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On infinite composition of affine mappings

by

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Abstract. Let $\{F_i : i = 1, \ldots, N\}$ be affine mappings of $\mathbb{R}^n$. It is well known that if there exists $j \le 1$ such that for every $\sigma_1, \ldots, \sigma_j \in \{1, \ldots, N\}$ the composition

(1) $$F_{\sigma_1} \circ \ldots \circ F_{\sigma_j}$$

is a contraction, then for any infinite sequence $\sigma_1, \sigma_2, \ldots \in \{1, \ldots, N\}$ and any $z \in \mathbb{R}^n$, the sequence

(2) $$F_{\sigma_1} \circ \ldots \circ F_{\sigma_n}(z)$$

is convergent and the limit is independent of z. We prove the following converse result: If (2) is convergent for any $z \in \mathbb{R}^n$ and any $\sigma = \{\sigma_1, \sigma_2, \ldots\}$ belonging to some subshift Σ of N symbols (and the limit is independent of z), then there exists $j \ge 1$ such that for every $\sigma = \{\sigma_1, \sigma_2, \ldots\} \in \Sigma$ the composition (1) is a contraction. This result can be considered as a generalization of the main theorem of Daubechies and Lagarias [1], p. 239. The proof involves some easy but non-trivial combinatorial considerations. The most important tool is a weighted version of the König Lemma for infinite trees in graph theory.

1. Throughout the paper we use the terminology of Lind and Marcus [2]. Let $\mathcal{J}$ be the set of infinite sequences of the N symbols $1, \ldots, N$ and s be the *shift operator*, that is,

$$s(\sigma_1 \ldots \sigma_n \ldots) = \sigma_2 \sigma_3 \ldots \sigma_{n+1} \ldots, \qquad \sigma_i \in \{1, \ldots, N\},\ i \in \mathbb{N}.$$

The dynamical system $[\mathcal{J}, s]$ thus obtained with the usual metric

$$d_c[\omega, \sigma] = \sum_{i=1}^{\infty} \frac{|\omega_i - \sigma_i|}{N^i}, \qquad \sigma, \omega \in \mathcal{J},$$

is called the *full N-shift*. Every full N-shift is compact. A *block* or *word* is a finite sequence $\sigma_i \sigma_{i+1} \ldots \sigma_j$. The set of *symbols* is also called the *alphabet*.

We consider in this paper a closed shift-invariant subspace $\mathcal{K}$ of the full N-shift $\mathcal{J}$ called a *sub-N-shift* or briefly *subshift*.

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If $B(\mathcal{K})$ is the set of words consisting of symbols from $\sigma \in \mathcal{K}$ then $w \in B(\mathcal{K})$ implies that every part of w, called a subword, also belongs to $B(\mathcal{K})$; moreover, for every $w \in B(\mathcal{K})$ there is $v \in B(\mathcal{K})$ such that the concatenation wv also belongs to $B(\mathcal{K})$. A word of the form $\{\sigma_1 \ldots \sigma_n\}$, i.e. a word obtained by truncation of $\sigma \in \mathcal{K}$, is called a *prefix* of $\sigma \in \mathcal{K}$.

We suppose that a positive *submultiplicative functional* Φ is defined on $B(\mathcal{K})$. That is, we assume that there is a function $\Phi : B(\mathcal{K}) \to \mathbb{R}$ with

$$\Phi(\sigma_1 \ldots \sigma_n) \le \Phi(\sigma_1 \ldots \sigma_j)\Phi(\sigma_{j+1}\sigma_{j+2} \ldots \sigma_n), \qquad 1 \le j < n. \tag{1.1}$$

A subshift $\mathcal{K}$ equipped with a positive Φ satisfying (1.1) is called a *subshift with weight* Φ or briefly a *weighted subshift*. For every weighted subshift $\mathcal{K}$ we define an average-like characteristic number $\Phi^* = \Phi^*(\mathcal{K}, \Phi)$ as follows:

$$\Phi^* = \lim_{k\to\infty} \Phi_k^{*1/k} \quad \text{where} \quad \Phi_k^* = \max\{\Phi(\sigma_1 \ldots \sigma_k) : \sigma \in \mathcal{K}\} \tag{1.2}$$

2. We begin with an observation on the distribution of the values of Φ on $B(\mathcal{K})$. Let

$$\mathcal{T} = \{\sigma_1 \ldots \sigma_n : \Phi(\sigma_1 \ldots \sigma_k) \ge \Phi^{*k} \text{ for every } k \le n\}, \tag{2.1}$$

i.e. a word $\sigma_1 \ldots \sigma_n$ in $B(\mathcal{K})$ belongs to $\mathcal{T}$ for *any* n if $\Phi(\sigma_1 \ldots \sigma_n)^{1/n}$ is not less than Φ^* and this is also valid for every prefix of $\sigma_1 \ldots \sigma_n$.

PROPOSITION 1. *$\mathcal{T}$ is an infinite subset of $B(\mathcal{K})$.*

Proof. $\mathcal{T}$ is not empty since (1.1) implies that

$$\max\{\Phi(\sigma_i) : \sigma_i \in \{1, \ldots, N\}\} \ge \Phi^*.$$

Suppose that $\mathcal{T}$ is finite. Define the *boundary* $\mathcal{C}$ of $\mathcal{T}$ as follows:

- for $k > 1$: $\sigma_1 \ldots \sigma_k \in \mathcal{C}$ if $\sigma_1 \ldots \sigma_k \notin \mathcal{T}$ and $\sigma_1 \ldots \sigma_{k-1} \in \mathcal{T}$,
- for $k = 1$: $\sigma_1 \in \mathcal{C}$ if $\sigma_1 \notin \mathcal{T}$.

Since $\mathcal{T}$ is finite, so is $\mathcal{C}$. Moreover

$$\Phi(\sigma_1 \ldots \sigma_k)^{1/k} < \Phi^* \quad \text{for } \sigma_1 \ldots \sigma_k \in \mathcal{C},$$

hence there exists an $\alpha > 0$ such that

$$\max\{\Phi(\sigma_1 \ldots \sigma_k)^{1/k} : \sigma_1 \ldots \sigma_k \in \mathcal{C}\} = \Phi^* - \alpha \tag{2.2}$$

since $\mathcal{C}$ is finite.

Let r be the length of the longest prefix in $\mathcal{C}$. Then every $\sigma \in \mathcal{K}$ is the concatenation of finite strings belonging to $\mathcal{C}$ with length at most r. Hence for every prefix $\sigma_1 \ldots \sigma_k$ of $\sigma \in \mathcal{K}$ with $k > r$ we have

$$\Phi(\sigma_1 \ldots \sigma_k) \le M^r(\Phi^* - \alpha)^w \tag{2.3}$$

where w is an integer in $(k - r, k]$ and

$$M = \max\{1, \Phi(\sigma_i) : \sigma_i \in \{1, \ldots, N\}\}. \tag{2.4}$$

It follows from (2.3), by taking the maximum over all prefixes in $\mathcal{K}$ with length k, that we also have

$$\Phi_k^* \le M^r(\Phi^* - \alpha)^w$$

and hence

$$\Phi^* = \lim_{k\to\infty} \Phi_k^{*1/k} \le \lim_{k\to\infty} M^{r/k} \lim_{k\to\infty} (\Phi^* - \alpha)^{w/k} = \Phi^* - \alpha.$$

This contradiction shows that $\mathcal{T}$ is infinite.

We now reformulate Proposition 1 in terms of graph theory. By combining the graph-theoretical formulation of Proposition 1 with the celebrated König Lemma for infinite trees, a weighted version of the König Lemma will be obtained.

Consider the infinite graph $\mathcal{G}$ with vertex set $B(\mathcal{K})$ and edges

$$[\sigma_1 \dots \sigma_k, \sigma_1 \dots \sigma_{k+1}] \qquad \text{for } k = 1, 2, \dots,$$

i.e. $[\sigma_1 \dots \sigma_k, \omega_1 \dots \omega_m]$ is an edge in $\mathcal{G}$ if $m = k + 1$ and $\omega_i = \sigma_i$ for $i \le k$.

If we add the symbol $\emptyset$ as a new vertex and $[\emptyset, \sigma_i]$ for $i = 1, \dots, N$ as new edges, then $\mathcal{G}$ is a rooted tree with root $\emptyset$. In fact, $\mathcal{G}$ is a weighted tree with weight $\Phi(\sigma_1 \dots \sigma_n)$ for the vertex $\sigma_1 \dots \sigma_n$.

It is easy to check that the following properties of $\mathcal{G}$ hold. The indegree of each vertex is 1 and the outdegree is at least 1 and at most N. There is a path between two vertices, say P and Q, iff the word P is a prefix of Q. Each $\sigma \in \mathcal{K}$ corresponds to an infinite path starting at $\emptyset$ and conversely, since $\mathcal{K}$ is closed.

Now the *Weighted König Lemma* is the following.

THEOREM 1. *There exists a $\sigma \in \mathcal{K}$ with*

$$\Phi(\sigma_1 \dots \sigma_k) \ge \Phi^{*k}, \qquad k = 1, 2, \dots$$

In other words, there is an infinite path in $\mathcal{G}$ with weights not less than Φ^{*k}.

Proof. It is easy to verify that the subgraph of $\mathcal{G}$ corresponding to $\mathcal{T}$ is also a rooted tree with root $\emptyset$. It follows from Proposition 1 that $\mathcal{T}$ is an infinite tree.

The König lemma says that in an infinite rooted tree with all vertices of finite degree, there is an infinite path starting from the root. Apply the König lemma to the subgraph $\mathcal{T}$.

REMARK. In the proof of Proposition 1 we did not use the fact that $\mathcal{K}$ is a subshift. We only needed that $\mathcal{K}$ is a shift-invariant subset of the full N-shift. However, for the Weighted König Lemma (Theorem 1) it is necessary that $\mathcal{K}$ be also a closed subset.

Theorem 1 implies

Corollary. *There is $\sigma \in \mathcal{K}$ such that*

$$\Phi^* \le \Phi(\sigma_1 \dots \sigma_k)^{1/k} \le \Phi_k^{*1/k}, \tag{2.5}$$

hence

$$\lim_{k\to\infty} \Phi(\sigma_1 \dots \sigma_k)^{1/k} = \Phi^* \quad \textit{and} \quad \Phi^* = \inf\{\Phi_k^{*1/k} : k = 1, 2, \dots\}.$$

Theorem 2. *The sequence*

$$\{\Phi(\sigma_1 \dots \sigma_k) : k = 1, 2, \dots\} \tag{2.6}$$

tends to zero for every $\sigma \in \mathcal{K}$ if and only if $\Phi^* < 1$. In particular, if it does then there is $0 < q < 1$ and an integer j such that for $n > j$,

$$\Phi(\sigma_1 \dots \sigma_n) \le M^j q^w, \qquad k = 1, 2, \dots, \tag{2.7}$$

where $M = \max\{1, \Phi(\sigma_i) : \sigma_i \in \{1, \dots, N\}\}$ and w is an integer in $(n-j, n]$.

Proof. It follows from Theorem 1 that if $\Phi^* \ge 1$ then there is $\sigma \in \mathcal{K}$ such that (2.6) does not tend to zero. Now let $\Phi^* < 1$. Then there is j such that $\Phi_j^{*1/j} < 1$ and hence there is $q < 1$ such that $\Phi(\sigma_1 \dots \sigma_j) \le q^j$ for every $\sigma \in \mathcal{K}$. It follows that

$$\Phi(\sigma_1 \dots \sigma_n) \le M^j q^w, \qquad k = 1, 2, \dots,$$

where M and w are as in the assertion.

Remark 1. There is an algorithmic view on the Weighted König Lemma by means of a walk in the graph $\mathcal{G}$ as follows.

Starting from $\emptyset$, walk along a path $\sigma \in \mathcal{K}$ until $\sigma_1 \dots \sigma_k \in \mathcal{C}$. Then the walk is continued to $\sigma_{k+1} \in \mathcal{G}$ (at the first level of $\mathcal{G}$) till $\sigma_{k+1}\sigma_{k+2} \dots \sigma_{k+m} \in \mathcal{C}$ and then we continue to $\sigma_{k+m+1} \in \mathcal{G}$ (at the first level of $\mathcal{G}$ again) etc.

We have the following cases considering the *outcome* of this algorithm:

1. For each vertex $\sigma_1 \dots \sigma_k$ of the path σ,

$$\Phi(\sigma_1 \dots \sigma_k) \ge \Phi^{*k}, \qquad k = 1, 2, \dots$$

2. There is a positive integer M such that

$$\Phi(\sigma_{M+1}\sigma_{M+2} \dots \sigma_{M+k}) \ge \Phi^{*k}, \qquad k = 1, 2, \dots$$

3. There exists k_1 such that $\sigma_1 \dots \sigma_{k_1} \in \mathcal{C}$ and a sequence $\{k_i : i = 1, 2, \dots\}$ such that $\sigma_{k_i}\sigma_{k_i+1} \dots \sigma_{k_{i+1}} \in \mathcal{C}$.

What we have shown in Theorem 1 is that there is $\sigma \in \mathcal{K}$ with property 1.

Remark 2. We cannot express the shift-invariance of $\mathcal{K}$ on the infinite tree model $\mathcal{G}$ in the language of graph theory. In a heuristic way, the shift-invariance means that if we delete a finite number of edges on an infinite path of $\mathcal{G}$ starting at $\emptyset$, the remaining infinite path appears also as an infinite path of $\mathcal{G}$ starting at $\emptyset$. This looks like a "self-similarity" of $\mathcal{G}$.

3. Let $\{F_i : i = 1, \ldots, N\}$ be affine mappings of $\mathbb{R}^n$ and $\mathcal{K}$ be a subshift of the full N-shift $\mathcal{J}$ and consider the sequences

$$\{F_{\sigma_1} \circ \ldots \circ F_{\sigma_n}(z) : \sigma \in \mathcal{K},\ z \in \mathbb{R}^n\}. \tag{$*$}$$

Our main objective in this section is to give a necessary and sufficient condition for every sequence $(*)$ to be convergent to a limit $x = x(\sigma)$ independent of z.

If F_i is an affine mapping of $\mathbb{R}^n$, i.e. $F_i(z) = A_i z + b$ where A_i is an $n \times n$ matrix and b is a column n-vector, then

$$F_{\sigma_1} \circ \ldots \circ F_{\sigma_n}(z) = A_{\sigma_1} \ldots A_{\sigma_n} z + (b_{\sigma_1} + A_{\sigma_1} b_{\sigma_2} + \ldots + A_{\sigma_1} \ldots A_{\sigma_{n-1}} b_{\sigma_n}).$$

It follows that if $(*)$ tends to the same $x(\sigma)$ for every $z \in \mathbb{R}^n$ then

$$F_{\sigma_1} \circ \ldots \circ F_{\sigma_n}(z') - F_{\sigma_1} \circ \ldots \circ F_{\sigma_n}(z'') = A_{\sigma_1} \ldots A_{\sigma_n}(z' - z'')$$

tends to zero for all $z', z'' \in \mathbb{R}^n$ and hence

$$\|A_{\sigma_1} \ldots A_{\sigma_n}\| \to 0. \tag{3.1}$$

Throughout this section

$$\Phi(\sigma_1 \ldots \sigma_n) = \|A_{\sigma_1} \ldots A_{\sigma_n}\| \tag{3.2}$$

and Φ_k^* resp. Φ^*, defined by (1.2), are computed via (3.2).

THEOREM 3. *Let $\mathcal{K}$ be a subshift of the full N-shift $\mathcal{J}$. Then the sequence $(*)$ is convergent for every $\sigma \in \mathcal{K}$ and $z \in \mathbb{R}^n$ to a limit $x(\sigma)$ independent of z if and only if $\Phi^* < 1$.*

Proof. It follows from (3.2) and Theorem 2 that $\Phi^* < 1$ if and only if (3.1) is satisfied for every $\sigma \in \mathcal{K}$, and hence $\Phi^* < 1$ is necessary.

Conversely, if $\Phi^* < 1$ then it follows from (3.2) and (2.7) that

$$\sum_{n=1}^{\infty} \|A_{\sigma_1} \ldots A_{\sigma_n}\| < \infty \tag{3.3}$$

and hence $A_{\sigma_1} \ldots A_{\sigma_n}(z) \to 0$ for every $z \in \mathbb{R}^n$; moreover,

$$\{b_{\sigma_1} + A_{\sigma_1} b_{\sigma_2} + \ldots + A_{\sigma_1} \ldots A_{\sigma_{n-1}} b_{\sigma_n} : n = 1, 2, \ldots\}$$

is also convergent.

COROLLARY. *If*

$$\{F_{\sigma_1} \circ \ldots \circ F_{\sigma_n}(z') - F_{\sigma_1} \circ \ldots \circ F_{\sigma_n}(z'') : n = 1, 2, \ldots\}$$

tends to 0 *for every $z', z'' \in \mathbb{R}^n$ and every $\sigma \in \mathcal{K}$, then the sequence $\{F_{\sigma_1} \circ \ldots \circ F_{\sigma_n}(z) : n = 1, 2, \ldots\}$ is convergent for every $z', z'' \in \mathbb{R}^n$ and every $\sigma \in \mathcal{K}$.*

Proof. Since

$$F_{\sigma_1} \circ \ldots \circ F_{\sigma_n}(z') - F_{\sigma_1} \circ \ldots \circ F_{\sigma_n}(z'') = A_{\sigma_1} \ldots A_{\sigma_n}(z' - z''),$$

it follows from (3.2) and Theorem 2 that there are $q < 1$ and integers w and j such that

$$\|A_{\sigma_1} \dots A_{\sigma_n}\| \leq M^j q^w$$

where $M = \max\{1, \|A_{\sigma_i}\| : \sigma_i \in \{1, \dots, N\}\}$. Hence (3.3) holds.

We call the set $\{F_i : i = 1, \dots, N\}$ of mappings a *joint contraction* on $\mathcal{K}$ if there is j such that

$$F_{\sigma_1} \circ \dots \circ F_{\sigma_j} \tag{3.4}$$

is a contraction for every $\sigma \in \mathcal{K}$.

THEOREM 4. *If $\Phi^* < 1$ then $\{F_i : i = 1, \dots, N\}$ is a joint contraction on $\mathcal{K}$.*

Proof. If $\Phi^* < 1$ then there is j such that $\Phi_j^{*1/j} < 1$ and hence there is $q < 1$ such that $\|A_{\sigma_1} \dots A_{\sigma_j}\| \leq q^j$ for every $\sigma \in \mathcal{K}$.

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