_n, \quad B(n) = B_1 \cap \ldots \cap B_n,$$

and by convention, we put $A(k) = \emptyset$ for $k > m$ and $B(k) = \emptyset$ for $k > n$. A collection $\mathbf{M}$ of subsets of X is an *m-system* if $\emptyset \in \mathbf{M}$ and whenever $A_1, \ldots, A_m$ and $B_1, \ldots, B_n$ are sets in $\mathbf{M}$ such that

$$(*) \qquad A(k+1) \subseteq B(k) \subseteq A(k) \quad \text{for all } k \geq 1,$$

then

$$(**) \qquad \bigcup_{k=1}^{N} [A(k) \setminus B(k)] \in \mathbf{M}, \quad \text{where } N \geq m, n.$$

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Clearly, every field is an m-system, and every m-system is a u-system. The converse implications do not hold, as is shown in an example given later. If $\mathbf{A}$ is a class of subsets of X, then $u(\mathbf{A})$ and $m(\mathbf{A})$ denote, respectively, the smallest u-system and m-system containing $\mathbf{A}$. Then $u(\mathbf{A}) \subseteq m(\mathbf{A})$.

Given a non-empty set X, let $\mathbb{Z}^X$ be the additive group of all functions from X to the integers $\mathbb{Z}$. If $A \subseteq X$, then the *indicator* of A is the function $1_A : X \to \mathbb{Z}$ such that $1_A(x) = 1$ if $x \in A$ and $1_A(x) = 0$ if $x \notin A$. Given a collection $\mathbf{A}$ of subsets of X, we define $S(\mathbf{A})$ as the subgroup of $\mathbb{Z}^X$ generated by all the indicators 1_A for $A \in \mathbf{A}$.

LEMMA 1. *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are collections of subsets of* X, *then* $S(\mathbf{A} \cup \mathbf{B}) = S(\mathbf{A}) + S(\mathbf{B})$.

LEMMA 2. *Let* $\mathbf{A}$ *be a collection of subsets of* X. *For any* $E \subseteq X$, *we have* $E \in m(\mathbf{A})$ *if and only if* $1_E \in S(\mathbf{A})$.

Proof. Suppose that $1_E \in S(\mathbf{A})$. Then there are sets $A_1, \ldots, A_m$ and $B_1, \ldots, B_n$ in $\mathbf{A}$ such that $1_E = 1_{A_1} + \ldots + 1_{A_m} - 1_{B_1} - \ldots - 1_{B_n}$. We see that the sets A_i and B_j satisfy condition $(*)$ in the definition of an m-system, so that E, which is the set in $(**)$, must belong to $m(\mathbf{A})$.

Now let $\mathbf{M}$ be the collection of all sets $F \subseteq X$ such that $1_F \in S(\mathbf{A})$. It is easy to verify that $\mathbf{M}$ is an m-system containing $\mathbf{A}$, so that $u(\mathbf{A}) \subseteq \mathbf{M}$. ∎

The proof gives indication of a useful alternative definition of m-system: if A_i and B_j are sets in $\mathbf{M}$, and $1_E = 1_{A_1} + \ldots + 1_{A_m} - 1_{B_1} - \ldots - 1_{B_n}$, then $E \in \mathbf{M}$.

LEMMA 3. *Let* $\mathbf{A}$ *be a collection of subsets of* X. *Then* $S(m(\mathbf{A})) = S(u(\mathbf{A})) = S(\mathbf{A})$.

Proof. Clearly, $S(\mathbf{A})] \subseteq S(u(\mathbf{A})) \subseteq S(m(\mathbf{A}))$. The inclusion $S(m(\mathbf{A})) \subseteq S(\mathbf{A})$ follows from the preceding lemma. ∎

EXAMPLE. We show that the concepts of u-system and m-system are in general distinct. Put

$$Y = \{0,1\}^3, \qquad X = \{(a_1, a_2, a_3) \in Y : a_1 + a_2 \geq a_3\},$$
$$A_i = \{(a_1, a_2, a_3) \in X : a_i = 1\} \quad \text{for } i = 1, 2, 3.$$

Then the collection

$$\mathbf{U} = \{\emptyset, A_1, A_2, A_3, X \setminus A_1, X \setminus A_2, X \setminus A_3, X\}$$

is a u-system, but $m(\mathbf{U})$ contains the additional set

$$E = \{(1,1,1), (1,0,0), (0,1,0)\};$$

we have $1_E = 1_{A_1} + 1_{A_2} - 1_{A_3}$.

Let $\mathbf{A}$ be a field of subsets of a set X and let G be an Abelian group. A function $\mu : \mathbf{A} \to G$ is a (G-valued) *charge* if $\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2)$

whenever A_1, A_2 are disjoint sets in $\mathbf{A}$. Every G-valued charge μ induces a unique homomorphism $\varphi : S(\mathbf{A}) \to G$ such that $\varphi(1_A) = \mu(A)$ for every $A \in \mathbf{A}$; using the same equation, we see that each homomorphism $\varphi : S(\mathbf{A}) \to G$ is induced by a charge $\mu : \mathbf{A} \to G$. Zero sets of group-valued charges are characterised in the

THEOREM. *Let* $\mathbf{M}$ *be a collection of subsets of a non-empty set* X *and define* $\mathbf{A} = a(\mathbf{M})$. *The following conditions are equivalent*:

(i) *there is an Abelian group* G *and a charge* $\mu : \mathbf{A} \to G$ *such that* $\mathbf{M} = \{A \in \mathbf{A} : \mu(A) = 0\}$;
(ii) $\mathbf{M}$ *is an* m*-system.*

Proof. (i)⇒(ii). Let $\varphi : S(\mathbf{A}) \to G$ be the homomorphism induced by μ. If A_i and B_j are sets in $\mathbf{A}$ with $\mu(A_i) = \mu(B_j) = 0$ and $1_E = 1_{A_1} + \ldots + 1_{A_m} - 1_{B_1} - \ldots - 1_{B_n}$, then $\mu(E) = \varphi(1_E) = 0$. The collection $\mathbf{M} = \{A \in \mathbf{A} : \mu(A) = 0\}$ is thus closed under the operation that defines m-systems.

(ii)⇒(i). Define $G = S(\mathbf{A})/S(\mathbf{M})$ and let $\varphi : S(\mathbf{A}) \to G$ be the standard projection onto the quotient. Define $\mu : \mathbf{A} \to G$ by $\mu(A) = \varphi(1_A)$. By Lemma 2, $\mathbf{M} = \{A \in \mathbf{A} : \mu(A) = 0\}$. ■

Quotient groups of the form $S(a(\mathbf{A} \cup \mathbf{B}))/[S(\mathbf{A}) + S(\mathbf{B})]$, where $\mathbf{A}$ and $\mathbf{B}$ are fields, arise naturally in and have been studied for their connection with the problem of joint extensions of group-valued charges (see [1], [2]). With this application in mind, we now prove that the u-system and the m-system generated by the union of two fields coincide.

THEOREM. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be fields of subsets of a set* X. *For* $E \subseteq X$, *we have* $1_E \in S(\mathbf{A}) + S(\mathbf{B})$ *if and only if* $E \in u(\mathbf{A} \cup \mathbf{B})$. *Then* $u(\mathbf{A} \cup \mathbf{B}) = m(\mathbf{A} \cup \mathbf{B})$.

Proof. From Lemma 2 and the inclusion $u(\mathbf{A} \cup \mathbf{B}) \subseteq m(\mathbf{A} \cup \mathbf{B})$, we see that $1_E \in S(\mathbf{A} \cup \mathbf{B}) = S(\mathbf{A}) + S(\mathbf{B})$ whenever $E \in u(\mathbf{A} \cup \mathbf{B})$. Now suppose that $1_E \in S(\mathbf{A} \cup \mathbf{B})$. Then $1_E = h + k$ for functions $h \in S(\mathbf{A})$ and $k \in S(\mathbf{B})$. Since constant functions in $\mathbb{Z}^X$ belong to $S(\mathbf{A}) \cap S(\mathbf{B})$, it involves no loss of generality to assume that $h \geq 0$ and $k \leq 0$. Then we have

$$E = \bigcup_{i=0}^{\infty} \{x : k(x) \geq -i\} \setminus \{x : h(x) \leq i\},$$

a finite disjoint union of proper differences of sets of $\mathbf{B}$ with sets of $\mathbf{A}$. Thus $E \in u(\mathbf{A} \cup \mathbf{B})$.

We have shown that $1_E \in S(\mathbf{A} \cup \mathbf{B})$ if and only if $E \in u(\mathbf{A} \cup \mathbf{B})$. Lemma 2 then implies that $u(\mathbf{A} \cup \mathbf{B}) = m(\mathbf{A} \cup \mathbf{B})$. ■

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