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COHEN–MACAULAY MODULES OVER TWO-DIMENSIONAL GRAPH ORDERS

BY

KLAUS W. ROGGENKAMP (STUTTGART)

Abstract. For a split graph order $\mathcal{L}$ over a complete local regular domain $\mathcal{O}$ of dimension 2 the indecomposable Cohen–Macaulay modules decompose—up to irreducible projectives—into a union of the indecomposable Cohen–Macaulay modules over graph orders of type $\bullet - \bullet$. There, the Cohen–Macaulay modules filtered by irreducible Cohen–Macaulay modules are in bijection to the homomorphisms $\phi : \overline{\mathcal{O}}^{(\mu)} \to \overline{\mathcal{O}}^{(\nu)}$ under the bi-action of the groups $(\mathrm{Gl}(\mu, \overline{\mathcal{O}}), \mathrm{Gl}(\nu, \overline{\mathcal{O}}))$, where $\overline{\mathcal{O}} = \mathcal{O}/\langle\pi\rangle$ for a prime π. This problem strongly depends on the nature of $\overline{\mathcal{O}}$. If $\overline{\mathcal{O}}$ is regular, then the category of indecomposable filtered Cohen–Macaulay modules is bounded. This latter condition is satisfied if $\mathcal{L}$ is the completion of the Hecke order of the dihedral group of order $2p$ with p an odd prime at the maximal ideal $\langle q-1, p\rangle$, and more generally of blocks of defect p of complete Hecke orders. If $\overline{\mathcal{O}}$ is not regular, then the category of indecomposable filtered Cohen–Macaulay modules is unbounded.

1. Introduction. Brauer tree algebras arise as modular blocks of cyclic defect and their indecomposable modules are classified purely combinatorially (cf. the paper [GaRi; 79]). Integral p-adic blocks with cyclic defect are Brauer tree orders and in case the defect is p the indecomposable lattices arise from the projective resolution of an irreducible lattice — Green's walk around the Brauer tree (cf. [Ro; 92], [Gr; 74]). Graph orders generalize the tree orders and have been used successfully to get information on modular blocks with dihedral defect (as done in [KaRo; 98]). Maximal deformations of blocks with cyclic defect and "blocks of cyclic defect" of p-adic Hecke orders are described as tree orders over the two-dimensional complete ring $\mathcal{O} := \mathbb{Z}[q, q^{-1}]_{\langle q-1,p\rangle}$ (1). Let $\mathcal{L}$ be such a tree order over $\mathcal{O}$. Then a description of the Cohen–Macaulay $\mathcal{L}$-modules (2) seems to be out of reach, since in general the epimorphic images of Cohen–Macaulay modules are not Cohen–Macaulay.

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(1) A *maximal deformation* is an $\mathcal{O}$-order such that for specializing q to 1 we obtain the p-adic block and for $q \in \mathcal{O} \setminus (1 + p\mathbb{Z}_p)$ we obtain a maximal $\mathbb{Z}_p$-order.

(2) These are the left $\mathcal{L}$-modules which are $\mathcal{O}$-free.

A good substitute seem to be those Cohen–Macaulay modules which have a filtration with irreducible Cohen–Macaulay modules ([3]) as sections. W. Rump has given an example of a Cohen–Macaulay module which is not filtered with respect to any ordering.

In the case of a tree order ([4]) over a one-dimensional base ring, the extension groups between the irreducible lattices are isomorphic to $\mathbb{F}_p$, and so there is essentially only one extension, the projective cover sequence.

In the two-dimensional situation, the extension groups are of the form $\overline{\mathcal{O}} := \mathcal{O}/\langle\pi\rangle$ for a height one — and hence principal — prime ideal $\langle\pi\rangle$. The structure of the category of filtered Cohen–Macaulay $\mathcal{L}$-modules now depends heavily on the arithmetical structure of the $\mathbb{Z}_{\langle p\rangle}$-order $\overline{\mathcal{O}}$. The ring $\overline{\mathcal{O}}$ is a subring of the ring R_p of p-adic integers in the algebraic number field $K_p := \mathbb{Q}[q, q^{-1}]/\langle\pi\rangle \cdot \mathbb{Q}[q, q^{-1}]$ in case $\mathcal{O}$ is $\mathbb{Z}[q, q^{-1}]_p$ and π is given by an irreducible polynomial.

In case $\overline{\mathcal{O}} = R_p$, i.e. $\overline{\mathcal{O}}$ is a complete discrete rank one valuation ring, the only indecomposable filtered Cohen–Macaulay modules are then given as follows: We start the walk around the tree from an end vertex. The walk then gives rise to a numbering of the rational idempotents and also to the projective resolution of the unique irreducible Cohen–Macaulay module M_0 corresponding to the chosen end vertex. The only irreducible Cohen–Macaulay modules are the syzygies $\Omega^i(M_0)$, $0 \le i \le s$, where $s+1$ is the number of vertices in our tree. The construction of Cohen–Macaulay modules is done as follows: We calculate $\mathrm{Ext}^1_{\mathcal{L}}(\Omega^{i-1}(M_0), \Omega^i(M_0)) \simeq \overline{\mathcal{O}}$ and so the non-isomorphic exact sequences are given as extensions

$$\mathbb{E}_\nu : 0 \to \Omega^i(M_0) \to X_\nu \to \Omega^{i-1}(M_0) \to 0,$$

which corresponds to the element $\varrho^\nu \in \mathrm{Ext}^1_{\mathcal{L}}(\Omega^{i-1}(M_0), \Omega^i(M_0))$, with $\langle\varrho\rangle = \mathrm{rad}(\overline{\mathcal{O}})$. It turns out that $\{X_\nu \mid \nu = 0, 1, \ldots\}$ are non-isomorphic indecomposable Cohen–Macaulay modules. The *main result* is that these are all indecomposable filtered Cohen–Macaulay $\mathcal{L}$-modules.

In case $\overline{\mathcal{O}} \neq R_p$, we can find in the category of filtered Cohen–Macaulay $\mathcal{L}$-modules modules of arbitrarily large $\mathcal{O}$-rank. This case will be considered in Section 7. Luckily, in all the examples I know of blocks of cyclic defect of Hecke orders and of deformations of blocks with defect p we are in the first situation, i.e. $\overline{\mathcal{O}} = R_p$.

We shall give here — as a demonstration — a description of the filtered Cohen–Macaulay modules for localized dihedral groups of order $2p$ for an odd prime p.

([3]) A Cohen–Macaulay module M for $\mathcal{L}$ is *irreducible* if it spans a simple module over the total ring of quotients of $\mathcal{L}$.

([4]) We assume here a tree order since the situation is more complex for a general graph order.

So let $\mathcal{H}_p = \mathcal{H}_{D_p} \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Z}[q,q^{-1}]_{\langle q-1,p\rangle}$ be the completion of the Hecke order $\mathcal{H}_{D_p}$ of the dihedral group of order $2 \cdot p$ for an odd prime p at the maximal ideal $\langle q-1,p\rangle$ of $\mathbb{Z}[q,q^{-1}]$. Then $\mathcal{H}_p$ is a tree order which is described as follows (cf. [Ro; 98]):

Let $R = \mathbb{Z}[\theta_p]^{C_2}$ be the ring of the integers in the fixed field under the cyclic group of order two of the pth cyclotomic field and define $\mathcal{O}_2 = R \otimes_{\mathbb{Z}} \mathbb{Z}[q,q^{-1}]_{\langle q-1,p\rangle}$. In $\mathcal{O}_2$ we have the prime ideal $\langle\pi\rangle = \langle (q-\theta_p)\cdot(q-\theta_p^{-1})\rangle$. We observe that $\overline{\mathcal{O}}_2 := \mathcal{O}_2/\langle\pi\rangle \simeq \mathbb{Z}_{\langle p\rangle}[\theta_p]$ is regular of dimension one, i.e. a rank one valuation ring. The order $\mathcal{L}_2$ is defined as

$$\mathcal{L}_2 := \begin{pmatrix} \mathcal{O}_2 & \mathcal{O}_2 \\ \langle\pi\rangle & \mathcal{O}_2 \end{pmatrix}.$$

We put $\mathcal{L}_1 = \mathbb{Z}[q,q^{-1}]_{\langle q-1,p\rangle} = \mathcal{L}_3$ and denote by $\Phi_p(q)$ the pth cyclotomic polynomial. The ring $\mathcal{H}_p$ is then obtained as the pull-back

$$\begin{tikzcd} \mathcal{H}_p \arrow[r] \arrow[d] & \mathcal{L}_1 \times \mathcal{L}_3 \arrow[d, "\phi"'] \\ \mathcal{L}_2 \arrow[r, "\psi"] & \mathbb{Z}_{\langle p\rangle}[\theta_p] \times \mathbb{Z}_{\langle p\rangle}[\theta_p] \end{tikzcd}$$

where ϕ is reduction modulo $\langle\Phi_p(q)\rangle \times \langle\Phi_p(q)\rangle$, and ψ is reduction modulo the ideal

$$\begin{pmatrix} \langle\pi\rangle & \mathcal{O}_2 \\ \langle\pi\rangle & \langle\pi\rangle \end{pmatrix}.$$

We label the rational central primitive idempotents of $\mathcal{A}$, the algebra spanned by $\mathcal{L} := \mathcal{H}_p$, as (e_1, e_2, e_3), where e_i is the unit element in $\mathcal{L}_i$. The non-isomorphic irreducible Cohen–Macaulay $\mathcal{H}_p$-modules are

$$\left\{ N_1 = \mathcal{L}_1,\ M_1 = \begin{pmatrix} \mathcal{O}_2 \\ \langle\pi\rangle \end{pmatrix},\ M_2 = \begin{pmatrix} \mathcal{O}_2 \\ \mathcal{O}_2 \end{pmatrix},\ N_3 = \mathcal{L}_3 \right\}.$$

Green's walk around the Brauer tree (cf. [Gr; 74]) gives the projective resolution

$$N_3 \to P \xrightarrow{\alpha_3} Q \xrightarrow{\alpha_2} Q \xrightarrow{\alpha_1} P \to N_3,$$

which is put together from the projective cover sequences

$$\mathbb{E}_1 : 0 \to M_1 \to P \xrightarrow{\phi_1} N_3 \to 0,$$

$$\mathbb{E}_2 : 0 \to N_1 \to Q \xrightarrow{\phi_2} M_1 \to 0,$$

$$\mathbb{E}_3 : 0 \to M_2 \to Q \xrightarrow{\phi_3} N_1 \to 0,$$

$$\mathbb{E}_4 : 0 \to N_3 \to P \xrightarrow{\phi_4} M_2 \to 0.$$

We now look at the modules X which have an (e_1, e_2, e_3)-*Cohen–Macaulay filtration*, i.e. $X_1 \subseteq X_2 \subseteq X_3 = X$ such that

$$X_1 \simeq N_1^{(s_1)}, \quad X_2/X_1 \simeq M_1^{(t_1)} \oplus M_2^{(t_2)}, \quad X_3/X_2 \simeq N_3^{(s_3)}.$$

THEOREM 1.1. *The indecomposable non-isomorphic Cohen–Macaulay $\mathcal{H}_p$-modules having an (e_1, e_2, e_3)-filtration are given as*:

- *the irreducible Cohen–Macaulay $\mathcal{H}_p$-modules* $\{N_1, M_1, M_2, N_3\}$,
- *the modules* $\phi_1^{-1}((p^\nu + \pi) \cdot N_3)$, $\nu = 0, 1, \ldots,$
- *the modules* $\phi_2^{-1}(((1 - \theta_p)^\nu + \pi) \cdot M_1)$, $\nu = 0, 1, \ldots$

If we look at the modules X which have an (e_2, e_1, e_3)-Cohen–Macaulay filtration, i.e. $X_1 \subseteq X_2 \subseteq X_3$ such that

$$X_1 \simeq M_1^{(t_1)} \oplus M_2^{(t_2)}, \quad X_2/X_1 \simeq N_1^{(s_1)}, \quad X_3/X_2 \simeq N_3^{(s_3)},$$

we obtain

THEOREM 1.2. *The indecomposable non-isomorphic Cohen–Macaulay $\mathcal{H}_p$-modules having an (e_2, e_1, e_3)-filtration are given as*:

- *the irreducible Cohen–Macaulay $\mathcal{H}_p$-modules* $\{N_1, M_1, M_2, N_3\}$,
- *the modules* $\phi_1^{-1}((p^\nu + \pi) \cdot N_3)$, $\nu = 0, 1, \ldots,$
- *the modules* $\phi_3^{-1}((p^\nu + \pi) \cdot N_1)$, $\nu = 0, 1, \ldots$

In the following sections we describe the situation in its most general setup.

2. d-Dimensional orders

DEFINITION 2.1. 1. *A locally regular noetherian integral domain $\mathcal{O}$* of dimension d is an integral domain $\mathcal{O}$ such that for every maximal ideal $\mathfrak{m}$ the completion $\mathcal{O}_\mathfrak{m}$ is regular of dimension d. (This is sometimes also called a *regular ring*, cf. [Na; 62], §28.)

2. Let $\mathcal{O}_i$ for $1 \leq i \leq s$ be locally regular noetherian integral domains of dimension d with field of fractions $\mathcal{K}_i$. By abuse of notation we shall write $\mathcal{O} = \{\mathcal{O}_i\}$ and treat it as a ring. Similarly, we put $\mathcal{K} = \{\mathcal{K}_i\}$. A finitely generated $\prod_{1 \leq i \leq s} \mathcal{K}_i$-module V is called a *finite-dimensional $\mathcal{K}$-vector space*, and a $\prod_{1 \leq i \leq s} \mathcal{O}_i$-module M is called a *Cohen–Macaulay $\mathcal{O}$-module* provided $M \simeq \bigoplus \wp_{i,j}^{(\nu_{i,j})}$, where $\wp_{ij}$ are projective $\mathcal{O}_i$-ideals; in particular, M is finitely generated.

3. Let $\mathcal{A} = \prod_{i=1}^s \mathcal{A} \cdot e_i$ be a $\mathcal{K}$-algebra, i.e. for each i the ring $\mathcal{A}_i := \mathcal{A} \cdot e_i$ is a separable $\mathcal{K}_i$-algebra; here $\{e_i\}$ are central idempotents in $\mathcal{A}$.

4. A *Cohen–Macaulay $\mathcal{O}$-order $\mathcal{L}$* in $\mathcal{A}$ is a subring of the $\mathcal{K}$-algebra $\mathcal{A}$ which is a Cohen–Macaulay $\mathcal{O}$-module spanning $\mathcal{A}$.

5. We denote by $\mathrm{CM}(\mathcal{L})$ the *category of Cohen–Macaulay modules over* $\mathcal{L}$, i.e. of left $\mathcal{L}$-module which are at the same time Cohen–Macaulay $\mathcal{O}$-modules.

6. The *Cohen–Macaulay radical*, $\mathrm{rad}_{\mathrm{CM}}(\mathcal{L})$, of the Cohen–Macaulay order $\mathcal{L}$ is defined as the two-sided ideal ϱ in $\mathcal{L}$ satisfying the following conditions:

(a) ϱ is a two-sided Cohen–Macaulay module.
(b) $\mathcal{L}/\varrho = \prod_{i=1}^{n} \overline{\mathcal{L}}_i$, where for each i the ring $\overline{\mathcal{L}}_i$ is Morita equivalent to an integral domain — not necessarily commutative — of Krull dimension $d-1$. Let $\phi : \mathcal{L} \to \overline{\mathcal{L}} := \mathcal{L}/\varrho \cdot \mathcal{L}$ be the natural map.
(c) If there is an epimorphism $\mathcal{L} \xrightarrow{\psi} X$ where X has Krull dimension $d-1$, then we have a factorization, i.e. a map $\chi : \overline{\mathcal{L}} \to X$ with $\psi = \phi \cdot \chi$.
(d) The ideal ϱ is unique with respect to these properties.

Remark 2.2. 1. Let $\mathcal{O} = R[q]$, where R is the ring of algebraic integers in an algebraic number field. If R is not a principal ideal domain, then $\mathcal{O}$ is not regular in the classical sense; however, all completions at maximal ideals are, since the maximal ideals have the form $\mathfrak{m} = \langle f, \wp \rangle$, where $\wp$ is a maximal ideal in R and $f \in R[q]$ is irreducible modulo $\wp$ (cf. [Mu; 88]). In the completion $\mathcal{O}_{\mathfrak{m}}$ the ideal $\wp$ becomes principal, since the completion $R_{\wp}$ of R at $\wp$ embeds into $\mathcal{O}_{\mathfrak{m}}$. These rings occur in connection with Hecke orders of dihedral groups of order $2p$ with p odd.

2. One has to be careful with quotients of $\mathcal{O}$ modulo height one prime ideals. In general, these quotients will not be locally regular; they will be of Krull dimension $d-1$, which is tantamount to being a Cohen–Macaulay order. In fact, let $\mathcal{O} = \mathbb{Z}[q]$ and let $f \in \mathcal{O}$ be an irreducible polynomial such that in the algebraic number field $K := \mathbb{Q}[q]/\langle f \rangle$ the ring of integers is strictly larger than $\overline{\mathcal{O}} := \mathcal{O}/\langle f \rangle$. Then $\overline{\mathcal{O}}$ is not regular; but it is a $\mathbb{Z}$-order, and so it has Krull dimension $d-1$ (here $d = 2$).

3. We have allowed in the definition of a Cohen–Macaulay order $\mathcal{L}$ the orders $\mathcal{L}_i := \mathcal{L} \cdot e_i$ to be defined over possibly different rings. This is not "l'art pour l'art"; the situation occurs in the structure of Hecke orders of dihedral groups (cf. [Ro; 98 I]). We give an example below.

4. In case $d = 1$, the notion of Cohen–Macaulay radical coincides with the Jacobson radical, since integral domains of Krull dimension 0 are skew fields.

5. We now give some examples which show that the Cohen–Macaulay radical may exist or may not exist. Let $\mathcal{O}$ be regular local of dimension d, and let

$$\mathcal{L} := \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathfrak{a} & \mathcal{O} \end{pmatrix}.$$

Then $\mathcal{L}$ is a Cohen–Macaulay order in our sense if $\mathfrak{a}$ is a projective ideal, i.e. $\mathfrak{a}$ is a product of height one prime ideals, and so locally free of height one.

- Let $\mathcal{O}$ be complete and let $\mathfrak{a} = \langle \pi^\nu \rangle$ for some principal height one prime ideal $\langle \pi \rangle$. Then

$$\mathrm{rad}_{\mathrm{CM}}(\mathcal{L}) = \begin{pmatrix} \pi\mathcal{O} & \mathcal{O} \\ \pi^\nu \cdot \mathcal{O} & \pi \cdot \mathcal{O} \end{pmatrix}, \tag{1}$$

$$\mathcal{L}/\mathrm{rad}_{\mathrm{CM}}(\mathcal{L}) = \begin{pmatrix} \overline{\mathcal{O}} & 0 \\ 0 & \overline{\mathcal{O}} \end{pmatrix}, \quad \text{where } \overline{\mathcal{O}} = \mathcal{O}/\langle \pi \rangle. \tag{2}$$

  We see that $\mathrm{rad}_{\mathrm{CM}}(\mathcal{L}) = \mathrm{rad}(\mathcal{L})$ is the Jacobson radical if $d = 1$.
- Let $\mathcal{O} = \mathbb{Z}[q]_{\langle q,p \rangle}$, where p is a rational prime, and let $\mathfrak{a} = \langle pq \rangle$. Then $\mathrm{rad}_{\mathrm{CM}}(\mathcal{L})$ does not exist. In fact, the ideals

$$J_p := \begin{pmatrix} p \cdot \mathcal{O} & \mathcal{O} \\ p \cdot q \cdot \mathcal{O} & p \cdot \mathcal{O} \end{pmatrix}, \tag{3}$$

$$J_q := \begin{pmatrix} q \cdot \mathcal{O} & \mathcal{O} \\ p \cdot q \cdot \mathcal{O} & q \cdot \mathcal{O} \end{pmatrix} \tag{4}$$

  have all the properties of the Cohen–Macaulay radical but there is not a unique minimal one. Moreover, for the intersection

$$J_{pq} = \begin{pmatrix} p \cdot q \cdot \mathcal{O} & \mathcal{O} \\ p \cdot q \cdot \mathcal{O} & p \cdot q \cdot \mathcal{O} \end{pmatrix}, \tag{5}$$

  we have

$$\mathcal{L}/J_{pq} = \begin{pmatrix} \overline{\mathcal{O}} & 0 \\ 0 & \overline{\mathcal{O}} \end{pmatrix}, \tag{6}$$

  where $\overline{\mathcal{O}}$ is the pull-back

$$\begin{tikzcd} \overline{\mathcal{O}} \arrow[r] \arrow[d] & \mathbb{F}_p[q]_{\langle q \rangle} \arrow[d] \\ \mathbb{Z}_{\langle p \rangle} \arrow[r] & \mathbb{F}_p \end{tikzcd}$$

  which is definitely not an integral domain.

There are, however, many instances where the Cohen–Macaulay radical does exist. We shall give several examples below as *graph orders*.

The category of Cohen–Macaulay modules is not a "good" category, since in general epimorphism images of Cohen–Macaulay modules need not be Cohen–Macaulay (examples will be given later on). Hence one cannot in general describe Cohen–Macaulay modules inductively via extensions.

In order to find a good substitute, we propose the following category:

DEFINITION 2.3. 1. A Cohen–Macaulay module M over $\mathcal{O}$ has a *Cohen–Macaulay filtration* provided it has a filtration

$$0 = M_0 \subset M_1 \subset \ldots \subset M_i \subset M_{i+1} \subset \ldots \subset M_n = M,$$

where the sections are Cohen–Macaulay modules over $\mathcal{O}$.

2. We say that a Cohen–Macaulay order (of dimension d) has a *Cohen–Macaulay filtration with respect to* $\mathcal{E}$ provided there exists a complete ([5]) ordered chain $\mathcal{E} := (e_1, \ldots, e_s)$ of central primitive orthogonal idempotents of $\mathcal{A}$ such that with $\varepsilon_i := \sum_{j=1}^{i} e_j$ we have a Cohen–Macaulay filtration

$$0 \subset \mathcal{L} \cdot \varepsilon_1 \subset \ldots \subset \mathcal{L} \cdot \varepsilon_s = \mathcal{L}.$$

3. For $M \in \mathrm{CM}(\mathcal{L})$ we put

$$M_i = M \cap \varepsilon_i \cdot \mathcal{A} \cdot M.$$

We note that the module M_i is pure in M_{i+1}; but in general, the quotient will not be Cohen–Macaulay. We say that M has a *Cohen–Macaulay filtration with respect to* $\mathcal{E}$ provided

$$0 \subseteq M_1 \subseteq \ldots \subseteq M_s = M$$

is a Cohen–Macaulay filtration of M.

4. We denote by $\mathrm{CM}^{\mathrm{f}}_{\mathcal{E}}(\mathcal{L})$ or briefly by $\mathrm{CM}^{\mathrm{f}}(\mathcal{L})$ the category of $\mathcal{L}$-modules which have a Cohen–Macaulay filtration with respect to $\mathcal{E}$.

In the case of the graph orders we shall describe various categories $\mathrm{CM}^{\mathrm{f}}_{\mathcal{E}}(\mathcal{L})$; in those cases, however, we do not know the full category $\mathrm{CM}(\mathcal{L})$ of all Cohen–Macaulay modules for $\mathcal{L}$. W. Rump has given an example where a Cohen–Macaulay $\mathcal{L}$-module has no Cohen–Macaulay filtration for any numbering of the central primitive idempotents.

3. "Hereditary" orders. Let $\mathcal{O}$ be a (single) complete regular local domain of Krull dimension d, maximal ideal $\mathfrak{m} := \langle \pi, \tau_1, \ldots, \tau_{d-1} \rangle$ and field of fractions $\mathcal{K}$. We note that $\mathcal{O}/\langle \pi \rangle$ is a complete local domain of Krull dimension $d-1$.

ASSUMPTION 3.1. Assume that Ω is a maximal Cohen–Macaulay $\mathcal{O}$-order in a skew field $\mathcal{D}$ with height one principal prime ideal $\langle \omega \rangle$ over $\langle \pi \rangle$, such that $\Omega/\langle \omega \rangle$ is an $\mathcal{O}/\langle \pi \rangle$-order of Krull dimension $d-1$, and such that the ring $\overline{\Omega} := \Omega/\langle \tau_1, \ldots, \tau_{d-1} \rangle$ is a maximal order in a skew field over the complete Dedekind domain $\mathcal{O}/\langle \tau_1, \ldots, \tau_{d-1} \rangle$.

Note that the last condition is a very strong one (cf. Remark 2.2).

([5]) I.e. $\sum_{j=1}^{s} e_j = 1$.

PROPOSITION 3.2. *Let*

$$\mathbb{H} := \begin{pmatrix} \Omega & \Omega & \Omega & \dots & \Omega & \Omega \\ \langle\omega\rangle & \Omega & \Omega & \dots & \Omega & \Omega \\ \langle\omega\rangle & \langle\omega\rangle & \Omega & \dots & \Omega & \Omega \\ & & & \ddots & & \\ \dots & \dots & \dots & & \dots & \dots \\ \langle\omega\rangle & \langle\omega\rangle & \langle\omega\rangle & \dots & \Omega & \Omega \\ \langle\omega\rangle & \langle\omega\rangle & \langle\omega\rangle & \dots & \langle\omega\rangle & \Omega \end{pmatrix}_n . \tag{7}$$

We call $\mathbb{H}$ *a Cohen–Macaulay hereditary order of size* n *over* Ω *with respect to* $\langle\omega\rangle$. *The category of finitely generated Cohen–Macaulay* $\mathbb{H}$*-modules*, $\mathrm{CM}(\mathbb{H})$, *coincides with the category of finitely generated projective* $\mathbb{H}$*-modules*, $\mathcal{P}^{\mathrm{f}}(\mathbb{H})$. *Moreover*,

$$\mathrm{rad}_{\mathrm{CM}}(\mathbb{H}) := \begin{pmatrix} \langle\omega\rangle & \Omega & \Omega & \dots & \Omega & \Omega \\ \langle\omega\rangle & \langle\omega\rangle & \Omega & \dots & \Omega & \Omega \\ \langle\omega\rangle & \langle\omega\rangle & \langle\omega\rangle & \dots & \Omega & \Omega \\ & & & \ddots & & \\ \dots & \dots & \dots & & \dots & \dots \\ \langle\omega\rangle & \langle\omega\rangle & \langle\omega\rangle & \dots & \langle\omega\rangle & \Omega \\ \langle\omega\rangle & \langle\omega\rangle & \langle\omega\rangle & \dots & \langle\omega\rangle & \langle\omega\rangle \end{pmatrix}_n \tag{8}$$

with quotient

$$\mathbb{H}/\mathrm{rad}_{\mathrm{CM}}(\mathbb{H}) \simeq \prod_1^n \Omega/\langle\omega\rangle.$$

P r o o f. This follows under some weaker conditions from the characterization of Cohen–Macaulay orders of finite Cohen–Macaulay type as isolated singularities by M. Auslander and I. Reiten in [AuRe; 89]. We shall, however, give a short proof which arose in a discussion with Dorin Popescu.

Because of our hypotheses, the order $\overline{\mathbb{H}} := \mathbb{H}/\langle\tau_1, \dots, \tau_{d-1}\rangle$ is a classical hereditary order, and hence every $\overline{\mathbb{H}}$-lattice is projective (6). We have the reduction functor

$$\mathcal{F} : \mathrm{CM}(\mathbb{H}) \to \mathrm{CM}(\overline{\mathbb{H}}), \qquad M \mapsto \overline{M} := M/\langle\tau_1, \dots, \tau_{d-1}\rangle \cdot M.$$

We shall use this functor to show that M is projective. Our hypothesis $M \in \mathrm{CM}(\mathbb{H})$ implies that $M \simeq_{\mathcal{O}} \mathcal{O}^{(n)}$ as $\mathcal{O}$-modules. Since $\overline{M}$ is projective, there exists a projective module $P \in \mathrm{CM}(\mathbb{H})$ which reduces to $\overline{M}$, i.e. $\overline{P} \simeq \overline{M}$. The reduction modulo $\langle\tau_1, \dots, \tau_{d-1}\rangle$ preserves the rank, and so we have

$$n := \mathrm{rank}_{\mathcal{O}}(M) = \mathrm{rank}_{\overline{\mathcal{O}}}(\overline{M}) = \mathrm{rank}_{\mathcal{O}}(P).$$

Let $\phi : M \to \overline{M}$ and $\psi : P \to \overline{M}$ be the reduction maps. Because of the projectivity of P there exists a homomorphism $\chi : P \to M$ with $\chi \circ \phi = \psi$.

(6) In our situation this is the same as Cohen–Macaulay.

Nakayama's lemma implies that $\chi : P \to M$ is surjective; but then the above rank considerations show that χ must also be injective. ∎

These orders will play the crucial role in the construction of our graph orders.

4. The order of a truncated graph. The definition has originally been given in [Ro; 96] in the one-dimensional case and also in Rump's paper [Ru; 98]. However, for the convenience of the reader we repeat it here in the generality needed.

Definition 4.1. A *truncated graph* G consists of a finite set of vertices $V = V(G) = \{v_1, \ldots, v_n\}$ and a set of *edges* $E = E(G) = E_1 \cup E_2$, where the edges in E_1 join two, not necessarily distinct, vertices, which are called the *genuine edges*, and the *truncated edges* in E_2 are associated to only one vertex, and have a free second vertex (not to be confused with loops).

Next we choose a fixed local embedding of G into the plane, i.e. every vertex v together with the germs of edges of v (local edges at v) is embedded into the plane.

We then call G a *locally embedded truncated graph.*

We always assume that the graphs under consideration are connected.

Note 4.2. 1. In G we allow loops — a loop is then automatically a genuine edge — and multiple edges.

2. Although not every finite graph can be embedded into the plane, every graph can be locally embedded into the plane.

3. A local embedding of a vertex v is given if we number the germs of the edges at v (i.e. the local edges) and the truncated edges at v as $\varepsilon(v) := (e_1(v), \ldots, e_{n_v}(v))$. We call n_v the *valency* of v.

4. A cyclic permutation of $\varepsilon(v)$ gives rise to the same local embedding.

5. If a local embedding is given, it gives rise to a unique numbering of the germs of the edges and truncated edges of v modulo a cyclic permutation. This observation is important for the uniqueness of our constructions later on.

We shall now construct an order $\mathcal{L}$ depending on the locally embedded graph G with truncated edges such that certain modules have the walks along the graph as projective resolutions. This will be done in two steps. First, with each locally embedded vertex we associate a Cohen–Macaulay hereditary order (cf. Proposition 3.2). We then use the genuine edges to define certain congruences between the orders corresponding to the respective vertices in order to construct the order $\mathcal{L} = \mathcal{L}(G)$.

Assumption 4.3. 1. Let G be a finite connected locally embedded truncated graph and let v be a fixed locally embedded vertex with local edges (genuine and truncated) $\varepsilon(v) = (e_1(v), \ldots, e_{n_v}(v))$.

2. Let $\mathcal{O}_v$ be a complete regular local integral domain of dimension d with field of fractions $\mathcal{K}_v$ and with a fixed principal height one prime ideal $\langle \pi(v) \rangle$ which is part of a regular sequence, i.e. the maximal ideal is

$$\mathfrak{m}_v = \langle \pi(v), \tau_1(v), \ldots, \tau_{d-1}(v) \rangle.$$

3. Let $\mathcal{D}_v$ be a finite-dimensional skew field over $\mathcal{K}_v$.

4. Assume that Ω_v is a Cohen–Macaulay $\mathcal{O}_v$-order in $\mathcal{D}_v$ with height one principal 2-sided prime ideal $\langle \omega_v \rangle$ over $\langle \pi(v) \rangle$ such that $\Omega_v / \langle \omega_v \rangle$ is an order of Krull dimension $d-1$.

5. We also assume that the rings Ω_v satisfy Assumption 3.1.

Definition 4.4 (The order associated with v). 1. The order associated with the locally embedded vertex v is then given by $\mathbb{H}_v$, which is the order $\mathbb{H}$ from Proposition 3.2 with $n = n_v$ and $\Omega := \Omega_v$ and $\omega = \omega_v$.

2. We denote by Ω_v^j the (j,j)-entry Ω_v in $\mathbb{H}_v$.

3. We put, for $1 \le j \le n_v$,

$$M_{v,j} := \begin{array}{c} \begin{pmatrix} \Omega_v \\ \ldots \\ \Omega_v \\ \langle \omega_v \rangle \\ \ldots \\ \langle \omega_v \rangle \end{pmatrix}_{n_v} \end{array} \begin{array}{l} \\ \\ j\text{th row} \\ \\ \\ \\ \end{array}. \tag{9}$$

Then the modules $\{M_{v,j} : 1 \le j \le n_v\}$ constitute a complete set of non-isomorphic indecomposable projective $\mathbb{H}_v$-modules (cf. Proposition 3.2) and hence also a complete set of indecomposable Cohen–Macaulay modules for $\mathbb{H}_v$.

4. We then have natural inclusions (except the last, which is right multiplication by $\omega(v)$):

$$M_{v,1} \to M_{v,2} \to \ldots \to M_{v,n_v-1} \to M_{v,n_v} \xrightarrow{\cdot\omega(v)} M_{v,1}. \tag{10}$$

5. We identify $M_{v,j}$ with the germ of the edge $e_j(v)$ at the vertex v; then the above chain of inclusions represents the cycle ε_v from Note 4.2; it is the clockwise walk around v starting at $e_1(v)$.

6. To simplify the notation we shall identify $M_{v,i}$ with $M_{v,i+n_v}$ but keeping in mind that multiplication by $\omega(v)$ is involved here.

7. Conjugation with the element

$$\underline{\omega}_v := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ \omega & 0 & 0 & \dots & 0 & 0 \end{pmatrix}_{n_v}$$

cyclically permutes the indecomposable projective $\mathbb{H}_v$-modules $\{M_{v,j}\}$; it induces an automorphism α_v of $\mathbb{H}_v$.

8. We have seen in Note 4.2 that the local embedding only determines ε_v up to a cyclic permutation; this can, however, be compensated by the automorphism α_v.

9. Hence this construction of $\mathbb{H}_v$ only depends on the local embedding of v, the order Ω_v, and the element ω_v.

Before we can finally define the order associated with G, we have to fix *some more notation*:

Assumption 4.5. 1. For each pair $\{v, w\}$ of vertices we have an isomorphism

$$\Omega_v/\langle\omega_v\rangle \simeq \Omega_w/\langle\omega_w\rangle.$$

2. We fix a ring $\overline{\Omega}$, which is isomorphic to Ω_v/ω_v for every $v \in V$.

3. By $\phi_v : \Omega_v \to \overline{\Omega}$ we denote a *fixed* epimorphism with kernel $\omega_v \cdot \Omega_v$. Note that there may be many different epimorphisms of this kind.

4. We use the abbreviation $\Omega_v \overset{\omega}{—} \Omega_w$ to denote the pull-back

$$\begin{tikzcd} \Omega_v \overset{\omega}{—} \Omega_w \arrow[r] \arrow[d] & \Omega_v \arrow[d, "\phi_v"] \\ \Omega_w \arrow[r, "\phi_w"] & \overline{\Omega} \end{tikzcd} \tag{12}$$

Note that $\Omega_v \overset{\omega}{—} \Omega_w$ changes — not only up to isomorphism — if the maps ϕ_v and ϕ_w are changed.

Definition 4.6 (The order associated with G). Put $\mathbb{H} := \prod_{v\in V} \mathbb{H}_v$. We now describe the order $\mathcal{L} := \mathcal{L}(G)$ as a subring of $\mathbb{H}$ spanning the same algebra. Let $v \in V$ be a vertex and let $e_i(v)$ be the germ of a genuine edge at v (cf. Note 4.2). Since $e_i(v)$ is a genuine edge, it is associated with a second vertex $e_j(w)$; note that $v = w$ is possible.

We now replace in $\mathbb{H}_v \times \mathbb{H}_w$ (in $\mathbb{H}_v$ if $v = w$) the product $\Omega_v^i \times \Omega_w^j$ by $\Omega_v \overset{\omega}{—} \Omega_w$ (cf. Definition 4.4). This means that we have identified the (i, i)-entry of $\mathbb{H}_v$ with the (j, j)-entry in $\mathbb{H}_w$ "modulo" ω. We do this for all genuine edges. Then $\mathcal{L} = \mathcal{L}(G)$ is called the *order associated with the graph* G (*with respect to* $\{\Omega_v\}$ *and* $\{\phi_v\}$).

Let us note some obvious properties of $\mathcal{L}(G)$:

NOTE 4.7. 1. For the graph order $\mathcal{L}(G)$ we have

$$\mathrm{rad}_{\mathrm{CM}}(\mathcal{L}(G)) = \prod_{v\in V} \mathrm{rad}_{\mathrm{CM}}(\mathbb{H}_v) \quad \text{(cf. Proposition 3.2)}.$$

2. The indecomposable projective Cohen–Macaulay $\mathcal{L}$-modules are in bijection to the "edges" — both genuine and truncated — of G, so we label them $\{P_e\}_{e\in E}$.

3. If $e = e_i(v)$ is a truncated edge with vertex v, then $P_e = M_{v,i}$ is a projective $\mathbb{H}_v$-module as well as a projective $\mathcal{L}$-module.

4. In this case we have a short exact sequence

$$0 \to M_{v,i-1} \to P_e = M_{v,i} \to \overline{\Omega} \to 0.$$

5. If e is a genuine edge, then P_e is the pull-back

$$\begin{tikzcd} P_e \arrow[r] \arrow[d] & M_{v,i} \arrow[d, "\phi_v^i"'] \\ M_{w,j} \arrow[r, "\phi_w^j"] & \overline{\Omega} \end{tikzcd} \tag{13}$$

(The case $v = w$ is not excluded.)

6. We have the following commutative diagram with exact rows and columns for a genuine edge:

$$\begin{tikzcd} & 0 & 0 & 0 & \\ 0 \arrow[r] & M_{v,i-1} \arrow[r] \arrow[u] & M_{v,i} \arrow[r] \arrow[u] & \overline{\Omega} \arrow[r] \arrow[u] & 0 \\ 0 \arrow[r] & M_{v,i-1} \arrow[r] \arrow[u] & P_e \arrow[r] \arrow[u] & M_{w,j} \arrow[r] \arrow[u] & 0 \\ 0 \arrow[r] & 0 \arrow[r] \arrow[u] & M_{w,j-1} \arrow[r] \arrow[u] & M_{w,j-1} \arrow[r] \arrow[u] & 0 \\ & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & \end{tikzcd} \tag{14}$$

the maps being self-explanatory.

7. It should be noted that the kernels of the projections $P_e \to M_{v,i}$ and $P_e \to M_{w,j}$ are projective Cohen–Macaulay $\mathbb{H}$-modules, and that they are

local $\mathcal{L}$-modules; this is the reason why we can interpret the "walks around the graph" as projective resolutions of some of the modules $\{M_{v,j}\}$ (cf. [Ro; 96]).

8. Actually, there is no reason to require that $\mathcal{A}$ is separable, the whole construction works if the rings Ω_v are finitely generated — as modules — local algebras.

5. Cohen–Macaulay modules over graph orders. Let $\mathcal{L} = \mathcal{L}(G)$ be the graph order of a locally embedded truncated graph as in Definition 4.1. According to Proposition 3.2 the only irreducible Cohen–Macaulay $\mathcal{L}$-modules ([7]) are the modules $\{M_{v,i} : v \in V,\ 1 \leq i \leq n_v\}$.

In order to describe the $\mathcal{L}$-modules with a suitable Cohen–Macaulay filtration, we first compute the extension groups between the irreducible Cohen–Macaulay $\mathcal{L}$-modules. (If there are loops, one has to take the filtration induced by the edges and the Cohen–Macaulay $\mathbb{H}$-modules.)

PROPOSITION 5.1. *We have* ([8]) *for* $M_{v,i}$ *not projective and* $M_{w,j-1}$ *not injective in* $\mathrm{CM}(\mathcal{L})$,

$$\mathrm{Ext}^1_{\mathcal{L}}(M_{v,i}, M_{u,k}) = \begin{cases} \overline{\Omega} & \text{if } M_{u,k} = \Omega^1(M_{v,i}), \\ 0 & \text{otherwise}. \end{cases}$$

P r o o f. We use the projective cover sequence from diagram (14),

$$\mathbb{E}_e : 0 \to M_{w,j-1} \xrightarrow{\psi_e} P_e \xrightarrow{\phi_e} M_{v,i} \to 0,$$

noting that e is a genuine edge, $M_{v,i}$ not being projective. Noting further that P_e is also an injective object in the category of Cohen–Macaulay $\mathcal{L}$-modules ([9]) we conclude that

$$\mathrm{Ext}^1_{\mathcal{L}}(M_{v,i}, M_{u,k}) \simeq \underline{\mathrm{Hom}_{\mathcal{L}}(M_{w,j-1}, M_{u,k})},$$

where the latter are the homomorphisms modulo those factoring via ψ_e. Hence if $u \neq w$, then $\mathrm{Ext}^1_{\mathcal{L}}(M_{v,i}, M_{u,j}) = 0$. So we may assume that $u = w$ and that $k = j + l$ with $-1 \leq j \leq n_w - 1$. We have the chain of "natural inclusions"

$$M_{w,j-1} \xrightarrow{\iota_{j-1}} M_{w,j} \xrightarrow{\iota_j} \ldots \to M_{w,j-2} \xrightarrow{\iota_{j-2}} M_{w,j-1}, \tag{15}$$

where the composition of all the maps is multiplication by ω_v, and moreover, ι_{j-1} already factorizes via ψ_e. Since $\Omega_v/\omega_v \simeq \overline{\Omega}$ the statement follows. ∎

([7]) These are the modules M such that $\mathcal{A} \cdot M$ is simple.

([8]) $\Omega^1(-)$ means the first syzygy.

([9]) It is easily seen that all indecomposable projective modules corresponding to genuine edges are also injective in the category of Cohen–Macaulay modules. The indecomposable projective modules corresponding to truncated edges are not injective in $\mathrm{CM}(\mathcal{L})$.

The next result is straightforward.

LEMMA 5.2. *We have*

$$\mathrm{Ext}^1_{\mathcal{L}}(M_{v,i}, M_{w,j-1}) \simeq \underline{\mathrm{End}_{\mathcal{L}}}(M_{v,i}) \simeq \underline{\mathrm{End}_{\mathcal{L}}}(M_{w,j-1}) \simeq \overline{\Omega}.$$

Moreover, $\mathrm{Ext}^1_{\mathcal{L}}(M_{v,i}, M_{w,j-1})$ *is an* (Ω_v, Ω_w)*-bimodule, and for* $x_v \in \Omega_v$ *and* $z_w \in \Omega_w$ *we have* ([10])

$$z_w \mathbb{E}_e \equiv \mathbb{E}_e x_v \Leftrightarrow x_v \phi_v = z_w \phi_w,$$

$(z_w + \omega_w \cdot z'_w)\mathbb{E}_e \equiv z_w \mathbb{E}_e$, *and similarly on the other side. In particular, for every* $z_w \in \Omega_w$ *there exist an* $x_v \in \Omega_v$ *such that* $z_w \mathbb{E}_e \equiv \mathbb{E}_e x_v$ *(cf. Assumption* 4.5 *for the definition of* ϕ_v *and* ϕ_w*).*

We next explicitly describe the extensions in $\mathrm{Ext}^1_{\mathcal{L}}(M_{v,i}, M_{w,j-1})$. Given $x_v \in \Omega_v$, we can then form the commutative diagram with exact rows via the pull-back (for brevity we write $M := M_{v,i}$ and $N := M_{w,j-1}$)

$$\begin{tikzcd}
\mathbb{E}_e : 0 \arrow[r] & N \arrow[r, "\psi_e"] \arrow[d, equal] & P_e \arrow[r, "\phi_e"] & M \arrow[r] & 0 \\
x_v\mathbb{E}_e : 0 \arrow[r] & N \arrow[r, "\psi_{x_v}"] & X_{x_v} \arrow[r, "\phi_{x_v}"] \arrow[u, "\beta_{x_v}"] & M \arrow[r] \arrow[u, "x_v"'] & 0
\end{tikzcd}$$

Then

$$X_{x_v} \simeq \phi_e^{-1}(M \cdot x_v). \tag{16}$$

According to the pull-back in (13), we may view

$$P_e = \{(a,b) : a \in M_{v,i},\ b \in M_{w,j},\ a\phi_v^i = b\phi_w^j\} \subset M_{v,i} \times M_{w,j},$$

and so we may also view $X_{x_v} \subset M_{v,i} \times M_{w,j}$. The map $\psi_e : N \to P_e$ is then given by $n \mapsto (n, 0)$. Similarly the map $\psi_{x_v} : N \to X_{x_v}$ is given by $n \mapsto (n, 0)$. As we have described P_e via a pull-back (cf. diagram (13)), we have a similar description for X_{x_v}. We now fix some notation:

- Let $M_{x_v} = M \cdot x_v$.
- Denote by $\overline{\Omega}_{x_v}$ the image of M_{x_v} under the map $M_{v,i} \to \overline{\Omega}$ with kernel $M_{v,i-1}$. Then X_{x_v} is the pull-back described as follows:

([10]) Recall that $\phi_v : \Omega_v \to \overline{\Omega}$ is the map which is incorporated into the definition of $\mathcal{L}$ (cf. Assumption 4.5).

$$
\begin{tikzcd}
& 0 & 0 & 0 & \\
0 \arrow[r] & M_{v,i-1} \cap M_{x_v} \arrow[r] \arrow[u] & M_{x_v} \arrow[r] \arrow[u] & \bar{\Omega}_{x_v} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & M_{v,i-1} \cap M_{x_v} \arrow[r] \arrow[u] & X_{x_v} \arrow[r] \arrow[u] & M_{w,j-1} + M_{w,j} z_w \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[u] & M_{w,j-1} \arrow[r] \arrow[u] & M_{w,j-1} \arrow[r] \arrow[u] & 0 \\
& 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}
\tag{17}
$$

where z_w is defined as follows: the ring $\mathrm{End}_{\mathcal{L}}(P_e)$ is given as a pull-back

$$
\begin{tikzcd}
\Omega_v \overset{\omega}{\text{———}} \Omega_w \arrow[r] \arrow[d] & \Omega_v \arrow[d, "\phi_v"] \\
\Omega_w \arrow[r, "\phi_w"] & \bar{\Omega}
\end{tikzcd}
$$

So we may choose $z_w \Omega_w$ such that $x_v \phi_v = w_z \phi_w$, which then as a pair (x_v, z_w) give rise to an endomorphism of P_e. Note, however, that the element z_w is not uniquely determined by x_v; it is unique in the stable category, i.e. in its action on the extension groups. We may thus view

$$X_{x_v} \subset M_{x_v} \oplus (M_{w,j-1} + M_{w,j} \cdot z_w).$$

Note that the latter module is in general *not a Cohen–Macaulay module*; it has, however, a Cohen–Macaulay filtration, as described by the extension $x_v \mathbb{E}_e$.

6. Examples of Cohen–Macaulay modules for graph orders. As pointed out in Remark 2.2 the reduction modulo principal prime ideals has to be done with great care. We shall elaborate on two examples. Let $\mathbb{Z}[q, q^{-1}]_{\langle \mathfrak{m} \rangle}$ be the completion with respect to a maximal ideal. We now consider the following two essentially different cases:

DEFINITION 6.1. • Let p be an odd rational prime and let $\pi_1 := \Phi_{p^n}(q)$ be the p^nth cyclotomic polynomial. In this case we choose $\mathfrak{m} = \langle q-1, p \rangle$. We denote by θ_{p^n} a primitive p^nth root of unity and put $\mathcal{O}_1 = \mathbb{Z}[q, q^{-1}]_{\mathfrak{m}}$, the completion at $\mathfrak{m}$. In this case $\bar{\mathcal{O}}_1 := \mathcal{O}_1 / \langle \pi_1 \rangle = \mathbb{Z}[\theta_{p^n}]_{\langle \theta_{p^n} - 1 \rangle}$ is a discrete rank one valuation ring with maximal ideal $\mathfrak{m}_1 := \langle \theta_{p^n} - 1 \rangle$, a principal prime ideal over $\langle p \rangle$.

• As a second example let $\pi_2 := f_0(q)$ be a monic irreducible polynomial over $\mathbb{Z}_{\langle p\rangle}$ and let $\mathfrak{m}$ be a maximal ideal of $\mathbb{Z}_{\langle p\rangle}[q,q^{-1}]$ such that $f_0(q)$ stays irreducible in the completion $\mathcal{O}_2 := \mathbb{Z}_{\langle p\rangle}[q,q^{-1}]_{\mathfrak{m}}$. Assume that $\overline{\mathcal{O}}_2 := \mathcal{O}_2/\langle f_0(q)\rangle$ is not the ring of p-adic integers in $\mathbb{Q}_{\langle p\rangle}[q,q^{-1}]/\langle f_0(q)\rangle$. In this case $\overline{\mathcal{O}}_2$ is a local $\mathbb{Z}_{\langle p\rangle}$-order which is not maximal, and so its maximal ideal $\mathfrak{m}_2 = \langle \alpha_1,\ldots,\alpha_m\rangle$ is not cyclic, say it is minimally generated by $m \geq 2$ elements.

We next consider the $\mathcal{O}_i$ graph orders for the graph $\bullet\!-\!\!-\!\bullet$:

Definition 6.2. Let $\phi_i : \mathcal{O}_i \to \overline{\mathcal{O}}_i$ be the reductions modulo π_i for $i = 1, 2$. We define the Cohen–Macaulay $\mathcal{O}_i$-order $\mathcal{L}_i$ as the pull-back

$$\begin{tikzcd} \mathcal{L}_i \arrow[r] \arrow[d] & \mathcal{O}_i \arrow[d, "\phi_i"'] \\ \mathcal{O}_i \arrow[r, "\phi_i"] & \overline{\mathcal{O}}_i \end{tikzcd}$$

Let $\mathcal{K}_i$ be the quotient field of $\mathcal{O}_i$ and let $\mathcal{A}_i$ be the $\mathcal{K}_i$-algebra generated by $\mathcal{O}_i$.

The structure of the Cohen–Macaulay $\mathcal{L}_i$-modules is completely different in case $i = 1$ and in case $i = 2$.

Let us collect some easy facts (cf. Section 5).

Note 6.3. • We have the projective resolutions

$$\mathbb{E}_i^l : 0 \to \pi_i \cdot N_i \to P_i \to M_i \to 0$$

and

$$\mathbb{E}_i^r : 0 \to \pi_i \cdot M_i \to P_i \to N_i \to 0;$$

moreover, $\mathrm{Hom}_{\mathcal{L}_i}(M_i, N_i) = \mathrm{Hom}_{\mathcal{L}_i}(N_i, M_i) = 0$.

• $\mathrm{Ext}^1_{\mathcal{L}_i}(M_i, N_i) = \overline{\Omega}_i = \mathrm{Ext}^1_{\mathcal{L}_i}(N_i, M_i)$.

• The corresponding extensions are described in Lemma 5.2.

• We shall write $\mathrm{CM}^{\mathrm{l}}(\mathcal{L}_i)$ ($\mathrm{CM}^{\mathrm{r}}(\mathcal{L}_i)$) for the category of Cohen–Macaulay $\mathcal{L}_i$ modules having a Cohen–Macaulay filtration with respect to (e_1, e_2) ((e_2, e_1) resp.), where e_1 is the primitive idempotent of $\mathcal{A}_i$ with $e_1 \cdot N_i \neq 0$ and e_2 is the primitive idempotent of $\mathcal{A}_i$ with $e_2 \cdot M_i \neq 0$.

Claim 6.4. *Let the following extension of Cohen–Macaulay $\mathcal{L}_i$-modules be given*:

$$0 \to N_i \oplus N_i \to E_i \to M_i \to 0.$$

• *If $i = 1$, then E_1 decomposes.*

• *If $i = 2$, then there exists an extension where E_2 is indecomposable.*

P r o o f. Because of the hypotheses from Note 6.3 (cf. [Ro; 70], Chap. X) the module E_i decomposes if and only if the corresponding matrix

$$\mathbb{E}_i \in (\mathrm{Ext}^1_{\mathcal{L}_i}(M_i, N_i), \mathrm{Ext}^1_{\mathcal{L}_i}(M_i, N_i))$$

decomposes under transformations of the form $A_i \cdot \mathbb{E}_i \cdot B_i$ with $A_i \in \mathrm{Gl}(2, \overline{\Omega}_i)$ and $B_i \in \mathrm{Gl}(2, \overline{\Omega}_i)$. The matrix $\mathbb{E}_i$ is given as $\mathbb{E}_i = (\alpha_i, \beta_i)$ with $\alpha_i, \beta_i \in \overline{\Omega}_i$. We now consider our two cases:

Case 1. Let $i = 1$. Then — if necessary after renumbering — there exists an element $\gamma_1 \in \overline{\Omega}_1$ with $\alpha_1 \cdot \gamma_1 = \beta_1$, since $\overline{\Omega}_1$ is a valuation ring. Hence E_1 decomposes.

Case 2. If $i = 2$, then there are elements α_2 and β_2 such that none of them is a multiple in $\overline{\Omega}_2$ of the other. Hence the matrix (α_2, β_2) cannot be decomposed and so the middle term of the corresponding extension $(\alpha_2, \beta_2) \in \mathrm{Ext}^1_{\mathcal{L}_1}(M_1, N_1^{(2)})$ is indecomposable. ∎

Proposition 6.5. • *In Case* 1 *let* $\overline{\tau}$ *be a generator of the radical of* $\overline{\Omega}_1$ *and let* τ *be a preimage in* Ω_1. *The non-isomorphic indecomposable Cohen–Macaulay* $\mathcal{L}_1$*-modules in* $\mathrm{CM}^{\mathrm{l}}(\mathcal{L}_1)$ *are the modules* E^{l}_ν *occurring in the middle term of the exact sequence* $\tau^\nu \cdot \mathbb{E}^{\mathrm{l}}_1$ *for* $\nu = 0, 1, \ldots$ *and the two irreducible modules* M *and* N.

The non-isomorphic indecomposable Cohen–Macaulay $\mathcal{L}_1$*-modules in* $\mathrm{CM}^{\mathrm{r}}(\mathcal{L}_1)$ *are the modules* E^{r}_ν *occurring in the middle term of the exact sequence* $\mathbb{E}^{\mathrm{r}}_1 \cdot \tau^\nu$ *for* $\nu = 0, 1, \ldots$ *and the two irreducible modules* M *and* N.

The indecomposable Cohen–Macaulay $\mathcal{L}_1$*-modules in* $\mathrm{CM}^{\mathrm{l}}(\mathcal{L}_1) \cap \mathrm{CM}^{\mathrm{r}}(\mathcal{L}_1)$ *are thus the modules* P_1, M_1, N_1 (11).

• *In Case* 2, *there are indecomposable modules of arbitrarily large rank and with an* (e_1, e_2)*-filtration.*

P r o o f. Case 1. Let $X \in \mathrm{CM}^{\mathrm{l}}(\mathcal{L}_1)$ be indecomposable. Since it has a Cohen–Macaulay (e_1, e_2)-filtration, we have an exact sequence

$$\mathbb{E} : 0 \to X' \to X \to X'' \to 0$$

with $e_1 \cdot X' = X'$ and $e_2 \cdot X'' = X''$. If N (resp. M) is the only indecomposable Cohen–Macaulay $\mathcal{L}_1 \cdot e_1$- (resp. $\mathcal{L}_1 \cdot e_2$-) module, we conclude that

$$X' \simeq N_1^{(s_1)} \quad \text{and} \quad X'' \simeq M_1^{(s_2)}.$$

Therefore the above sequence $\mathbb{E}$ corresponds to a matrix $E \in \mathrm{Mat}(s_2 \times s_1, \overline{\Omega}_1)$ which is operated on by $\mathrm{Gl}(s_2, \overline{\Omega}_1)$ from the left and by $\mathrm{Gl}(s_2, \overline{\Omega}_1)$ from the right. Since $\overline{\Omega}_1$ is a principal ideal domain, we can reduce E under this operation to the diagonal form; i.e. the sequence $\mathbb{E}$ decomposes unless

(11) I do not know whether there are indecomposable Cohen–Macaulay $\mathcal{L}_1$-modules of arbitrarily large rank which then cannot have a Cohen–Macaulay filtration.

$s_1 = 1 = s_2$. Thus $\mathbb{E} = \varrho \cdot \mathbb{E}_1^1$ for some $\varrho \in \overline{\Omega}_1$. Since modifying with a unit does not change the isomorphism class of X, we may replace ϱ by τ^ν, provided ϱ has value ν. The case of $\mathrm{CM}^{\mathrm{r}}(\mathcal{L}_1)$ is dealt with similarly.

CASE 2. We generalize the example from Claim 6.4: Let $\mathfrak{m}_2 = \{\alpha, \beta, \ldots\}$. Then the matrix

$$\mathbb{E}_n := \begin{pmatrix} \alpha & \beta & 0 & \ldots & 0 & 0 \\ 0 & \alpha & \beta & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & 0 & 0 & \alpha \end{pmatrix}_n$$

gives an indecomposable filtered Cohen–Macaulay module as an exact sequence in $\mathrm{Ext}^1_{\mathcal{L}_2}(M_2^{(n)}, N_2^{(n)})$. ∎

7. The classification of Cohen–Macaulay modules with a Cohen–Macaulay filtration for graph orders. We let $\mathcal{L}$ be the $\mathcal{O}$-order corresponding to the graph G with underlying hereditary $\mathbb{H} = \prod_{v \in V} (\mathcal{O})_{n_v}$ with parameter $\langle \pi \rangle = \langle \pi_v \rangle = \langle \omega_v \rangle$ for every vertex v being a height one prime ideal. We put $\overline{\mathcal{O}} := \mathcal{O}/\langle \pi \rangle$ with field of fractions $\overline{K}$. For brevity we write $J := \mathrm{rad}_{\mathrm{CM}}(\mathcal{L})$ for the Cohen–Macaulay radical of $\mathcal{L}$, which is at the same time the Cohen–Macaulay radical of $\mathbb{H}$.

REMARK 7.1. In case $\mathcal{O} = \mathbb{Z}[q]_{\mathfrak{m}}$ is the completion at a maximal ideal with $\langle p \rangle = \mathbb{Z} \cap \mathcal{O}$ and $\pi = f$ an irreducible monic polynomial over $\mathbb{Z}_{\langle p \rangle}$, we conclude that $\mathcal{O}/\langle \pi \rangle$ is an integral domain of Krull dimension 1 with $|\mathcal{O} : \mathbb{Z}_{\langle p \rangle}| < \infty$; as a matter of fact, this degree is the degree of f.

NOTE 7.2. We have

$$\begin{aligned} \overline{\mathbb{H}} &:= \mathbb{H}/J \simeq \prod_{2 \cdot |E_1|} \overline{\mathcal{O}} \times \prod_{|E_2|} \overline{\mathcal{O}}, \\ \overline{\mathcal{L}} &:= \mathcal{L}/J \simeq \prod_{|E_1|} \overline{\mathcal{O}} \times \prod_{|E_2|} \overline{\mathcal{O}}, \end{aligned} \tag{18}$$

where the embedding of a copy of $\overline{\mathcal{O}}$ corresponding to the genuine edge $e \in E_1$ (cf. Definition 4.1) from $\overline{\mathcal{L}}$ into two copies of $\overline{\mathcal{O}}$ corresponding to the endpoints of e is the diagonal embedding. For a truncated edge $e \in E_2$ (again Definition 4.1) we have the identity as identification.

This implies that the embedding problem $\overline{\mathcal{L}} \subset \overline{\mathbb{H}}$ decomposes as

$$\begin{aligned} (\overline{\mathcal{L}} \subset \overline{\mathbb{H}}) = \Big(\prod_{e \in E_1} \overline{\mathcal{O}}_e \xrightarrow{(\alpha,\beta)} \overline{\mathcal{O}}_{v,i} \times \overline{\mathcal{O}}_{w,j} \Big) \\ \times \Big(\prod_{e \in E_2} \overline{\mathcal{O}}_e \to \overline{\mathcal{O}}_{v,i} \Big), \end{aligned} \tag{19}$$

where the edge $e \in E_1$ has local edges v, i and w, j and the truncated edge $e \in E_2$ has (only) the local edge v, i.

As mentioned in the introduction, the Cohen–Macaulay representation type of $\mathcal{L}$ strongly depends on $\overline{\mathcal{O}}$. We now demonstrate this.

Recall that for a finitely generated $\mathcal{O}$-torsion-free $\mathcal{O}$-module T we denote by $T^{**} = \mathrm{Hom}_{\mathcal{O}}(\mathrm{Hom}_{\mathcal{O}}(T, \mathcal{O})\mathcal{O})$ its Cohen–Macaulay-fication.

PROPOSITION 7.3. *Let M be an indecomposable not irreducible Cohen–Macaulay module for $\mathcal{L}$. Then there are exactly two vertices v and w together with an edge $e = v_i - w_j$ such that*

$$M_{\mathbb{H}} := (\mathbb{H} \cdot M)^{**} \simeq M_{v,i}^{\nu} \oplus M_{w,j}^{\mu}$$

with $\mu, \nu \in \mathbb{N} \setminus \{0\}$ [12].

For an edge $e = v_i - w_j$ we put

$$\mathrm{CM}(\mathcal{L})_e := \{M \in \mathrm{CM}(\mathcal{L}) : M_{\mathbb{H}} \simeq M_{v,i}^{\nu} \oplus M_{w,j}^{\mu},\ \mu, \nu \in \mathbb{N}\}.$$

Note that here we allow $\mu = 0$ and $\nu = 0$.

The category $\mathrm{CM}(\mathcal{L})_e$ is equivalent to $\mathrm{CM}(\mathcal{L}(\bullet - \bullet))$ with respect to $\Omega_1 = \mathcal{O}$, $\Omega_2 = \mathcal{O}$ and $\omega_1 = \omega_2 = \langle \pi \rangle$.

Before we come to the proof, we note:

CLAIM 7.4. *For a Cohen–Macaulay $\mathcal{L}$-module M we have a chain of inclusions*:

$$J \cdot M_{\mathbb{H}} = J \cdot (\mathbb{H} \cdot M)^{**} \subset M \subseteq \mathbb{H} \cdot M \subseteq (\mathbb{H} \cdot M)^{**} = M_{\mathbb{H}}. \tag{20}$$

Proof. Let

$$0 \to X \to Y \to T \to 0$$

be an exact sequence of finitely generated $\mathcal{O}$-modules with X and Y torsion-free for $\mathcal{O}$ and T a torsion module. Since $T^* = 0$, we get the exact sequence

$$0 \to Y^* \to X^* \to T_1 \to 0,$$

where T_1 is again an $\mathcal{O}$-torsion module. Thus we obtain an exact sequence

$$0 \to X^{**} \to Y^{**} \to T_2 \to 0,$$

where T_2 is again $\mathcal{O}$-torsion. Hence

$$X^{**} \subset Y^{**}.$$

We have the inclusion $J \cdot M = J \cdot \mathbb{H} \cdot M \subset M$ with torsion cokernel, and hence we conclude that $(J \cdot \mathbb{H} \cdot M)^{**} \subset M^{**} = M$ [13]. However, since every Cohen–Macaulay module for $\mathbb{H}$ is projective and since J is a Cohen–Macaulay module for $\mathbb{H}$, we conclude that $(J \cdot \mathbb{H} \cdot M)^{**} = J \cdot (\mathbb{H} \cdot M)^{**}) = J \cdot M_{\mathbb{H}}$. ∎

[12] This is so, since M is not irreducible.

[13] We have identified M with M^{**}, since M is Cohen–Macaulay.

Definition 7.5. Let $\mathbb{D}_0$ be the category whose objects are pairs $(\overline{X}, \overline{Y})$ of a finitely generated torsion-free $\overline{\mathcal{L}}$-module $\overline{X}$ embedded into a finitely generated projective $\overline{\mathbb{H}}$-module $\overline{Y}$. A morphism $\alpha : (\overline{X}, \overline{Y}) \to (\overline{X}_1, \overline{Y}_1)$ is an $\overline{\mathbb{H}}$-homomorphism $\alpha : \overline{Y} \to \overline{Y}_1$ such that $\alpha{\downarrow_{\overline{X}}} : \overline{X} \to \overline{X}_1$.

Note 7.6. Because of equation (19) the category $\mathbb{D}_0$ decomposes into the product of the categories $\mathbb{D}_e$ over all edges $e = v_i — w_j$ and categories which correspond to the identity embedding $\overline{\mathcal{O}} \to \overline{\mathcal{O}}$ corresponding to the truncated edges.

Since the truncated edges only give one indecomposable projective representation, we shall omit them in our considerations.

Claim 7.7. *Given a Cohen–Macaulay $\mathcal{L}$-module M, the functor $\mathbb{F}$ induced from*

$$\mathbb{F} : \mathrm{CM}(\mathcal{L}) \to \mathbb{D}_0 : M \to (\overline{M} := M/J \cdot M_{\mathbb{H}}, \overline{M}_{\mathbb{H}} := M_{\mathbb{H}}/J \cdot M_{\mathbb{H}}).$$

preserves indecomposability, i.e. M is an indecomposable Cohen–Macaulay $\mathcal{L}$-module if and only if $\mathbb{F}(M)$ is an indecomposable object in $\mathbb{D}$ ([14]).

Proof. Assume that $\overline{M} \xrightarrow{\alpha} \overline{M}_{\mathbb{H}}$ decomposes, i.e. $\overline{M} = \overline{Z}_1 \oplus \overline{Z}_2, \overline{M}_{\mathbb{H}} = \overline{Y}_1 \oplus \overline{Y}_2$ and $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_i : Z_i \to Y_i$. Since Z_i are projective $\overline{\mathbb{H}}$-modules, they can be lifted to projective $\mathbb{H}$-modules $Y_i \xrightarrow{\sigma_i} \overline{Y}_i$. We now define $X_i = \sigma^{-1}(\mathrm{Im}(\alpha_i))$. Then $M = X_1 \oplus X_2$, since simultaneously with $M_{\mathbb{H}}$ also $J \cdot M_{\mathbb{H}}$ decomposes. ∎

Combining this with the observation in Note 7.2 we get:

Corollary 7.8. *The indecomposable Cohen–Macaulay modules for $\mathcal{L}$ are in bijection with $\prod_{e\in E} \mathrm{ind}(\mathrm{CM}(\mathcal{L}(\bullet — \bullet)))$ with respect to $(\mathcal{O}, \langle\pi\rangle)$.*

This also completes the proof of Proposition 7.3. ∎

Because of the above results we can make the following

Assumption 7.9. From now on we assume that the underlying graph is $\bullet — \bullet$. (This is legitimate since truncated edges only contribute one projective irreducible Cohen–Macaulay module.) The order $\mathbb{H}$ is then $\mathcal{O} \times \mathcal{O} = \mathbb{H} \cdot e_1 \times \mathbb{H} \cdot e_2$, and $\mathcal{L}$ is the diagonal modulo $\langle\pi\rangle$.

Moreover, we shall only study the Cohen–Macaulay modules which have a Cohen–Macaulay filtration. There are exactly two filtrations, namely (e_1, e_2) and (e_2, e_1). Since the associated categories of filtered Cohen–Macaulay modules are isomorphic, we restrict attention to the category $\mathrm{CM}^{\mathrm{f}}(\mathcal{L})$ of (e_1, e_2)-filtered Cohen–Macaulay modules.

([14]) Here we do allow truncated edges.

We consider representations of the *embedding problem*

$$\overline{\mathcal{L}} \xrightarrow{\Delta} \overline{\mathbb{H}} = \overline{\mathcal{O}}_1 \times \overline{\mathcal{O}}_2 \tag{21}$$

where $\overline{\mathcal{O}}_i \simeq_{\text{rings}} \overline{\mathcal{O}}$ and $\overline{\mathcal{L}} \simeq_{\text{rings}} \overline{\mathcal{O}}$; moreover, Δ is the diagonal.

The objects under consideration are quintuples $(X, Y_1, Y_2, \phi_1, \phi_2)$, where X is a torsion-free $\overline{\mathcal{L}}$-module of finite rank and Y_i is a free $\overline{\mathcal{O}}_i$-module of finite rank. $\phi_i : X \to Y_i$ is an $\overline{\mathcal{L}}$-homomorphism for $i = 1, 2$. Morphisms are triples $(\alpha : X \to X',\ \beta_i : Y_i \to Y_i',\ i = 1, 2)$ such that $\phi_i \cdot \beta_i = \alpha \cdot \phi_i',\ i = 1, 2$. We shall, however, modify our functor $\mathbb{F}$ as follows. Recall that we are considering only (e_1, e_2)-filtered Cohen–Macaulay modules for $\mathcal{L}$. Given such a module M we have associated with it two exact sequences:

$$\begin{aligned} &0 \to N_1 \to M \to Q_2 \to 0, \\ &0 \to N_2 \to M \to L_1 \to 0, \end{aligned} \tag{22}$$

where Q_2 and N_2 are Cohen–Macaulay modules for $\mathcal{O}_2$ and N_1 is a Cohen–Macaulay module for $\mathcal{O}_1$; note that although L_1 is not a Cohen–Macaulay module in general, it is a submodule of the Cohen–Macaulay module L_2^{**}, and thus N_2 is Cohen–Macaulay; because of the Cohen–Macaulay filtration Q_2 is Cohen–Macaulay for $\mathcal{O}_2$, and so N_1 is Cohen–Macaulay. Using Claim 7.4 we conclude that $Q_2 := \pi_1^{-1} \cdot N_1 \supset L_1$. We can thus modify our above sequences as follows:

$$\begin{aligned} &0 \to \pi_1 \cdot Q_1 \to M \xrightarrow{\widetilde{\phi}_1} Q_2 \to 0, \\ &0 \to N_2 \to M \xrightarrow{\widetilde{\phi}_2} Q_2. \end{aligned} \tag{23}$$

We now consider the functor $\mathbb{G}$ from $\mathrm{CM}^{\mathrm{f}}(\mathcal{L})$ to the representations of the embedding problem (21):

$$M/(\pi_1 \cdot Q_1 \oplus \pi_2 \cdot Q_2) \xrightarrow{(\phi_1, \phi_2)} Q_1/\pi_1 \cdot Q_2 \oplus Q_2/\pi_2 \cdot Q_2. \tag{24}$$

As above in Proposition 7.3 and Claim 7.7 the functor $\mathbb{G}$ recovers isomorphism and decomposability. We thus have to determine the image under $\mathbb{G}$ of $\mathrm{CM}^{\mathrm{f}}(\mathcal{L})$.

PROPOSITION 7.10. • *The image under the functor* $\mathbb{G}$ *of a module* $M \in \mathrm{CM}^{\mathrm{f}}(\mathcal{L})$ *is of the form*

$$\mathcal{X} := (X, Y_1, Y_2, \psi_1 : X \to Y_1, \psi_2 : X \to Y_2),$$

where $X = \overline{M} := M/(\pi_1 \cdot Q_1 \oplus \pi_2 \cdot Q_2) \simeq \overline{\mathcal{L}}^{(\mu_M)}$, *the module* $Y_1 \simeq \overline{\mathcal{O}}_1^{(\nu_M)}$ *and* $Y_2 \simeq \overline{\mathcal{O}}_2^{(\mu_M)}$; *i.e.* $X \simeq Y_2$ *as* $\overline{\mathcal{O}}$-*modules, and moreover*, ψ_2 *is an* $\overline{\mathcal{O}}$-*linear map and* ψ_2 *is an isomorphism, which can then be taken as the identity.*

• *Conversely, every object* $\mathcal{X} = (\overline{\mathcal{O}}^{\mu}, \overline{\mathcal{O}}^{\nu}, \overline{\mathcal{O}}^{\mu}, \psi, \mathrm{id})$ *as above is the image of a Cohen–Macaulay module in* $\mathrm{CM}^{\mathrm{f}}(\mathcal{O})$.

• *$M \simeq M'$ if and only if there are $\alpha \in \mathrm{Gl}(\mu_M, \overline{\mathcal{O}})$ and $\beta \in \mathrm{Gl}(\nu_M, \overline{\mathcal{O}})$ with $\psi_M \cdot \beta = \alpha \cdot \psi_{M'}$.*

• *M is indecomposable if and only if $\overline{\mathcal{O}}^{\mu_M} \xrightarrow{\psi} \overline{\mathcal{O}}^{\nu_M}$ is indecomposable.*

P r o o f. Because of the exact sequences in (23) the map ψ_2 is an isomorphism and can thus be taken to be the identity. Hence the representations in the image of $\mathbb{G}$ have the form described above.

Since on the object $\mathcal{X}$ every modification on X can be compensated for — via ψ_2 — by a modification of Y_2, the statements about the morphism and decompositions are immediate.

It thus remains to determine the image of $\mathbb{G}$.

So let an object $X \xrightarrow{(\psi_1,\mathrm{id})} Y_1 \oplus Y_2$ be given with Y_i free of finite rank over $\overline{\mathcal{O}}$. This object then gives rise to the short exact sequence

$$0 \to X \xrightarrow{(\psi_1,\mathrm{id})} Y_1 \oplus Y_2 \to Y_1 \to 0. \tag{25}$$

We can lift Y_i to a projective $\mathbb{H}\cdot e_i$-module Q_i, and we obtain an epimorphism $\alpha := \alpha_1 \oplus \alpha_2 : Q_1 \oplus Q_2 \to Y_1 \oplus Y_2$. If we now let $M := \alpha^{-1}(\mathrm{Im}(\psi_1, \mathrm{id}))$, then M is a finitely generated torsion-free $\mathcal{L}$-module, and we get the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & X \arrow[u] \arrow[r, "\psi_1{,}\mathrm{id}"] & Y_1 \oplus Y_2 \arrow[u] \arrow[r] & Y_1 \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & M \arrow[u] \arrow[r, "\chi_1{,}\chi_2"] & Q_1 \oplus Q_2 \arrow[u] \arrow[r] & Y_1 \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & \pi_1 \cdot Q_1 \oplus \pi_2 \cdot Q_2 \arrow[u] \arrow[r] & \pi_1 \cdot Q_1 \oplus \pi_2 \cdot Q_2 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd} \tag{26}$$

Because of Nakayama's Lemma, the map $\chi_2 : M \to Q_2$ is surjective.

The next claim is the crucial observation.

CLAIM 7.1. *We have an exact sequence*

$$0 \to \pi_1 \cdot Q_1 \to M_1 \to Q_2 \to 0.$$

P r o o f. From the diagram (26) we get the short exact sequence

$$0 \to \pi_1 \cdot Q_1 \to M/\pi_2 \cdot Q_2 \to X \simeq Y_2 \to 0. \tag{27}$$

This uses heavily the equality $X = Y_2$. We now obtain from M the exact sequence from (22),

$$0 \to N_1 \to M \to Q_2 \to 0, \tag{28}$$

which induces the exact sequence

$$0 \to N_1 \to M/\pi_2 \cdot Q_2 \to Q_2/\pi_2 \cdot Q_2 = Y_2 \to 0.$$

Comparing this with the exact sequence from (27) and noting that the right-hand maps are the same, we conclude that $N_1 = \pi_1 \cdot Q_1$. Thus the sequence (28) is $\mathcal{O}$-split, and so $M \in \mathrm{CM}^{\mathrm{f}}(\mathcal{L})$. ∎

Let us summarize the result as follows:

THEOREM 7.12. *Let $\mathcal{L}(G)$ be the graph order of a truncated graph with $\Omega_v = \mathcal{O}$ and $\omega_v = \pi$. Let $(e_1, \ldots, e_m)$ be an ordering of the primitive idempotents of the underlying algebra such that genuine edges have neighboring idempotents.*

1. *Apart from the projective irreducible Cohen–Macaulay modules the category of the modules in* $\mathrm{CM}^{\mathrm{f}}(\mathcal{L}(G))$, *i.e. filtered with respect to the above sequence of idempotents, decomposes into the disjoint union of categories* $\mathrm{CM}^{\mathrm{f}}(\mathcal{L})_e$, *one for each genuine edge e.*
2. *The categories* $\mathrm{CM}^{\mathrm{f}}(\mathcal{L})_e$ *are all isomorphic. They are all equivalent to* $\mathrm{CM}^{\mathrm{f}}(\mathcal{L})$, *where $\mathcal{L}$ is the graph order for $\bullet$ — $\bullet$ with respect to the above data.*
3. *The objects in* $\mathrm{CM}^{\mathrm{f}}(\mathcal{L})$ *are given by the homomorphisms $\overline{\mathcal{O}}^{\mu} \to \overline{\mathcal{O}}^{\nu}$ with the bi-action of* $(\mathrm{Gl}(\mu, \overline{\mathcal{O}}), \mathrm{GL}(\nu, \overline{\mathcal{O}}))$, *where $\overline{\mathcal{O}} = \mathcal{O}/\langle\pi\rangle$.*
4. *If $\overline{\mathcal{O}}$ has infinite lattice type* (*note that this is a ring of Krull dimension one*), *then* $\mathrm{CM}^{\mathrm{f}}(\mathcal{L})$ *has infinite type.* (*This is a very coarse condition. Finer distinctions for infinite type may be derived from the arguments in the proof of Proposition* 6.5.)

Proof. For the construction of the ordered chain $(e_1, \ldots, e_m)$ of primitive idempotents we note that every indecomposable projective P of $\mathcal{L}(G)$ which does not correspond to a truncated edge projects onto exactly two different primitive idempotents ε and η (unless there are loops). Moreover, no other indecomposable projective projects onto either ε or η. Hence an ordering as claimed above does exist.

Now, only the last statement needs verification. Let L be an indecomposable $\overline{\mathcal{O}}$-lattice and let $\overline{\mathcal{O}}^{\mu}$ be a projective cover of L. Embed L into $\overline{\mathcal{O}}^{\nu}$ with torsion cokernel. Then the object $\overline{\mathcal{O}}^{\mu} \to L \to \overline{\mathcal{O}}^{\nu}$ is indecomposable, as is easily seen. ∎

COROLLARY 7.13. *In the case when $\overline{\mathcal{O}}$ is regular*, *the filtered Cohen–Macaulay modules are given as in Section* 7.

REMARK 7.14. The above results also hold if the rings $\mathcal{O}_v$ at the various vertices are not the same.

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Mathematisches Institut B, Universität Stuttgart
Pfaffenwaldring 57
D-70550 Stuttgart, Germany
E-mail: Roggenkamp@mathematik.uni-stuttgart.de

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