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High-dimensional knots corresponding to the fractional Fibonacci groups

by

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Abstract. We prove that the natural HNN-extensions of the fractional Fibonacci groups are the fundamental groups of high-dimensional knot complements. We also give some characterization and interpretation of these knots. In particular we show that some of them are 2-knots.

1. Introduction. The *fractional Fibonacci groups* $F^{k/l}(2,m)$ were introduced in [6] for integers k, l, m such that $m \geq 3$ and $(k,l)=1$:

$$(1) \qquad F^{k/l}(2,m) = \langle a_1,\ldots,a_m \mid a_i^l\, a_{i+1}^k = a_{i+2}^l,\ i=1,\ldots,m\rangle$$

with all subscripts reduced modulo m. In the case $k=l=1$ we get the well-known Fibonacci groups $F(2,m)$ introduced by J. Conway in 1965. The topological and geometric properties of three-dimensional closed orientable manifolds uniformized by the groups $F(2,2n)$ and $F^{k/l}(2,2n)$ were studied in [2] and [6], respectively. Almost all of these groups (and manifolds) are hyperbolic. Concerning the case of m odd, H. Helling pointed out to us that the groups $F(2,m)$ have torsion, and it was shown in [7] that they cannot be fundamental groups of hyperbolic 3-manifolds or hyperbolic 3-orbifolds.

The investigation of HNN-extensions of the Fibonacci groups as high-dimensional knot groups was started by J. Hillman in [3], where he considered the natural extension of the group $F(2,6)$ that is the fundamental group of the Euclidean Hantzsche–Wendt manifold. This study was continued in [11], where it was proven that the same HNN-extensions of the

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Fibonacci groups $F(2,m)$ also arise as high-dimensional knot groups. In the present paper we demonstrate that natural HNN-extensions of the fractional Fibonacci groups $F^{k/l}(2,m)$ are high-dimensional knot groups if and only if $k=1$.

We recall that a group G is said to be an *n-dimensional knot group*, for $n \geq 1$, if $G = \pi_1(S^{n+2} \setminus K(S^n))$ for some n-dimensional knot $K : S^n \to S^{n+2}$. Kervaire [10, Section 11D] obtained the following algebraic characterization of such groups: *A group G is a 3-knot group* (*and so*, *an n-knot group for any $n \geq 3$*) *if and only if it is finitely presentable*, $H_1(G,\mathbb{Z}) = \mathbb{Z}$, $H_2(G,\mathbb{Z}) = 0$ *and G has weight* 1 (*that is*, *G is a normal closure of some single element*). In dimensions $n = 1, 2$ the above conditions are necessary but no longer sufficient. Our study of the group extensions of $F^{k/l}(2,m)$ will be based on this criterion and on Zeeman's twist spun construction of knots [12].

2. The natural HNN-extension. The group $F^{k/l}(2,m)$ with the presentation (1) is an example of a cyclically presented group in sense of [5]. We recall that a group G is *cyclically presented* if for some n and w it has a presentation

$$G = \langle x_1, \ldots, x_m \mid w, \eta(w), \ldots \eta^{m-1}(w) \rangle, \tag{2}$$

where $\eta : \mathbb{F}_m \to \mathbb{F}_m$ is an automorphism of the free group $\mathbb{F}_m = \langle x_1, \ldots, x_m \rangle$ of rank m given by $\eta(x_i) = x_{i+1}$, $i = 1, \ldots, m$, and $w \in \mathbb{F}_m$ is a cyclically reduced word. The fractional Fibonacci groups with the presentation (1) arise in this construction for $w = x_1^l x_2^k x_3^{-l}$.

Obviously, η induces an automorphism $\Phi : G \to G$ given by $\Phi(x_i) = x_{i+1}$, $i = 1, \ldots, m$. Let us define the *natural HNN-extension* $\mathcal{G}$ of a cyclically presented group G:

$$\mathcal{G} = \{G, t \mid tgt^{-1} = \Phi(g),\ g \in G\}. \tag{3}$$

Thus for the group $F^{k/l}(2,m)$ with the presentation (1) we consider an automorphism $\Phi : F^{k/l}(2,m) \to F^{k/l}(2,m)$ given by $\Phi(a_i) = a_{i+1}$, $i = 1, \ldots, m$, and the natural HNN-extension

$$\mathcal{F}_m^{k/l} = \{F^{k/l}(2,m), t \mid tft^{-1} = \Phi(f),\ f \in F^{k/l}(2,m)\} \tag{4}$$

which is defined for the same k, l, m as the group $F^{k/l}(2,m)$ was. We have

THEOREM. *The group $\mathcal{F}_m^{k/l}$ is a 3-knot group if and only if $k = 1$.*

Proof. We check the conditions of Kervaire's criterion.

(1) First we prove that Φ is a *meridional* automorphism if and only if $k = 1$; in other words, that the normal closure $A^{k/l}(2,m)$ in $F^{k/l}(2,m)$ of $\{f^{-1}\Phi(f) \mid f \in F^{k/l}(2,m)\}$ is $F^{k/l}(2,m)$ only for $k = 1$ (see [3, p. 123]). In

fact, there is an obvious epimorphism

$$h : F^{k/l}(2,m) \to \mathbb{Z}_k = \langle \gamma \mid \gamma^k = 1\rangle$$

given by $h(a_i) = \gamma$ for all i. This epimorphism induces an epimorphism of the abelianization:

$$h^{\mathrm{ab}} : F^{k/l}(2,m)^{\mathrm{ab}} \to \mathbb{Z}_k.$$

Thus $H_1(\mathcal{F}_m^{k/l}) = \mathbb{Z}$ if and only if $k = 1$. So, from now on we assume that $k = 1$ in our considerations.

(2) We show that $\mathcal{F}_m^{1/l}$ is a normal closure of the element $b = t^{-1}a_1$. In fact,

$$t^{-1}a_2 = t^{-1}(ta_1t^{-1}) = a_1t^{-1} = t(t^{-1}a_1)t^{-1} = tbt^{-1},$$

and for $i = 1,\ldots,m$ we get

$$\begin{aligned} t^{-1}a_i &= t^{-1}(t^{i-1}a_1t^{-(i-1)}) = t^{i-2}a_1t^{-(i-1)} \\ &= t^{i-1}(t^{-1}a_1)t^{-(i-1)} = t^{i-1}bt^{-(i-1)}. \end{aligned}$$

Let $\mathcal{B}$ be the normal closure of b in $\mathcal{F}_m^{1/l}$. Consider the canonical projection

$$\varrho : \mathcal{F}_m^{1/l} \to \mathcal{F}_m^{1/l}/\mathcal{B},$$

and write $\varrho(a) = \overline{a}$ for $a \in \mathcal{F}_m^{1/l}$. Then $\overline{a}_1 = \ldots = \overline{a}_m = \overline{t}$. For any i we get $\overline{a}_i^l\overline{a}_{i+1} = \overline{a}_{i+2}^l$, hence $\overline{t} = e$ is the neutral element. Therefore $\overline{a}_1 = \ldots = \overline{a}_m = \overline{t} = e$ and $\mathcal{F}_m^{1/l} = \mathcal{B}$. Thus $\mathcal{F}_m^{1/l}$ has weight 1.

(3) From the short exact sequence of groups

$$1 \to F^{k/l}(2,m) \to \mathcal{F}_m^{k/l} \to \mathbb{Z} \to 1$$

we have the Hochschild–Serre spectral sequence

$$E^2_{p,q} = H_p(\mathbb{Z}, H_q(F^{k/l}(2,m)))$$

[1, p. 171]. So, it is enough to prove that

$$\begin{aligned} &E^2_{2,0} = H_2(\mathbb{Z}, H_0(F^{k/l}(2,m))) = 0, \\ &E^2_{1,1} = H_1(\mathbb{Z}, H_1(F^{k/l}(2,m))) = H_1(F^{k/l}(2,m))_{\mathbb{Z}} = 0, \\ &E^2_{0,2} = H_0(\mathbb{Z}, H_2(F(2,n))) = 0. \end{aligned}$$

The first equality is obvious. For the proof of the next one we use Hopf's formula to get a 5-term exact sequence [1, p. 47]

$$H_2(\mathcal{F}_m^{k/l}) \to H_2(\mathbb{Z}) \to H_1(F^{k/l}(2,m))_{\mathbb{Z}} \to H_1(\mathcal{F}_m^{k/l}) \to H_1(\mathbb{Z}) \to 0,$$

where the $\mathbb{Z}$-action on $H_1(F^{k/l}(2,m))$ is induced by the conjugation action of $\mathcal{F}_m^{k/l}$ on $F^{k/l}(2,m)$. We have $H_2(\mathbb{Z}) = 0$, $H_1(\mathbb{Z}) = \mathbb{Z}$. Suppose $k = 1$; then by (1), $H_1(\mathcal{F}_m^{1/l}) = \mathbb{Z}$. Thus

$$0 \to H_1(F^{1/l}(2,m))_{\mathbb{Z}} \to \mathbb{Z} \to \mathbb{Z} \to 0,$$

and so $H_1(F^{1/l}(2,m))_{\mathbb{Z}} = 0$.

The abelianization $F^{k/l}(2,m)^{\mathrm{ab}}$ has the following property.

LEMMA. *For any k, l, m the group $F^{k/l}(2,m)^{\mathrm{ab}}$ is finite of order*

$$|F^{k/l}(2,m)^{\mathrm{ab}}| = d_m^{k/l} = \begin{cases} c_m^{k/l} + l^2 c_m^{k/l}, & m \text{ odd}, \\ c_m^{k/l} + l^2 c_{m-2}^{k/l} - 2l^m, & m \text{ even}, \end{cases} \tag{5}$$

where

$$c_m^{k/l} = k c_{m-1}^{k/l} + l^2 c_{m-2}^{k/l}, \tag{6}$$

with $c_0^{k/l} = 1$ and $c_1^{k/l} = k$.

Proof. The lemma is obvious by direct computation of the determinant of the exponential sum matrix [8]. ∎

In particular, from (5) and (6) we get

$$d_5^{k/l} = k^5 + 5k^3l^2 + 5kl^4, \quad d_6^{k/l} = k^6 + 6k^4l^2 + 9k^2l^4.$$

Returning to the proof of the theorem, we note that $H_2(F^{k/l}(2,m)) = 0$. In fact, from Hopf's formula [1, p. 46] the number of generators of the group $H_2(F^{k/l}(2,m))$ is $r - g + w$. Here g is the number of generators and r the number of relations of the group $F^{k/l}(2,m)$, and $w = \operatorname{rank}(H_1(F^{k/l}(2,m)))$. In our case $g = r = m$, and by the lemma above, $w = 0$. So, $H_2(F^{k/l}(2,m)) = 0$ and $E_{0,2}^2 = 0$. Summing up we have shown that $H_2(\mathcal{F}_m^{1/l}) = 0$. ∎

3. The fibred 2-knots. As we remarked in the introduction the Kervaire conditions of the high-dimensional knot groups are also necessary when $n = 1$ or 2, but are then no longer sufficient. However, it is proven in [4, p. 34] that if a group G is a torsion free 3-knot group such that G' is the fundamental group of a closed orientable 3-manifold M whose factors are Haken, hyperbolic or Seifert fibred, then G is the group of a fibred 2-knot with closed fibre M.

We recall [6] that the fractional Fibonacci group $F^{1/l}(2,2n)$, $l \ge 1$, $n \ge 3$, is the fundamental group of a closed orientable 3-manifold $\mathcal{M}_n^{1/l}$, which can be obtained as an n-fold cyclic covering of the two-bridge knot $\left(2l + \frac{1}{2l}\right)$. The manifolds $\mathcal{M}_n^{1/l}$ are hyperbolic with one exceptional case $\mathcal{M}_3^1$, which is the Euclidean Hantzsche–Wendt manifold that is Seifert fibred. Thus we have the following corollary of the above theorem.

PROPOSITION 1. *For $l \ge 1$ and $n \ge 3$ the group $\mathcal{F}_{2n}^{1/l}$ is a fibred 2-knot group with closed fibre $\mathcal{M}_n^{1/l}$.*

Recall that by the definition (4) the automorphism Φ corresponding to the group $\mathcal{F}_{2n}^{1/l}$ is of order $2n$.

We now show that there are other HNN-extensions of the group $F^{1/l}(2,2n)$. Indeed, the n-fold cyclic covering $\mathcal{M}_n^{1/l}$ of the 2-bridge knot $\left(2l+\frac{1}{2l}\right)$ is induced by the automorphism $\Psi : F^{1/l}(2,2n) \to F^{1/l}(2,2n)$ given by $\Psi(a_i) = a_{i+2}$, $i = 1,\ldots,2n$.

Define the HNN-extension

$$\mathcal{G}_n^{l,p} = \{F^{1/l}(2,2n), t \mid tft^{-1} = \Psi^p(f),\ f \in F^{1/l}(2,2n)\}. \tag{7}$$

By a general construction of a twist spun knot ([9], [10, Section 3L]) we see that for $0 < p < n$, $(p,n) = 1$, the group $\mathcal{G}_n^{l,p}$ is a fundamental group of a p-twist spun of a $\left(2l+\frac{1}{2l}\right)$-knot, which can be described as a fibred knot with fibre $\operatorname{punc}(\mathcal{M}_n^{1/l})$. We denote this 2-knot by $\mathcal{K}_n^{l,p}$. Here $\operatorname{punc}(\mathcal{M}_n^{1/l})$ denotes a manifold $\mathcal{M}_n^{1/l}$ minus an open ball around the branch set. As a corollary of these considerations, we get

PROPOSITION 2. *For $n \geq 3$, $l \geq 2$, $0 < p < n$, $(p,n) = 1$ the group $\mathcal{G}_n^{l,p}$ is a fibred 2-knot group with fibre* $\operatorname{punc}(\mathcal{M}_n^{1/l})$.

For integers p, q satisfying $0 < p,q < n$, $(p,n) = (q,n) = 1$ we can distinguish knots $\mathcal{K}_n^{l,p}$ and $\mathcal{K}_n^{l,q}$ in virtue of [9]. The groups $\mathcal{G}_n^{l,p}$ and $\mathcal{G}_n^{l,q}$ are isomorphic if and only if Ψ^p is conjugate to $\Psi^{\pm q}$ in $\operatorname{Out}(F^{1/l}(2,2n))$. The knot exteriors are homeomorphic only if Ψ^p is conjugate to $\Psi^{\pm q}$ in $\operatorname{Aut}(F^{1/l}(2,2n))$.

We remark that the extension (7) can also be regarded as a natural HNN-extension of a cyclically presented group. Indeed, if we suppose $y_i = a_{2i}$ for $i = 1,\ldots,n$, then $a_{2i+1} = y_i^{-l} y_{i+2}^l$. Therefore we get the following cyclic presentation for the fractional Fibonacci group:

$$F^{1/l}(2,2n) = \langle y_1,\ldots,y_n \mid (y_i^{-l} y_{i+1}^l)^l y_{i+1} = (y_{i+1}^{-l} y_{i+2}^l)^l,\ i = 1,\ldots,n\rangle, \tag{8}$$

and the automorphism $\Psi(y_i) = y_{i+1}$. Thus the group $\mathcal{G}_n^{l,1}$ is the natural HNN-extension of the cyclically presented group (8).

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