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Embedding lattices in the Kleene degrees

by

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Abstract. Under ZFC+CH, we prove that some lattices whose cardinalities do not exceed $\aleph_1$ can be embedded in some local structures of Kleene degrees.

0. We denote by ${}^2E$ the existential integer quantifier and by χ_A the characteristic function of A, i.e. $x \in A \Leftrightarrow \chi_A(x) = 1$, and $x \notin A \Leftrightarrow \chi_A(x) = 0$. Kleene reducibility is defined as follows: for $A, B \subseteq {}^\omega\omega$, $A \le_{\mathcal{K}} B$ iff there is $a \in {}^\omega\omega$ such that χ_A is recursive in a, χ_B, and ${}^2E$.

We introduce the following notations. $\mathcal{K}$ denotes the upper semilattice of all Kleene degrees with the order induced by $\le_{\mathcal{K}}$. For $X, Y \subseteq {}^\omega\omega$, we set $X \oplus Y = \{\langle 0\rangle * x \mid x \in X\} \cup \{\langle 1\rangle * x \mid x \in Y\}$. Then $\deg(X \oplus Y)$ is the supremum of $\deg(X)$ and $\deg(Y)$. The *superjump* of X is the set $X^{\mathrm{SJ}} = \{\langle e\rangle * x \in {}^\omega\omega \mid \{e\}((x)_0, (x)_1, \chi_X, {}^2E)\downarrow\}$. Here, $\langle e\rangle * x$ is the real such that $(\langle e\rangle * x)(0) = e$ and $(\langle e\rangle * x)(n+1) = x(n)$ for $n \in \omega$. More generally, for $m \in \omega$, $\langle e_0, \ldots, e_m\rangle * x$ is the real such that $(\langle e_0, \ldots, e_m\rangle * x)(n) = e_n$ for $n \le m$ and $(\langle e_0, \ldots, e_m\rangle * x)(n + m + 1) = x(n)$ for $n \in \omega$. Further, $(x)_0 = \lambda n.x(2n)$ and $(x)_1 = \lambda n.x(2n + 1)$. We identify $\langle (x)_0, (x)_1\rangle$ with x. An X-admissible set is closed under $\lambda x.\omega_1^{X;x}$ iff it is X^{SJ}-admissible.

The following conditions (1) and (2) are equivalent to $A \le_{\mathcal{K}} B$ ([8]).

(1) There is $y \in {}^\omega\omega$ such that A is uniformly Δ_1-definable over all $(B; y)$-admissible sets; i.e. there are $\Sigma_1(\dot{B})$ formulas φ_0 and φ_1 such that for any $(B; y)$-admissible set M and for all $x \in {}^\omega\omega \cap M$,

$$x \in A \Leftrightarrow M \models \varphi_0(x, y) \Leftrightarrow M \models \neg\varphi_1(x, y).$$

(2) There are $y \in {}^\omega\omega$ and $\Sigma_1(\dot{B})$ formulas φ_0 and φ_1 such that for all $x \in {}^\omega\omega$,

$$x \in A \Leftrightarrow L_{\omega_1^{B;x,y}}[B; x, y] \models \varphi_0(x, y) \Leftrightarrow L_{\omega_1^{B;x,y}}[B; x, y] \models \neg\varphi_1(x, y).$$

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Here, we are thinking of the language of set theory with an additional unary predicate symbol $\dot{B}$. A set M is said to be $(B;y)$-*admissible* iff the structure $\langle M, \in, B\cap M\rangle$ is admissible and $y \in M$. Next, $L_\alpha[B;y]$ denotes the αth stage of the hierarchy constructible from $\{y\}$ relative to a unary predicate B, and $\omega_1^{B;y}$ denotes the least $(B;y)$-admissible ordinal.

For $K, K' \subseteq {}^\omega\omega$, we set $\mathcal{K}[K,K'] = \{\deg(X) \mid K \le_{\mathcal{K}} X \le_{\mathcal{K}} K'\}$. In §3, we will prove that under ZFC+CH, for some $K \subseteq {}^\omega\omega$, lattices whose fields $\subseteq {}^\omega\omega$ and which are Kleene recursive in K^{SJ} can be embedded in $\mathcal{K}[K, K^{\mathrm{SJ}}]$. Without CH, it is unknown whether our Theorem can be proved or not.

1. Similarly to [3] and [6], we use lattice tables (lattice representations in [6]), on which lattices are represented by dual lattices of equivalence relations. For every lattice $\mathcal{L}$ with cardinality $\le 2^{\aleph_0}$, we denote the field of $\mathcal{L}$ also by $\mathcal{L}$ and regard $\mathcal{L} \subseteq {}^\omega\omega$. We denote by $\mathbf{0}$ the identically 0 function from ω to ω.

Definition. Let $\mathcal{L}$ be a lattice with relations $\le_{\mathcal{L}}$, $\vee_{\mathcal{L}}$, and $\wedge^{\mathcal{L}}$. For $a, b \in {}^{\mathcal{L}}({}^\omega\omega)$ and $l \in \mathcal{L}$, we define $a \equiv_l b$ by $a(l) = b(l)$. $\Theta \subseteq {}^{\mathcal{L}}({}^\omega\omega)$ is called an *upper semilattice table* of $\mathcal{L}$ iff Θ satisfies:

(R.0) If there is the least element $0_{\mathcal{L}}$ of $\mathcal{L}$, then for all $a \in \Theta$, $a(0_{\mathcal{L}}) = \mathbf{0}$.

(R.1) (Ordering) For all $a, b \in \Theta$ and $i, j \in \mathcal{L}$, if $i \le_{\mathcal{L}} j$ and $a \equiv_j b$, then $a \equiv_i b$.

(R.2) (Non-ordering) For all $i, j \in \mathcal{L}$, if $i \not\le_{\mathcal{L}} j$, then there are $a, b \in \Theta$ such that $a \equiv_j b$ and $a \not\equiv_i b$.

(R.3) (Join) For all $a, b \in \Theta$ and $i, j, k \in \mathcal{L}$, if $i \vee_{\mathcal{L}} j = k$, $a \equiv_i b$, and $a \equiv_j b$, then $a \equiv_k b$.

In addition, if Θ satisfies (R.4) below, then Θ is called a *lattice table* of $\mathcal{L}$:

(R.4) (Meet) For all $a, b \in \Theta$ and $i, j, k \in \mathcal{L}$, if $i \wedge^{\mathcal{L}} j = k$ and $a \equiv_k b$, then there are $c_0, c_1, c_2 \in \Theta$ such that $a \equiv_i c_0 \equiv_j c_1 \equiv_i c_2 \equiv_j b$.

For every lattice $\mathcal{L}$ with relations $\le_{\mathcal{L}}$, $\vee_{\mathcal{L}}$, $\wedge^{\mathcal{L}}$, and $\mathcal{L} \subseteq {}^\omega\omega$, we say that $(\mathcal{L}, \le_{\mathcal{L}}, \vee_{\mathcal{L}}, \wedge^{\mathcal{L}})$ is *Kleene recursive* in $X \subseteq {}^\omega\omega$ iff $\mathcal{L} \oplus \{\langle i,j\rangle \mid i \le_{\mathcal{L}} j\} \oplus \{\langle i,j,k\rangle \mid i \vee_{\mathcal{L}} j = k\} \oplus \{\langle i,j,k\rangle \mid i \wedge^{\mathcal{L}} j = k\} \le_{\mathcal{K}} X$.

In this paper, we need suitable restrictions in (R.2) and (R.4).

Proposition 1.1. *Let $\mathcal{L}$ be a lattice with relations $\le_{\mathcal{L}}$, $\vee_{\mathcal{L}}$, $\wedge^{\mathcal{L}}$, and $\mathcal{L} \subseteq {}^\omega\omega$. Let $X \subseteq {}^\omega\omega$. If $(\mathcal{L}, \le_{\mathcal{L}}, \vee_{\mathcal{L}}, \wedge^{\mathcal{L}})$ is Kleene recursive in X, then there are a lattice table Θ of $\mathcal{L}$ and $F \subseteq {}^\omega\omega \times \mathcal{L} \times {}^\omega\omega$ such that $\Theta = \{F^{[x]} \mid x \in {}^\omega\omega\}$, $F \le_{\mathcal{K}} X$, and F satisfies*:

(R.2*) *For all $i, j \in \mathcal{L}$, if $i \not\le_{\mathcal{L}} j$, then there are $a, b \in {}^\omega\omega \cap L_{\omega_1^{i,j}}[i,j]$ such that $F^{[a]} \equiv_j F^{[b]}$ and $F^{[a]} \not\equiv_i F^{[b]}$.*

(R.4*) *For all* $a, b \in {}^{\omega}\omega$ *and* $i, j, k \in \mathcal{L}$, *if* $i \wedge^{\mathcal{L}} j = k$ *and* $F^{[a]} \equiv_k F^{[b]}$, *then there are* $c_0, c_1, c_2 \in {}^{\omega}\omega \cap L_{\omega_1^{a,b,i,j,k}}[a, b, i, j, k]$ *such that* $F^{[a]} \equiv_i F^{[c_0]} \equiv_j F^{[c_1]} \equiv_i F^{[c_2]} \equiv_j F^{[b]}$.

(R.5) *For all* $a \in {}^{\omega}\omega$, $\mathrm{Rng}(F^{[a]}) \subseteq L_{\omega_1^a}[a]$.

Here, for $x \in {}^{\omega}\omega$, *we set* $F^{[x]} = \{\langle l, y\rangle \mid \langle x, l, y\rangle \in F\}$ *and regard* $F^{[x]} : \mathcal{L} \to {}^{\omega}\omega$.

Proof. We fix X and $\mathcal{L}$ as in the proposition. We assume that there is the least element $0_{\mathcal{L}}$ of $\mathcal{L}$. We will construct Θ and F with the required properties.

For $x \in {}^{\omega}\omega$ and $m \in \omega$, we define the function $f^{\langle 0,m\rangle * x} : \mathcal{L} \to {}^{\omega}\omega$ as follows: If $x \notin \mathcal{L}$ or $m \neq 2$, then

$$f^{\langle 0,m\rangle * x}(l) = \begin{cases} \mathbf{0} & \text{if } l = 0_{\mathcal{L}}, \\ \langle 0, m\rangle * x & \text{otherwise.} \end{cases}$$

If $x \in \mathcal{L}$ and $m = 2$, then

$$f^{\langle 0,2\rangle * x}(l) = \begin{cases} \mathbf{0} & \text{if } l = 0_{\mathcal{L}}, \\ \langle 0, 1\rangle * x & \text{if } 0_{\mathcal{L}} \neq l \leq_{\mathcal{L}} x, \\ \langle 0, 2\rangle * x & \text{otherwise.} \end{cases}$$

For $x \in {}^{\omega}\omega$ and $n, m \in \omega$, we define the function $f^{\langle n+1,m\rangle * x} : \mathcal{L} \to {}^{\omega}\omega$ inductively as follows: If $x = \langle a, b, i, j, k\rangle$, $a \neq b$, $\max\{a(0), b(0)\} = n$, $i, j, k \in \mathcal{L}$, $i \wedge^{\mathcal{L}} j = k$, $i \not\leq_{\mathcal{L}} j$, $j \not\leq_{\mathcal{L}} i$, $f^a(k) = f^b(k)$, and $m \leq 2$, then

$$f^{\langle n+1,0\rangle * x}(l) = \begin{cases} f^a(l) & \text{if } l \leq_{\mathcal{L}} i, \\ \langle n+1, 0\rangle * x & \text{otherwise,} \end{cases}$$

$$f^{\langle n+1,1\rangle * x}(l) = \begin{cases} f^{\langle n+1,0\rangle * x}(l) & \text{if } l \leq_{\mathcal{L}} j, \\ \langle n+1, 1\rangle * x & \text{if } l \leq_{\mathcal{L}} i \text{ and } l \not\leq_{\mathcal{L}} j, \\ \langle n+1, 2\rangle * x & \text{otherwise,} \end{cases}$$

$$f^{\langle n+1,2\rangle * x}(l) = \begin{cases} f^b(l) & \text{if } l \leq_{\mathcal{L}} j, \\ \langle n+1, 1\rangle * x & \text{if } l \leq_{\mathcal{L}} i \text{ and } l \not\leq_{\mathcal{L}} j, \\ \langle n+1, 3\rangle * x & \text{otherwise.} \end{cases}$$

In the other case,

$$f^{\langle n+1,m\rangle * x}(l) = \begin{cases} \mathbf{0} & \text{if } l = 0_{\mathcal{L}}, \\ \langle n+1, m+1\rangle * x & \text{otherwise.} \end{cases}$$

We set $\Theta = \{f^x \mid x \in {}^{\omega}\omega\}$ and $F = \{\langle x, l, y\rangle \in {}^{\omega}\omega \times \mathcal{L} \times {}^{\omega}\omega \mid f^x(l) = y\}$. Then $F^{[x]} = f^x$ for $x \in {}^{\omega}\omega$. (To define f^x for all $x \in {}^{\omega}\omega$, we make Θ contain some excess elements.)

We prove that Θ and F have the required properties. By definition, $\Theta = \{F^{[x]} \mid x \in {}^{\omega}\omega\}$, $F \leq_{\mathcal{K}} X$, and F satisfies (R.5).

For $n \in \omega$, we set $\Theta_n = \{f^x \mid x \in {}^{\omega}\omega \wedge x(0) \leq n\}$.

Lemma 1.2. (1) *Θ_0 is an upper semilattice table of $\mathcal{L}$.*

(2) *F satisfies* (R.2*).

P r o o f. (1) We check that Θ_0 satisfies (R.0)–(R.3).

(R.0) By definition, for all $f^x \in \Theta_0$, $f^x(0_{\mathcal{L}}) = \mathbf{0}$.

(R.1) Suppose $f^{\langle 0,m\rangle * x}, f^{\langle 0,m'\rangle * x'} \in \Theta_0$ and $i, j \in \mathcal{L}$ satisfy $i \leq_{\mathcal{L}} j$ and $f^{\langle 0,m\rangle * x}(j) = f^{\langle 0,m'\rangle * x'}(j)$. If $f^{\langle 0,m\rangle * x} = f^{\langle 0,m'\rangle * x'}$ or $i = 0_{\mathcal{L}}$, then clearly $f^{\langle 0,m\rangle * x}(i) = f^{\langle 0,m'\rangle * x'}(i)$. Suppose $f^{\langle 0,m\rangle * x} \neq f^{\langle 0,m'\rangle * x'}$ and $i \neq 0_{\mathcal{L}}$. Clearly $j \neq 0_{\mathcal{L}}$. By definition and $f^{\langle 0,m\rangle * x}(j) = f^{\langle 0,m'\rangle * x'}(j)$, we have $\{m, m'\} = \{1, 2\}$, $x = x' \in \mathcal{L}$, and $j \leq_{\mathcal{L}} x$ (moreover, $f^{\langle 0,m\rangle * x}(j) = f^{\langle 0,m'\rangle * x'}(j) = \langle 0, 1\rangle * x$). Hence, $i \leq_{\mathcal{L}} x$ and so $f^{\langle 0,m\rangle * x}(i) = \langle 0, 1\rangle * x = f^{\langle 0,m'\rangle * x'}(i)$ by definition.

(R.2) Let $i, j \in \mathcal{L}$ and $i \not\leq_{\mathcal{L}} j$. We choose $f^{\langle 0,1\rangle * j}$ and $f^{\langle 0,2\rangle * j}$ in Θ_0. Since $i \not\leq_{\mathcal{L}} j$, we have $f^{\langle 0,1\rangle * j}(i) = \langle 0, 1\rangle * j \neq \langle 0, 2\rangle * j = f^{\langle 0,2\rangle * j}(i)$. If $j = 0_{\mathcal{L}}$, then $f^{\langle 0,1\rangle * j}(j) = \mathbf{0} = f^{\langle 0,2\rangle * j}(j)$, and if $j \neq 0_{\mathcal{L}}$, then $f^{\langle 0,1\rangle * j}(j) = \langle 0, 1\rangle * j = f^{\langle 0,2\rangle * j}(j)$.

(R.3) Suppose $f^{\langle 0,m\rangle * x}, f^{\langle 0,m'\rangle * x'} \in \Theta_0$ and $i, j, k \in \mathcal{L}$ satisfy $i \vee_{\mathcal{L}} j = k$, $f^{\langle 0,m\rangle * x}(i) = f^{\langle 0,m'\rangle * x'}(i)$, and $f^{\langle 0,m\rangle * x}(j) = f^{\langle 0,m'\rangle * x'}(j)$. We may suppose $f^{\langle 0,m\rangle * x} \neq f^{\langle 0,m'\rangle * x'}$ and $k \neq 0_{\mathcal{L}}$. By definition, we have $\{m, m'\} = \{1, 2\}$, $x = x' \in \mathcal{L}$, and $i, j \leq_{\mathcal{L}} x$. Hence, $k \leq_{\mathcal{L}} x$ and so $f^{\langle 0,m\rangle * x}(k) = \langle 0, 1\rangle * x = f^{\langle 0,m'\rangle * x'}(k)$ by definition.

(2) Since $\langle 0, 1\rangle * j, \langle 0, 2\rangle * j \in L_{\omega_1^{i,j}}[i, j]$, (2) is clear from the proof of (R.2) in (1). ∎

Lemma 1.3. *For all $n \in \omega$, if Θ_n is an upper semilattice table of $\mathcal{L}$, then Θ_{n+1} is an upper semilattice table of $\mathcal{L}$.*

P r o o f. By definition, Θ_{n+1} satisfies (R.0). Since $\Theta_n \subseteq \Theta_{n+1}$, Θ_{n+1} satisfies (R.2). It is routine to check that Θ_{n+1} satisfies (R.1) and (R.3). Below, we check (R.1) in a few cases, and leave the check of (R.1) in the other cases and of (R.3) to the reader.

Suppose $f^{\langle m_0,m_1\rangle * x}, f^{\langle m_0',m_1'\rangle * x'} \in \Theta_{n+1}$ and $l, l' \in \mathcal{L}$ satisfy $l \leq_{\mathcal{L}} l'$ and $f^{\langle m_0,m_1\rangle * x}(l') = f^{\langle m_0',m_1'\rangle * x'}(l')$. We may assume $f^{\langle m_0,m_1\rangle * x} \neq f^{\langle m_0',m_1'\rangle * x'}$ and $l \neq 0_{\mathcal{L}}$. Since Θ_n is an upper semilattice table of $\mathcal{L}$, we may also assume that $f^{\langle m_0,m_1\rangle * x} \notin \Theta_n$ or $f^{\langle m_0',m_1'\rangle * x'} \notin \Theta_n$. We notice that if $f^{\langle m_0,m_1\rangle * x}$ or $f^{\langle m_0',m_1'\rangle * x'}$ is defined by "In the other case" in the construction of Θ_{n+1}, then $f^{\langle m_0,m_1\rangle * x}(l') = f^{\langle m_0',m_1'\rangle * x'}(l')$ does not occur.

Case 1: $f^{\langle m_0',m_1'\rangle * x'} \in \Theta_n$ and there are $a, b \in {}^{\omega}\omega$ and $i, j, k \in \mathcal{L}$ such that $m_0 = n + 1$, $m_1 = 1$, $x = \langle a, b, i, j, k\rangle$, $a \neq b$, $\max\{a(0), b(0)\} = n$, $i \wedge^{\mathcal{L}} j = k$, $i \not\leq_{\mathcal{L}} j$, $j \not\leq_{\mathcal{L}} i$, and $f^a(k) = f^b(k)$.

Since $f^{\langle m_0',m_1'\rangle * x'} \in \Theta_n$, it follows that $f^{\langle m_0',m_1'\rangle * x'}(l')(0) \leq n$ and so $f^{\langle n+1,1\rangle * x}(l')(0) \leq n$. Then, by definition, $l' \leq_{\mathcal{L}} j$, $l' \leq_{\mathcal{L}} i$, and $f^{\langle n+1,1\rangle * x}(l') = f^{\langle n+1,0\rangle * x}(l') = f^a(l')$. Hence $f^a(l') = f^{\langle m_0',m_1'\rangle * x'}(l')$. Since $f^a \in \Theta_n$

and Θ_n satisfies (R.1), $f^a(l) = f^{\langle m'_0, m'_1\rangle * x'}(l)$. Clearly, $l \leq_{\mathcal{L}} i \wedge^{\mathcal{L}} j$, hence $f^{\langle n+1,1\rangle * x}(l) = f^{\langle n+1,0\rangle * x}(l) = f^a(l) = f^{\langle m'_0, m'_1\rangle * x'}(l)$.

CASE 2: There are $a, b, a', b' \in {}^\omega\omega$ and $i, j, k, i', j', k' \in \mathcal{L}$ such that $m_0 = m'_0 = n+1$, $m_1 = 1$, $m'_1 = 2$, $x = \langle a, b, i, j, k\rangle$, $x' = \langle a', b', i', j', k'\rangle$, $a \neq b$, $a' \neq b'$, $\max\{a(0), b(0)\} = \max\{a'(0), b'(0)\} = n$, $i \wedge^{\mathcal{L}} j = k$, $i' \wedge^{\mathcal{L}} j' = k'$, $i \not\leq_{\mathcal{L}} j$, $j \not\leq_{\mathcal{L}} i$, $i' \not\leq_{\mathcal{L}} j'$, $j' \not\leq_{\mathcal{L}} i'$, $f^a(k) = f^b(k)$, and $f^{a'}(k') = f^{b'}(k')$.

By definition, we have two subcases.

SUBCASE 2.1: $l' \leq_{\mathcal{L}} i \wedge^{\mathcal{L}} j \wedge^{\mathcal{L}} j'$ and $f^{\langle n+1,1\rangle * x}(l') = f^{\langle n+1,0\rangle * x}(l') = f^a(l') = f^{b'}(l') = f^{\langle n+1,2\rangle * x'}(l')$. Then, similarly to Case 1, we obtain $f^{\langle n+1,1\rangle * x}(l) = f^a(l) = f^{b'}(l) = f^{\langle n+1,2\rangle * x'}(l)$.

SUBCASE 2.2: $l' \leq_{\mathcal{L}} i$, $l' \not\leq_{\mathcal{L}} j$, $x = x'$, and $f^{\langle n+1,1\rangle * x}(l') = \langle n+1, 1\rangle * x = f^{\langle n+1,2\rangle * x'}(l')$. Then $i = i'$, $j = j'$, $k = k'$, $a = a'$, and $b = b'$ clearly. If $l \not\leq_{\mathcal{L}} j$, then $f^{\langle n+1,1\rangle * x}(l) = \langle n+1, 1\rangle * x = f^{\langle n+1,2\rangle * x'}(l)$. Suppose $l \leq_{\mathcal{L}} j$. Since $l \leq_{\mathcal{L}} i \wedge^{\mathcal{L}} j$, $f^{\langle n+1,1\rangle * x}(l) = f^a(l)$ and $f^{\langle n+1,2\rangle * x'}(l) = f^b(l)$. Since $i \wedge^{\mathcal{L}} j = k$, $f^a(k) = f^b(k)$, and Θ_n satisfies (R.1), we have $f^a(l) = f^b(l)$. Hence, $f^{\langle n+1,1\rangle * x}(l) = f^{\langle n+1,2\rangle * x'}(l)$. ∎

By Lemmas 1.2 and 1.3, Θ is an upper semilattice table of $\mathcal{L}$.

LEMMA 1.4. *F satisfies* (R.4*). *Hence, Θ is a lattice table of $\mathcal{L}$.*

Proof. Suppose $a, b \in {}^\omega\omega$ and $i, j, k \in \mathcal{L}$ satisfy $i \wedge^{\mathcal{L}} j = k$ and $f^a(k) = f^b(k)$. In the case of $i \leq_{\mathcal{L}} j$ or $j \leq_{\mathcal{L}} i$, we set $c_0 = c_1 = c_2 = b$ or $c_0 = c_1 = c_2 = a$, and then c_0, c_1, c_2 have the required properties. Suppose $i \not\leq_{\mathcal{L}} j$, $j \not\leq_{\mathcal{L}} i$, and $a \neq b$. We set $n = \max\{a(0), b(0)\}$ and $c_m = \langle n+1, m\rangle * \langle a, b, i, j, k\rangle$ for $m \leq 2$. Then $c_0, c_1, c_2 \in L_{\omega_1^{a,b,i,j,k}}[a, b, i, j, k]$. By definition, $f^a \equiv_i f^{c_0} \equiv_j f^{c_1}$ and $f^{c_2} \equiv_j f^b$. Since $i \not\leq_{\mathcal{L}} j$, we have $f^{c_1} \equiv_i f^{c_2}$. ∎

This completes the proof of Proposition 1.1. ∎

2. We start this section with

LEMMA 2.1 (ZFC+CH). *There is $S \subseteq \aleph_1$ such that ${}^\omega\omega \subseteq L_{\aleph_1}[S]$.*

Proof. We take a bijection $f : \aleph_1 \to {}^\omega\omega$ and set

$$S = \{\xi \in \aleph_1 \mid \exists \gamma \leq \xi \exists m, n \in \omega (\xi = \omega \cdot \gamma + 2^m \cdot 3^n \wedge f(\gamma)(m) = n)\}.$$

Notice that for all $\xi < \aleph_1$, there are unique $\gamma \leq \xi$ and unique $k \in \omega$ such that $\xi = \omega \cdot \gamma + k$. Let $x \in {}^\omega\omega$ be arbitrary. We choose $\gamma \in \aleph_1$ such that $f(\gamma) = x$; then $x(m) = n \Leftrightarrow \omega \cdot \gamma + 2^m \cdot 3^n \in S$ for all $m, n \in \omega$. Hence, $x \in L_{\aleph_1}[S]$. ∎

We fix $S \subseteq \aleph_1$ such that ${}^\omega\omega \subseteq L_{\aleph_1}[S]$. We define the function rk : ${}^\omega\omega \to \aleph_1$ by $\mathrm{rk}(x) = \min\{\alpha \in \aleph_1 \mid x \in L_{\alpha+1}[S]\}$ for $x \in {}^\omega\omega$. We set $K_0 = \{x \in \mathrm{WO} \mid \mathrm{o.t.}(x) \in S\}$ and

$$K_1 = \{\langle m,n\rangle * x \in {}^\omega\omega \mid \exists w \in \mathrm{WO}(\mathrm{rk}(x) = \mathrm{o.t.}(w) \wedge \forall w' \in \mathrm{WO}(w' <_{L[S]} w \Rightarrow \mathrm{o.t.}(w') \neq \mathrm{rk}(x)) \wedge w(m) = n)\}.$$

Here, WO denotes the set of all $x \in {}^\omega\omega$ which code a well-ordering relation on ω, and o.t.(w) denotes the order type of w.

If e.g. $\mathbf{\Delta}^1_n$-determinacy ($2 \le n \in \omega$) is assumed, then by the localization of the theorem of Solovay [7], for any $\mathbf{\Delta}^1_n$ set $K \subseteq {}^\omega\omega$, $\mathcal{K}[K, K^{\mathrm{SJ}}] = \{\deg(K), \deg(K^{\mathrm{SJ}})\}$. Under ZFC+CH (even if some determinacy axiom is assumed), if $K_0 \le_{\mathcal{K}} K \subseteq {}^\omega\omega$, then $\mathcal{K}[K, K^{\mathrm{SJ}}] \neq \{\deg(K), \deg(K^{\mathrm{SJ}})\}$ ([5]; in fact we can prove that $\mathcal{K}[K, K^{\mathrm{SJ}}]$ contains many elements). To prove the Theorem in §3, we use K_1 in addition to K_0. We note that under ZFC+CH, $\{\mathbf{d} \in \mathcal{K} \mid \deg(K_0 \oplus K_1) \le_{\mathcal{K}} \mathbf{d}\}$ is dense, which can be proved similarly to [2] and [4].

Lemma 2.2 (ZFC+CH). *Let $K_0 \oplus K_1 \le_{\mathcal{K}} K \subseteq {}^\omega\omega$ and $T = S \cup K$.*

(1) *For all $x \in {}^\omega\omega$, $L_{\omega_1^{K;x}}[K; x]$ is S-admissible, and so T-admissible.*

(2) *If M is K-admissible, then for all $x \in {}^\omega\omega \cap M$, $\mathrm{rk}(x) \in M$.*

(3) *For all $x \in {}^\omega\omega$, $x \in L_{\omega_1^{T;x}}[T]$, hence $L_{\omega_1^{T;x}}[T; x] = L_{\omega_1^{T;x}}[T]$.*

(4) *If M is T-admissible and $\mathrm{On} \cap M = \alpha$, then ${}^\omega\omega \cap M = \{x \in {}^\omega\omega \mid \mathrm{rk}(x) < \alpha\}$.*

Proof. (1) It is sufficient to prove that S is Δ_1 over $L_{\omega_1^{K;x}}[K; x]$. For all $\xi \in \omega_1^{K;x}$, since there is an injection from ξ to ω in $L_{\omega_1^{K;x}}[K; x]$, there is $w \in \mathrm{WO} \cap L_{\omega_1^{K;x}}[K; x]$ which codes a well-ordering of order type ξ. Hence, for all $\xi \in \omega_1^{K;x}$,

$$\begin{aligned}\xi \in S &\Leftrightarrow L_{\omega_1^{K;x}}[K; x] \models \text{“}\exists w \in K_0(\mathrm{o.t.}(w) = \xi)\text{”}\\ &\Leftrightarrow L_{\omega_1^{K;x}}[K; x] \models \text{“}\forall w \in \mathrm{WO}(\mathrm{o.t.}(w) = \xi \Rightarrow w \in K_0)\text{”}.\end{aligned}$$

Therefore, S is Σ_1 and Π_1 over $L_{\omega_1^{K;x}}[K; x]$.

(2) Let w be the $\le_{L[S]}$-least element of WO such that o.t.$(w) = \mathrm{rk}(x)$. By definition, for all $m, n \in \omega$, $w(m) = n \Leftrightarrow \langle m, n\rangle * x \in K_1$. Since M is K_1-admissible, $w \in M$ and hence $\mathrm{rk}(x) = \mathrm{o.t.}(w) \in M$.

(3) Since $x \in L_{\omega_1^{T;x}}[T; x]$ and $L_{\omega_1^{T;x}}[T; x]$ is K-admissible, $\mathrm{rk}(x) < \omega_1^{T;x}$ by (2). Since $L_{\omega_1^{T;x}}[T]$ is S-admissible, $L_{\mathrm{rk}(x)+1}[S] \subseteq L_{\omega_1^{T;x}}[T]$. By definition, $x \in L_{\mathrm{rk}(x)+1}[S]$, hence $x \in L_{\omega_1^{T;x}}[T]$.

(4) Suppose $x \in {}^\omega\omega$ and $\mathrm{rk}(x) < \alpha$. Since M is S-admissible, $L_{\mathrm{rk}(x)+1}[S] \subseteq M$, hence $x \in M$. Conversely, if $x \in {}^\omega\omega \cap M$, then since M is K-admissible, $\mathrm{rk}(x) < \alpha$ by (2). ∎

3. Let S, rk, K_0, and K_1 be as in §2.

THEOREM (ZFC+CH). *Let $K_0 \oplus K_1 \leq_{\mathcal{K}} K \subseteq {}^\omega\omega$. For any lattice $\mathcal{L}$, if $\mathcal{L} \subseteq {}^\omega\omega$ and $(\mathcal{L}, \leq_{\mathcal{L}}, \vee_{\mathcal{L}}, \wedge^{\mathcal{L}})$ is Kleene recursive in K^{SJ}, then $\mathcal{L}$ can be embedded in $\mathcal{K}[K, K^{\mathrm{SJ}}]$.*

This section is entirely devoted to proving the Theorem. We use AC and CH without notice in the proof.

We fix $K \subseteq {}^\omega\omega$ such that $K_0 \oplus K_1 \leq_{\mathcal{K}} K$, and a lattice $\mathcal{L}$ such that $\mathcal{L} \subseteq {}^\omega\omega$ and $(\mathcal{L}, \leq_{\mathcal{L}}, \vee_{\mathcal{L}}, \wedge^{\mathcal{L}})$ is Kleene recursive in K^{SJ}. We set $T = S \cup K$. Then every T-admissible set is S-admissible and K-admissible, and ${}^\omega\omega \subseteq L_{\aleph_1}[T]$. We fix a lattice table Θ of $\mathcal{L}$ and $F \subseteq {}^\omega\omega \times \mathcal{L} \times {}^\omega\omega$ which are obtained by Proposition 1.1. For simplicity, we assume that $(\mathcal{L}, \leq_{\mathcal{L}}, \vee_{\mathcal{L}}, \wedge^{\mathcal{L}})$ is Kleene recursive in K^{SJ} with no additional real parameter and $F \leq_{\mathcal{K}} K^{\mathrm{SJ}}$ with no additional real parameter. For $x \in {}^\omega\omega$, we denote $F^{[x]}$ by f^x as in the proof of Proposition 1.1. We may assume that $f^{\mathbf{0}}$ is identically $\mathbf{0}$ on $\mathcal{L}$ and $\mathbf{0}$ is the $\leq_{L[T]}$-least real.

For every total or partial function p from ${}^\omega\omega$ to ${}^\omega\omega$, we define the projections of p by

$$P_l = \{\langle x, f^{p(x)}(l)\rangle \mid x \in \mathrm{Dom}(p)\} \quad \text{for } l \in \mathcal{L}.$$

We will construct a total function $g : {}^\omega\omega \to {}^\omega\omega$ such that $l \in \mathcal{L} \mapsto \deg(K \oplus G_l) \in \mathcal{K}[K, K^{\mathrm{SJ}}]$ is a lattice embedding. Recall that G_l denotes the projection of g on the coordinate l.

By recursion, we define a strictly increasing sequence $\langle \tau_\alpha \mid \alpha \in \aleph_1\rangle$ of countable ordinals which satisfies:

(T.1) $\tau_{\alpha+1}$ is the least T-admissible ordinal such that ${}^\omega\omega \cap (L_{\tau_{\alpha+1}}[T] - L_{\tau_\alpha}[T])$ is not empty.

(T.2) If α is a limit ordinal, then $\tau_\alpha = \bigcup_{\beta\in\alpha} \tau_\beta$.

The following is proved by routine work.

LEMMA 3.1. (1) *The graph of $\langle \tau_\alpha \mid \alpha \in \aleph_1\rangle$ is uniformly $\Sigma_1(T)$-definable over all T-admissible sets.*

(2) *For any T-admissible set M, if $\alpha \in \aleph_1 \cap M$ and $\langle \tau_\beta \mid \beta \in \alpha\rangle \subseteq M$, then $\langle \tau_\beta \mid \beta \in \alpha\rangle \in M$.*

LEMMA 3.2. *For all $\alpha \in \aleph_1$ and $x \in {}^\omega\omega \cap (L_{\tau_{\alpha+1}}[T] - L_{\tau_\alpha}[T])$, we have $L_{\tau_{\alpha+1}}[T] = L_{\omega_1^{K;x}}[K; x]$.*

Proof. By Lemma 2.2, $x \in L_{\omega_1^{T;x}}[T]$, hence it follows by the definition of $\tau_{\alpha+1}$ that $\tau_{\alpha+1} \leq \omega_1^{T;x}$. Since $L_{\omega_1^{K;x}}[K; x]$ is T-admissible by Lemma 2.2, $L_{\tau_{\alpha+1}}[T] \subseteq L_{\omega_1^{T;x}}[T] \subseteq L_{\omega_1^{K;x}}[K; x]$. Conversely, since $L_{\tau_{\alpha+1}}[T]$ is $(K; x)$-admissible, we have $L_{\omega_1^{K;x}}[K; x] \subseteq L_{\tau_{\alpha+1}}[T]$. ∎

Remember that for any K-admissible set N, N is closed under $\lambda x.\omega_1^{K;x}$ iff N is K^{SJ}-admissible, and moreover N is closed under $\lambda x.\omega_1^{K;x}$ iff $\forall x \in {}^{\omega}\omega \cap N \exists \alpha \in \mathrm{On} \cap N (L_\alpha[K;x]$ is $(K;x)$-admissible$)^N$. Hence the quantifiers in the statement "N is K^{SJ}-admissible" are bounded by N. Moreover, note that F is uniformly Δ_1 over all K^{SJ}-admissible sets, since $F \leq_{\mathcal{K}} K^{\mathrm{SJ}}$.

LEMMA 3.3. *Let p be a partial function from ${}^{\omega}\omega$ to ${}^{\omega}\omega$, M be a T-admissible set, $p \in M$ and $l \in \mathcal{L} \cap M$. If for all $x \in \mathrm{Dom}(p)$, there is $\sigma \in \mathrm{On} \cap M$ such that $L_\sigma[T]$ is K^{SJ}-admissible and $p(x), l \in L_\sigma[T]$, then $P_l \in M$.*

Proof. By Σ_1-collection, there exists $\gamma \in \mathrm{On} \cap M$ such that for all $x \in \mathrm{Dom}(p)$ there is $\sigma < \gamma$ such that $L_\sigma[T]$ is K^{SJ}-admissible and $p(x), l \in L_\sigma[T]$ (moreover $f^{p(x)}(l) \in L_\sigma[T]$ by (R.5)). Then for all $x, y \in {}^{\omega}\omega$ we have

$$\begin{aligned}\langle x, y\rangle \in P_l \Leftrightarrow M \models \text{“}&x \in \mathrm{Dom}(p) \wedge y \in L_\gamma[T]\\ &\wedge \exists \sigma < \gamma \exists z \in L_\sigma[T](L_\sigma[T] \text{ is } K^{\mathrm{SJ}}\text{-admissible}\\ &\wedge l, y \in L_\sigma[T] \wedge z = p(x) \wedge (\langle z, l, y\rangle \in F)^{L_\sigma[T]})\text{”}.\end{aligned}$$

Hence, $P_l \in M$ by Δ_1-separation. ∎

We construct g^α ($\alpha \in \aleph_1$) of the parts of g as follows:

STAGE 0. We set $g^0 = \emptyset$.

STAGE α LIMIT. We set $g^\alpha = \bigcup_{\beta \in \alpha} g^\beta$.

STAGE $\alpha + 1$.

CASE 1: There is $t \in {}^{\omega}\omega \cap L_{\tau_\alpha}[T]$ which satisfies (G.1) or (G.2) below:

(G.1) There are $e \in \omega$, $v \in {}^{\omega}\omega$, $i, j \in \mathcal{L}$, and $\sigma \leq \tau_\alpha$ such that $t = \langle 0, e\rangle * \langle v, i, j\rangle$, $i \not\leq_{\mathcal{L}} j$, $L_\sigma[T]$ is K^{SJ}-admissible, $t \in L_\sigma[T]$, and $\forall x \in {}^{\omega}\omega \cap L_{\tau_\alpha}[T](\chi_{G_i^\alpha}(x) \cong \{e\}(x, v, \chi_{K \oplus G_j^\alpha}, {}^2E))$.

(G.2) There are $e_0, e_1 \in \omega$, $v_0, v_1 \in {}^{\omega}\omega$, $i, j, k \in \mathcal{L}$, and $\sigma \leq \tau_\alpha$ such that $t = \langle 1, e_0, e_1\rangle * \langle v_0, v_1, i, j, k\rangle$, $i \wedge^{\mathcal{L}} j = k$, $L_\sigma[T]$ is K^{SJ}-admissible, $t \in L_\sigma[T]$, $\forall x \in {}^{\omega}\omega \cap L_{\tau_\alpha}[T](\{e_0\}(x, v_0, \chi_{K \oplus G_i^\alpha}, {}^2E) \cong \{e_1\}(x, v_1, \chi_{K \oplus G_j^\alpha}, {}^2E))$, and there is a partial function $p \in L_{\tau_{\alpha+1}}[T]$ from ${}^{\omega}\omega$ to ${}^{\omega}\omega$ such that $g^\alpha \subseteq p$, $\mathrm{Rng}(p - g^\alpha) \subseteq L_\sigma[T]$, and $\exists x \in {}^{\omega}\omega \cap L_{\tau_{\alpha+1}}[T](\{e_0\}(x, v_0, \chi_{K \oplus P_i * \mathbf{0}}, {}^2E) \not\cong \{e_1\}(x, v_1, \chi_{K \oplus P_j * \mathbf{0}}, {}^2E))$. Here, $P_l * \mathbf{0} = P_l \cup \{\langle y, \mathbf{0}\rangle \mid y \in {}^{\omega}\omega - \mathrm{Dom}(p)\}$ for $l \in \mathcal{L}$.

We choose the $\leq_{L[T]}$-least $t \in {}^{\omega}\omega \cap L_{\tau_\alpha}[T]$ which satisfies (G.1) or (G.2) and distinguish two subcases.

SUBCASE 1.1: t satisfies (G.1). We choose the $\leq_{L[T]}$-least $z \in {}^{\omega}\omega \cap (L_{\tau_{\alpha+1}}[T] - L_{\tau_\alpha}[T])$ and the $\leq_{L[T]}$-least $\langle a, b\rangle \in {}^{\omega}\omega \times {}^{\omega}\omega$ such that $f^a(j) = f^b(j)$ and $f^a(i) \neq f^b(i)$ by (R.2). Notice that if σ is as in (G.1), then

$a, b, f^a(i) \in L_\sigma[T]$ by (R.2*) and (R.5). We set $z' = \langle z, f^a(i)\rangle$ and define partial functions p^a, p^b by

$$p^a(x)\ (p^b(x)\ \text{resp.}) = \begin{cases} g^\alpha(x) & \text{if } x \in \mathrm{Dom}(g^\alpha),\\ a\ (b\ \text{resp.}) & \text{if } x = z. \end{cases}$$

Then $P^a_j = P^b_j$, $z' \in P^a_i$, and $z' \notin P^b_i$. If $\{e\}(z', v, \chi_{K\oplus P^a_j * \mathbf{0}}, {}^2E) \cong 0$, then we define

$$g^{\alpha+1}(x) = \begin{cases} p^a(x) & \text{if } x \in \mathrm{Dom}(p^a),\\ \mathbf{0} & \text{if } x \in {}^\omega\omega \cap L_{\tau_{\alpha+1}}[T] - \mathrm{Dom}(p^a), \end{cases}$$

and if $\{e\}(z', v, \chi_{K\oplus P^a_j * \mathbf{0}}, {}^2E) \not\cong 0$, then we define

$$g^{\alpha+1}(x) = \begin{cases} p^b(x) & \text{if } x \in \mathrm{Dom}(p^b),\\ \mathbf{0} & \text{if } x \in {}^\omega\omega \cap L_{\tau_{\alpha+1}}[T] - \mathrm{Dom}(p^b). \end{cases}$$

SUBCASE 1.2: t satisfies (G.2). We choose the $\le_{L[T]}$-least partial function $p \in L_{\tau_{\alpha+1}}[T]$ as in (G.2) and define

$$g^{\alpha+1}(x) = \begin{cases} p(x) & \text{if } x \in \mathrm{Dom}(p),\\ \mathbf{0} & \text{if } x \in {}^\omega\omega \cap L_{\tau_{\alpha+1}}[T] - \mathrm{Dom}(p). \end{cases}$$

CASE 2: Otherwise. We define

$$g^{\alpha+1}(x) = \begin{cases} g^\alpha(x) & \text{if } x \in \mathrm{Dom}(g^\alpha),\\ \mathbf{0} & \text{if } x \in {}^\omega\omega \cap L_{\tau_{\alpha+1}}[T] - \mathrm{Dom}(g^\alpha). \end{cases}$$

In the construction at Stage $\alpha + 1$ above, notice that for $l \in \mathcal{L}$, $G^{\alpha+1}_l = P^a_l * \mathbf{0} \cap L_{\tau_{\alpha+1}}[T]$ or $= P^b_l * \mathbf{0} \cap L_{\tau_{\alpha+1}}[T]$ (Subcase 1.1), or $= P_l * \mathbf{0} \cap L_{\tau_{\alpha+1}}[T]$ (Subcase 1.2), or $= G^\alpha_l * \mathbf{0} \cap L_{\tau_{\alpha+1}}[T]$ (Case 2) respectively.

We define $g = \bigcup_{\alpha\in\aleph_1} g^\alpha$. Then, for all $\alpha \in \aleph_1$, $g\lceil {}^\omega\omega \cap L_{\tau_\alpha}[T] = g^\alpha$ and $g^\alpha : {}^\omega\omega \cap L_{\tau_\alpha}[T] \to {}^\omega\omega \cap L_{\tau_\alpha}[T]$. Moreover $g^{\alpha+1} : {}^\omega\omega \cap L_{\tau_{\alpha+1}}[T] \to {}^\omega\omega \cap L_{\tau_\alpha}[T]$ by definition. If there is no $\sigma \le \tau_\alpha$ such that $L_\sigma[T]$ is K^{SJ}-admissible, then $\mathrm{Rng}(g^{\alpha+1}) = \{\mathbf{0}\}$. As for projections, for all $\alpha \in \aleph_1$ and $l \in \mathcal{L} \cap L_{\tau_\alpha}[T]$, we have $G_l \cap L_{\tau_\alpha}[T] = G^\alpha_l$.

LEMMA 3.4. *Let $\varrho \in \aleph_1$ and $L_\varrho[T]$ be K^{SJ}-admissible.*

(1) *For all $\alpha < \aleph_1$, if $\varrho \le \tau_\alpha$, then there is $\sigma \le \tau_\alpha$ such that $L_\sigma[T]$ is K^{SJ}-admissible and* $\mathrm{Rng}(g^{\alpha+1} - g^\alpha) \subseteq L_\sigma[T]$.

(2) *For all $x \in {}^\omega\omega$, there is $\sigma \le \max\{\mathrm{rk}(x), \varrho\}$ such that $L_\sigma[T]$ is K^{SJ}-admissible and $g(x) \in L_\sigma[T]$.*

Proof. (1) We distinguish three cases at Stage $\alpha + 1$.

CASE 1: $g^{\alpha+1}$ is constructed in Subcase 1.1 at Stage $\alpha + 1$. We choose σ as in (G.1). By definition, there is $c \in {}^\omega\omega \cap L_\sigma[T]$ ($c = a$ or $= b$ in Subcase

1.1) such that $\mathrm{Rng}(g^{\alpha+1} - g^{\alpha}) = \{c, \mathbf{0}\}$. Since $\mathbf{0} \in L_\sigma[T]$, $\mathrm{Rng}(g^{\alpha+1} - g^{\alpha}) \subseteq L_\sigma[T]$.

Case 2: $g^{\alpha+1}$ is constructed in Subcase 1.2 at Stage $\alpha+1$. We choose the $\le_{L[T]}$-least partial function p and σ as in (G.2). By (G.2), $\mathrm{Rng}(p - g^{\alpha}) \subseteq L_\sigma[T]$, hence $\mathrm{Rng}(g^{\alpha+1} - g^{\alpha}) \subseteq L_\sigma[T]$.

Case 3: $g^{\alpha+1}$ is constructed in Case 2 at Stage $\alpha + 1$. By definition, $\mathrm{Rng}(g^{\alpha+1} - g^{\alpha}) = \{\mathbf{0}\} \subseteq L_\varrho[T]$.

(2) We choose $\alpha < \aleph_1$ such that $x \in L_{\tau_{\alpha+1}}[T] - L_{\tau_\alpha}[T]$. By Lemma 2.2, $\tau_\alpha \le \mathrm{rk}(x)$. If $\varrho \le \tau_\alpha$, then by (1) there is $\sigma \le \mathrm{rk}(x)$ such that $L_\sigma[T]$ is K^{SJ}-admissible and $g(x) = g^{\alpha+1}(x) \in L_\sigma[T]$. If $\tau_\alpha < \varrho$, then since $\mathrm{Rng}(g^{\alpha+1}) \subseteq L_{\tau_\alpha}[T]$, we have $g(x) \in L_\varrho[T]$. ∎

Since $L_{\aleph_1}[T]$ is K^{SJ}-admissible and ${}^{\omega}\omega \subseteq L_{\aleph_1}[T]$, for all $x \in {}^{\omega}\omega$ there exists $\varrho < \aleph_1$ such that $L_\varrho[T]$ is K^{SJ}-admissible and $x \in L_\varrho[T]$ (using the Löwenheim–Skolem Theorem). For $x \in {}^{\omega}\omega$, we set $\varrho(x) = \min\{\sigma < \aleph_1 \mid L_\sigma[T]$ is K^{SJ}-admissible and $x \in L_\sigma[T]\}$.

Lemma 3.5. *Let $\alpha \in \aleph_1$ and $l \in \mathcal{L}$.*

(1) *For any T-admissible set M, if $\tau_\alpha \in M$, then $g^\alpha \in M$.*
(2) *For any T-admissible set M, if $\tau_\alpha, \varrho(l) \in M$, then $G_l^\alpha \in M$.*
(3) *If $\varrho(l) < \tau_{\alpha+1}$, then $L_{\tau_{\alpha+1}}[T]$ is G_l-admissible.*

Proof. (1) We prove

$$\forall \alpha \in \aleph_1 \forall M : T\text{-admissible set } (\tau_\alpha \in M \Rightarrow \langle g^\beta \mid \beta \le \alpha\rangle \in M)$$

by induction.

If $\alpha = 0$, then this is clear.

Let $0 < \alpha \in \aleph_1$. We assume that for all $\beta \in \alpha$ and every T-admissible set M we have $(\tau_\beta \in M \Rightarrow \langle g^\gamma \mid \gamma \le \beta\rangle \in M)$. Let M be a T-admissible set and $\tau_\alpha \in M$.

Let $\alpha = \beta + 1$ for some β. By assumption, $g^\beta \in L_{\tau_\alpha}[T]$. In the construction at Stage $\beta+1$, p^a, p^b in Subcase 1.1 and p in Subcase 1.2 are elements of $L_{\tau_\alpha}[T]$. Since $L_{\tau_\alpha}[T] \in M$, by definition $g^{\beta+1} \in M$. Hence $\langle g^\beta \mid \beta \le \alpha\rangle \in M$.

Let α be a limit ordinal. For every limit ordinal $\beta \in \alpha$, since $\langle g^\gamma \mid \gamma \le \beta\rangle \in L_{\tau_{\beta+1}}[T]$, the construction at Stage β can be expressed over $L_{\tau_{\beta+1}}[T]$. And for every $\beta + 1 \in \alpha$, since the conditions of every case at Stage $\beta + 1$ can be expressed over $L_{\tau_{\beta+1}}[T]$ (notice that if $t = \langle \ldots\rangle * \langle \ldots, i, j, \ldots\rangle$ and $\varrho(t) \le \tau_\beta$, then $G_i^\beta, G_j^\beta \in L_{\tau_{\beta+1}}[T]$ by Lemmas 3.4 and 3.3, hence we can express (G.1) (G.2); otherwise, we proceed to Case 2 immediately), the construction at Stage $\beta + 1$ can be expressed over $L_{\tau_{\beta+2}}[T]$. Thus, $\langle g^\beta \mid \beta \in \alpha\rangle$ is Δ_1-definable over M with parameter $\langle \tau_\beta \mid \beta \le \alpha\rangle$, hence $\langle g^\beta \mid \beta \in \alpha\rangle \in M$. (By

Lemma 3.1, $\langle \tau_\beta \mid \beta \le \alpha\rangle \in M$.) Therefore, by definition, $g^\alpha \in M$, and so $\langle g^\beta \mid \beta \le \alpha\rangle \in M$.

(2) By (1), $g^\alpha \in M$. For all $x \in \mathrm{Dom}(g^\alpha)$, since $\mathrm{rk}(x) \in M$, there is $\sigma \in \mathrm{On} \cap M$ such that $L_\sigma[T]$ is K^{SJ}-admissible and $g^\alpha(x), l \in L_\sigma[T]$ by Lemma 3.4. Hence, $G_l^\alpha \in M$ by Lemma 3.3.

(3) By (2), $G_l^\alpha \in L_{\tau_{\alpha+1}}[T]$. In the construction at Stage $\alpha+1$, p^a, p^b in Subcase 1.1 and p in Subcase 1.2 are elements of $L_{\tau_{\alpha+1}}[T]$, hence similarly to (2), $P_l^a, P_l^b, P_l \in L_{\tau_{\alpha+1}}[T]$ by Lemma 3.3. Since $G_l^{\alpha+1} = P_l^a * \mathbf{0} \cap L_{\tau_{\alpha+1}}[T]$ or $= P_l^b * \mathbf{0} \cap L_{\tau_{\alpha+1}}[T]$ or $= P_l * \mathbf{0} \cap L_{\tau_{\alpha+1}}[T]$ or $= G_l^\alpha * \mathbf{0} \cap L_{\tau_{\alpha+1}}[T]$, we see that $L_{\tau_{\alpha+1}}[T]$ is $G_l^{\alpha+1}$-admissible and so G_l-admissible. ∎

LEMMA 3.6. *For all $l \in \mathcal{L}$, $G_l \le_{\mathcal{K}} K^{\mathrm{SJ}}$, hence* $\deg(K \oplus G_l) \in \mathcal{K}[K, K^{\mathrm{SJ}}]$.

Proof. For $\alpha \in \aleph_1$, similarly to Lemma 3.5, the construction of g^α (i.e. constructions till Stage α) and the conditions of every case at Stage $\alpha+1$ can be expressed over $L_{\tau_{\alpha+1}}[T]$. Hence, there are formulas ψ_1 and ψ_2 such that:

$L_{\tau_{\alpha+1}}[T] \models \psi_1(p,\alpha)$

$\Leftrightarrow$ There is $t \in {}^\omega\omega \cap L_{\tau_\alpha}[T]$ which satisfies (G.1) or (G.2) at Stage $\alpha+1$ and let t be the $\le_{L[T]}$-least such real,

if $t = \langle 0, e\rangle * \langle v, i, j\rangle$ satisfies (G.1) and z, a, b, p^a, p^b are as in Subcase 1.1

then $\{e\}(\langle z, f^a(i)\rangle, v, \chi_{K\oplus P_j^a}, {}^2E) \cong 0 \wedge p = p^a$

or $\{e\}(\langle z, f^a(i)\rangle, v, \chi_{K\oplus P_j^a}, {}^2E) \not\cong 0 \wedge p = p^b$,

and if $t = \langle 1, e_0, e_1\rangle * \langle v_0, v_1, i, j, k\rangle$ satisfies (G.2),

then p is the $\le_{L[T]}$-least partial function as in (G.2).

$L_{\tau_{\alpha+1}}[T] \models \psi_2(p,\alpha)$

$\Leftrightarrow$ There is no $t \in {}^\omega\omega \cap L_{\tau_\alpha}[T]$ which satisfies (G.1) or (G.2) at Stage $\alpha+1$ and $p = g^\alpha$.

Here, ψ_1 and ψ_2 correspond to Case 1 and Case 2 respectively.

We choose $r \in \mathrm{WO}$ such that $\mathrm{o.t.}(r) = \varrho(l)$. We prove $G_l \le_{\mathcal{K}} K^{\mathrm{SJ}}$ via r using (2) of §0. Let $x, y \in {}^\omega\omega$ be arbitrary and $M = L_{\omega_1^{K^{\mathrm{SJ}};x,y,r}}[K^{\mathrm{SJ}}; x, y, r]$. Notice that if $x \in L_{\tau_{\alpha+1}}[T] - L_{\tau_\alpha}[T]$, then by Lemma 3.2 and K^{SJ}-admissibility of M, we have $L_{\tau_{\alpha+1}}[T] = L_{\omega_1^{K;x}}[K; x] \in M$. By Lemma 3.4, there is $\sigma \le \max\{\mathrm{rk}(x), \varrho(l)\}$ such that $L_\sigma[T]$ is K^{SJ}-admissible and $g(x), l \in L_\sigma[T]$; moreover, $f^{g(x)}(l) \in L_\sigma[T]$. Hence,

$$\begin{aligned}\langle x,y\rangle\in G_l \Leftrightarrow M\models \text{“}&\exists\alpha\in\omega_1^{K;x}\exists p\in L_{\omega_1^{K;x}}[K;x]\\ &(L_{\omega_1^{K;x}}[K;x]=L_{\tau_{\alpha+1}}[T]\wedge x\notin L_{\tau_\alpha}[T]\\ &\wedge L_{\tau_{\alpha+1}}[T]\models\psi_1(p,\alpha)\vee\psi_2(p,\alpha)\\ &\wedge(\exists\sigma\le\max\{\mathrm{rk}(x),\varrho(l)\}(x\in\mathrm{Dom}(p)\wedge p(x),\ l\in L_\sigma[T]\\ &\wedge L_\sigma[T]\text{ is }K^{\mathrm{SJ}}\text{-admissible}\wedge(y=f^{p(x)}(l))^{L_\sigma[T]})\\ &\vee(x\notin\mathrm{Dom}(p)\wedge y=\mathbf{0})))\text{”}.\end{aligned}$$

Notice that the quantifiers in the statement "$\omega_1^{K;x}=\tau_{\alpha+1}$" are bounded by $L_{\omega_1^{K;x}}[K;x]$, since $\omega_1^{K;x}=\tau_{\alpha+1}$ iff $\neg\exists\tau\in\omega_1^{K;x}(\tau_\alpha<\tau\wedge\tau$ satisfies (T.1)$)^{L_{\omega_1^{K;x}}[K;x]}$. Hence "$\langle x,y\rangle\in G_l$" is Δ_1 over M. Therefore, $G_l\le_{\mathcal K}K^{\mathrm{SJ}}$. ∎

LEMMA 3.7. (1) $G_{0_{\mathcal L}}\equiv_{\mathcal K}\emptyset$.

(2) *For all* $i,j\in\mathcal L$, *if* $i\le_{\mathcal L}j$, *then* $K\oplus G_i\le_{\mathcal K}K\oplus G_j$.

(3) *For all* $i,j,k\in\mathcal L$, *if* $i\vee_{\mathcal L}j=k$, *then* $(K\oplus G_i)\oplus(K\oplus G_j)\equiv_{\mathcal K}K\oplus G_k$.

Proof. (1) By definition, $G_{0_{\mathcal L}}=\{\langle x,f^{g(x)}(0_{\mathcal L})\rangle\mid x\in{}^\omega\omega\}=\{\langle x,\mathbf{0}\rangle\mid x\in{}^\omega\omega\}\equiv_{\mathcal K}\emptyset$.

(2) We choose $r\in\mathrm{WO}$ such that o.t.$(r)=\varrho(i,j)$. To prove $K\oplus G_i\le_{\mathcal K}K\oplus G_j$, it is sufficient to prove that for all $x,y\in{}^\omega\omega$,

$$\begin{aligned}\langle x,y\rangle\in G_i\Leftrightarrow M\models\text{“}&\exists\sigma\le\max\{\mathrm{rk}(x),\varrho(i,j)\}\exists a,z\in L_\sigma[T]\\ &(L_\sigma[T]\text{ is }K^{\mathrm{SJ}}\text{-admissible}\wedge i,j\in L_\sigma[T]\\ &\wedge\langle x,z\rangle\in G_j\wedge(f^a(j)=z\wedge f^a(i)=y)^{L_\sigma[T]})\text{”},\end{aligned}$$

where $M=L_{\omega_1^{K\oplus G_j;i,j,x,y,r}}[K\oplus G_j;i,j,x,y,r]$.

Suppose $\langle x,y\rangle\in G_i$. By Lemma 2.2, $\mathrm{rk}(x)\in M$. By Lemma 3.4, there is $\sigma\le\max\{\mathrm{rk}(x),\varrho(i,j)\}$ such that $L_\sigma[T]$ is K^{SJ}-admissible and $g(x)$, $i,j\in L_\sigma[T]$. By (R.5), we have $f^{g(x)}(i),f^{g(x)}(j)\in L_\sigma[T]$. Thus, if we set $a=g(x)$ and $z=f^a(j)$, then since $y=f^a(i)$ and $F\le_{\mathcal K}K^{\mathrm{SJ}}$, the right-hand side holds. Conversely, suppose that $x,y\in{}^\omega\omega$ satisfy the right-hand side. Let a,z be as in the right-hand side. By $\langle x,z\rangle\in G_j$, $f^{g(x)}(j)=z=f^a(j)$. Then, by (R.1), $f^{g(x)}(i)=f^a(i)$. Hence, $y=f^{g(x)}(i)$, and so $\langle x,y\rangle\in G_i$.

(3) By (2), $K\oplus G_i\oplus G_j\le_{\mathcal K}K\oplus G_k$. We choose $r\in\mathrm{WO}$ such that o.t.$(r)=\varrho(i,j,k)$. To prove $K\oplus G_k\le_{\mathcal K}K\oplus G_i\oplus G_j$, it is sufficient to prove that for all $x,y\in{}^\omega\omega$,

$$\begin{aligned}\langle x,y\rangle\in G_k\Leftrightarrow M\models\text{“}&\exists\sigma\le\max\{\mathrm{rk}(x),\varrho(i,j,k)\}\exists a,z,z'\in L_\sigma[T]\\ &(L_\sigma[T]\text{ is }K^{\mathrm{SJ}}\text{-admissible}\wedge i,j,k\in L_\sigma[T]\\ &\wedge\langle x,z\rangle\in G_i\wedge\langle x,z'\rangle\in G_j\\ &\wedge(f^a(i)=z\wedge f^a(j)=z'\wedge f^a(k)=y)^{L_\sigma[T]})\text{”},\end{aligned}$$

where $M=L_{\omega_1^{K\oplus G_i\oplus G_j;i,j,k,x,y,r}}[K\oplus G_i\oplus G_j;i,j,k,x,y,r]$.

Suppose $\langle x,y\rangle \in G_k$. Similarly to (2), we set $a = g(x)$, $z = f^a(i)$, $z' = f^a(j)$ and choose $\sigma \le \max\{\mathrm{rk}(x), \varrho(i,j,k)\}$ such that $L_\sigma[T]$ is K^{SJ}-admissible and $g(x), i, j, k \in L_\sigma[T]$. Then the right-hand side holds. Conversely, suppose that $x, y \in {}^\omega\omega$ satisfy the right-hand side. Let a, z, z' be as in the right-hand side. Similarly to (2), we have $f^{g(x)}(k) = f^a(k) = y$ by (R.3), and so $\langle x,y\rangle \in G_k$. ■

LEMMA 3.8. *Let $\alpha \in \aleph_1$ and $t \in {}^\omega\omega \cap L_{\tau_\alpha}[T]$ be the $\le_{L[T]}$-least real which satisfies* (G.1) *or* (G.2) *at Stage $\alpha+1$.*

(1) *If $t = \langle 0, e\rangle * \langle v, i, j\rangle$ satisfies* (G.1), *then there is $x \in {}^\omega\omega \cap L_{\tau_{\alpha+1}}[T]$ such that*

$$\chi_{G_i^{\alpha+1}}(x) \not\cong \{e\}(x, v, \chi_{K\oplus G_j^{\alpha+1}}, {}^2E)$$

and so $\chi_{G_i}(x) \not\cong \{e\}(x, v, \chi_{K\oplus G_j}, {}^2E)$.

(2) *If $t = \langle 1, e_0, e_1\rangle * \langle v_0, v_1, i, j, k\rangle$ satisfies* (G.2), *then there is $x \in {}^\omega\omega \cap L_{\tau_{\alpha+1}}[T]$ such that*

$$\{e_0\}(x, v_0, \chi_{K\oplus G_i^{\alpha+1}}, {}^2E) \not\cong \{e_1\}(x, v_1, \chi_{K\oplus G_j^{\alpha+1}}, {}^2E)$$

and so $\{e_0\}(x, v_0, \chi_{K\oplus G_i}, {}^2E) \not\cong \{e_1\}(x, v_1, \chi_{K\oplus G_j}, {}^2E)$.

Proof. Both in (1) and in (2) (i.e. in (G.1) and in (G.2)), since $\varrho(t) \le \tau_\alpha$, $L_{\tau_{\alpha+1}}[T]$ is G_i-admissible and G_j-admissible by Lemma 3.5.

(1) We choose the $\le_{L[T]}$-least $z \in {}^\omega\omega \cap (L_{\tau_{\alpha+1}}[T] - L_{\tau_\alpha}[T])$ and the $\le_{L[T]}$-least $\langle a,b\rangle \in {}^\omega\omega \times {}^\omega\omega$ such that $f^a(j) = f^b(j) \wedge f^a(i) \ne f^b(i)$. We set $z' = \langle z, f^a(i)\rangle$. Then $z' \in L_{\tau_{\alpha+1}}[T]$. Let p^a and p^b be as in Subcase 1.1 at Stage $\alpha+1$.

CASE 1: $\{e\}(z', v, \chi_{K\oplus P_j^a * \mathbf{0}}, {}^2E) \cong 0$. Then, for $l \in \{i,j\}$, $G_l \cap L_{\tau_{\alpha+1}}[T] = G_l^{\alpha+1} = P_l^a * \mathbf{0} \cap L_{\tau_{\alpha+1}}[T]$ by definition. Since $L_{\tau_{\alpha+1}}[T]$ is $(G_j; v, z')$-admissible, $\{e\}(z', v, \chi_{K\oplus G_j}, {}^2E) \cong \{e\}(z', v, \chi_{K\oplus G_j^{\alpha+1}}, {}^2E) \cong 0$. By definition, $z' \in G_i^{\alpha+1} \subseteq G_i$. Hence,

$$\{e\}(z', v, \chi_{K\oplus G_j^{\alpha+1}}, {}^2E) \not\cong 1 \cong \chi_{G_i^{\alpha+1}}(z')$$

and $\{e\}(z', v, \chi_{K\oplus G_j}, {}^2E) \not\cong \chi_{G_i}(z')$.

CASE 2: $\{e\}(z', v, \chi_{K\oplus P_j^a * \mathbf{0}}, {}^2E) \not\cong 0$. Similarly to Case 1,

$$\{e\}(z', v, \chi_{K\oplus G_j}, {}^2E) \cong \{e\}(z', v, \chi_{K\oplus G_j^{\alpha+1}}, {}^2E) \not\cong 0.$$

Since $g(z) = g^{\alpha+1}(z) = b$ and $f^b(i) \ne f^a(i)$, we have $z' \notin G_i^{\alpha+1}$ and $z' \notin G_i$. Hence,

$$\{e\}(z', v, \chi_{K\oplus G_j^{\alpha+1}}, {}^2E) \not\cong 0 \cong \chi_{G_i^{\alpha+1}}(z')$$

and $\{e\}(z', v, \chi_{K\oplus G_j}, {}^2E) \not\cong \chi_{G_i}(z')$.

(2) We choose the $\leq_{L[T]}$-least partial function $p \in L_{\tau_{\alpha+1}}[T]$ from ${}^{\omega}\omega$ to ${}^{\omega}\omega$ as in (G.2). Then, for $l \in \{i,j\}$, $G_l \cap L_{\tau_{\alpha+1}}[T] = G_l^{\alpha+1} = P_l * \mathbf{0} \cap L_{\tau_{\alpha+1}}[T]$. Hence, by (G.2), there is $x \in {}^{\omega}\omega \cap L_{\tau_{\alpha+1}}[T]$ such that

$$\{e_0\}(x, v_0, \chi_{K\oplus G_i^{\alpha+1}}, {}^2E) \not\cong \{e_1\}(x, v_1, \chi_{K\oplus G_j^{\alpha+1}}, {}^2E)$$

and hence $\{e_0\}(x, v_0, \chi_{K\oplus G_i}, {}^2E) \not\cong \{e_1\}(x, v_1, \chi_{K\oplus G_j}, {}^2E)$. ∎

LEMMA 3.9. *For all $t \in {}^{\omega}\omega$, $\{\alpha \in \aleph_1 \mid t$ satisfies (G.1) or (G.2) at Stage $\alpha+1\}$ is countable. Hence $\bigcup_{t <_{L[T]} s}\{\alpha \in \aleph_1 \mid t$ satisfies (G.1) or (G.2) at Stage $\alpha+1\}$ is countable and so bounded for all $s \in {}^{\omega}\omega$ (since $\{t \in {}^{\omega}\omega \mid t <_{L[T]} s\}$ is countable).*

Proof. We set $X_t = \{\alpha \in \aleph_1 \mid t$ satisfies (G.1) or (G.2) at Stage $\alpha+1\}$ for $t \in {}^{\omega}\omega$. We prove that for all $t \in {}^{\omega}\omega$, X_t is countable by induction on t.

Let $t \in {}^{\omega}\omega$ and assume that for all $u \in {}^{\omega}\omega$, if $u <_{L[T]} t$ then X_u is countable. Suppose that, on the contrary, X_t is uncountable. By the inductive assumption $\bigcup_{u <_{L[T]} t} X_u$ is countable, hence we can take $\beta \in X_t - \bigcup_{u <_{L[T]} t} X_u$. Then t is the $<_{L[T]}$-least real which satisfies (G.1) or (G.2) at Stage $\beta+1$. Since X_t is uncountable, there is $\alpha \in X_t$ such that $\beta+1 \leq \alpha$.

CASE 1: t satisfies (G.1) at Stage $\beta+1$. There are $e \in \omega$, $v \in {}^{\omega}\omega$, and $i,j \in \mathcal{L}$ such that $t = \langle 0, e\rangle * \langle v, i, j\rangle$. By Lemma 3.8, there is $x \in {}^{\omega}\omega \cap L_{\tau_{\beta+1}}[T]$ ($\subseteq L_{\tau_\alpha}[T]$) such that $\chi_{G_i^{\beta+1}}(x) \not\cong \{e\}(x, v, \chi_{K\oplus G_j^{\beta+1}}, {}^2E)$. Then, similarly to the proof of Lemma 3.8, since $G_l^{\alpha} \cap L_{\tau_{\beta+1}}[T] = G_l^{\beta+1}$ for $l \in \{i,j\}$ and $L_{\tau_{\beta+1}}[T]$ is G_j-admissible, we have $\chi_{G_i^{\alpha}}(x) \not\cong \{e\}(x, v, \chi_{K\oplus G_j^{\alpha}}, {}^2E)$. Hence, t does not satisfy (G.1) at Stage $\alpha+1$. Moreover, since $t(0) = 0$, t does not satisfy (G.2) at Stage $\alpha+1$. This contradicts $\alpha \in X_t$.

CASE 2: t satisfies (G.2) at Stage $\beta+1$. There are $e_0, e_1 \in \omega$, $v_0, v_1 \in {}^{\omega}\omega$, and $i,j,k \in \mathcal{L}$ such that $t = \langle 1, e_0, e_1\rangle * \langle v_0, v_1, i, j, k\rangle$. Similarly to Case 1, there is $x \in {}^{\omega}\omega \cap L_{\tau_{\beta+1}}[T]$ such that $\{e_0\}(x, v_0, \chi_{K\oplus G_i^{\alpha}}, {}^2E) \not\cong \{e_1\}(x, v_1, \chi_{K\oplus G_j^{\alpha}}, {}^2E)$. Hence, t does not satisfy (G.2) at Stage $\alpha+1$. Moreover, since $t(0) = 1$, t does not satisfy (G.1) at Stage $\alpha+1$. This contradicts $\alpha \in X_t$. ∎

LEMMA 3.10. *For all $i,j \in \mathcal{L}$, if $i \not\leq_{\mathcal{L}} j$, then $K \oplus G_i \not\leq_{\mathcal{K}} K \oplus G_j$.*

Proof. Assume $i \not\leq_{\mathcal{L}} j$ and $G_i \leq_{\mathcal{K}} K \oplus G_j$. We choose $e \in \omega$ and $v \in {}^{\omega}\omega$ such that for all $x \in {}^{\omega}\omega$, $\chi_{G_i}(x) \cong \{e\}(x, v, \chi_{K\oplus G_j}, {}^2E)$. We set $t = \langle 0, e\rangle * \langle v, i, j\rangle$. By Lemma 3.9, we can choose $\alpha \in \aleph_1$ such that for all $u <_{L[T]} t$, u does not satisfy (G.1) or (G.2) (taking u in place of t) at Stage $\alpha+1$. Choosing α sufficiently large, we may assume that there is $\alpha' < \alpha$ such that $\alpha = \alpha'+1$ and $\varrho(t) \leq \tau_{\alpha'}$. Then, by Lemma 3.5, $L_{\tau_\alpha}[T]$

is G_j-admissible, and so by the choice of e, v, for all $x \in {}^\omega\omega \cap L_{\tau_\alpha}[T]$, we have $\chi_{G_i^\alpha}(x) \cong \{e\}(x, v, \chi_{K \oplus G_j^\alpha}, {}^2E)$. Hence, t satsfies (G.1) at Stage $\alpha + 1$. Moreover, t is the $\leq_{L[T]}$-least real which satisfies (G.1) or (G.2) at Stage $\alpha + 1$. Therefore, by Lemma 3.8, there is $x \in {}^\omega\omega \cap L_{\tau_{\alpha+1}}[T]$ such that $\chi_{G_i}(x) \not\cong \{e\}(x, v, \chi_{K \oplus G_j}, {}^2E)$. This is a contradiction. ∎

LEMMA 3.11. *Let $i, j, k \in \mathcal{L}$, $i \wedge^{\mathcal{L}} j = k$, $\alpha \in \aleph_1$, $e_0, e_1 \in \omega$, and $v_0, v_1 \in {}^\omega\omega$. Assume that there are partial functions $p, p' \in L_{\tau_{\alpha+1}}[T]$ from ${}^\omega\omega$ to ${}^\omega\omega$, $\sigma \leq \tau_\alpha$, and $x \in {}^\omega\omega$ such that $g^\alpha \subseteq p, p'$, $\mathrm{Dom}(p) = \mathrm{Dom}(p')$, $P_k = P'_k$, $L_\sigma[T]$ is K^{SJ}-admissible, $i, j, k \in L_\sigma[T]$, $\mathrm{Rng}(p - g^\alpha), \mathrm{Rng}(p' - g^\alpha) \subseteq L_\sigma[T]$, and $\{e_0\}(x, v_0, \chi_{K \oplus P_i * \mathbf{0}}, {}^2E) \not\cong \{e_1\}(x, v_1, \chi_{K \oplus P'_j * \mathbf{0}}, {}^2E)$. Then there is a partial function $p'' \in L_{\tau_{\alpha+1}}[T]$ from ${}^\omega\omega$ to ${}^\omega\omega$ such that $g^\alpha \subseteq p''$, $\mathrm{Rng}(p'' - g^\alpha) \subseteq L_\sigma[T]$, and $\{e_0\}(x, v_0, \chi_{K \oplus P''_i * \mathbf{0}}, {}^2E) \not\cong \{e_1\}(x, v_1, \chi_{K \oplus P''_j * \mathbf{0}}, {}^2E)$.*

P r o o f. We set $D = \mathrm{Dom}(p) - \mathrm{Dom}(g^\alpha)$. Since $P_k = P'_k$, for all $y \in D$, $f^{p(y)}(k) = f^{p'(y)}(k)$. By (R.4*), for all $y \in D$ there are $c_0^y, c_1^y, c_2^y \in {}^\omega\omega \cap L_\sigma[T]$ such that $f^{p(y)} \equiv_i f^{c_0^y} \equiv_j f^{c_1^y} \equiv_i f^{c_2^y} \equiv_j f^{p'(y)}$. Since $p, p', D, L_\sigma[T] \in L_{\tau_{\alpha+1}}[T]$ and $F \leq_{\mathcal{K}} K^{\mathrm{SJ}}$, there exists $\langle \langle c_0^y, c_1^y, c_2^y \rangle \mid y \in D \rangle \in L_{\tau_{\alpha+1}}[T]$ such that for all $y \in D$, $c_0^y, c_1^y, c_2^y \in {}^\omega\omega \cap L_\sigma[T]$ and $f^{p(y)} \equiv_i f^{c_0^y} \equiv_j f^{c_1^y} \equiv_i f^{c_2^y} \equiv_j f^{p'(y)}$ by Δ_1-separation. We define $p^n : \mathrm{Dom}(p) \to {}^\omega\omega$ $(n \in 3)$ by

$$p^n(y) = \begin{cases} g^\alpha(y) & \text{if } y \in \mathrm{Dom}(g^\alpha), \\ c_n^y & \text{if } y \in D. \end{cases}$$

Then $p^n \in L_{\tau_{\alpha+1}}[T]$ and $\mathrm{Rng}(p^n - g^\alpha) \subseteq L_\sigma[T]$ for $n \in 3$. By definition, $P_i = P_i^0$, $P_j^0 = P_j^1$, $P_i^1 = P_i^2$, and $P_j^2 = P'_j$. If we assume that for all $n \in 3$, $\{e_0\}(x, v_0, \chi_{K \oplus P_i^n * \mathbf{0}}, {}^2E) \cong \{e_1\}(x, v_1, \chi_{K \oplus P_j^n * \mathbf{0}}, {}^2E)$, then we obtain $\{e_0\}(x, v_0, \chi_{K \oplus P_i * \mathbf{0}}, {}^2E) \cong \{e_1\}(x, v_1, \chi_{K \oplus P'_j * \mathbf{0}}, {}^2E)$, a contradiction. So there is $n \in 3$ such that $\{e_0\}(x, v_0, \chi_{K \oplus P_i^n}, {}^2E) \not\cong \{e_1\}(x, v_1, \chi_{K \oplus P_j^n}, {}^2E)$. ∎

LEMMA 3.12. *For all $i, j, k \in \mathcal{L}$, if $i \wedge^{\mathcal{L}} j = k$, then $\deg(K \oplus G_k)$ is the $\leq_{\mathcal{K}}$-infimum of $\deg(K \oplus G_i)$ and $\deg(K \oplus G_j)$.*

P r o o f. It is sufficient to prove that for all $X \subseteq {}^\omega\omega$, if $X \leq_{\mathcal{K}} K \oplus G_i$ and $X \leq_{\mathcal{K}} K \oplus G_j$, then $X \leq_{\mathcal{K}} K \oplus G_k$. We fix $X \subseteq {}^\omega\omega$ such that $X \leq_{\mathcal{K}} K \oplus G_i$ and $X \leq_{\mathcal{K}} K \oplus G_j$, and choose $e_0, e_1 \in \omega$ and $v_0, v_1 \in {}^\omega\omega$ such that for all $x \in {}^\omega\omega$, $\chi_X(x) \cong \{e_0\}(x, v_0, \chi_{K \oplus G_i}, {}^2E) \cong \{e_1\}(x, v_1, \chi_{K \oplus G_j}, {}^2E)$. We set $t = \langle 1, e_0, e_1 \rangle * \langle v_0, v_1, i, j, k \rangle$. By Lemma 3.9, we choose $\gamma \in \aleph_1$ such that $\sup(\bigcup_{u <_{L[T]} t} \{\alpha \in \aleph_1 \mid u \text{ satisfies (G.1) or (G.2) at Stage } \alpha + 1\}) < \gamma$ and $\varrho(t) \leq \tau_\gamma$.

CLAIM 1. *For all $\alpha \in \aleph_1$, if $\gamma \leq \alpha$, then there is no partial function $p \in L_{\tau_{\alpha+1}}[T]$ from ${}^\omega\omega$ to ${}^\omega\omega$ as in (G.2) at Stage $\alpha + 1$.*

Proof. Assume $\gamma \le \alpha \in \aleph_1$ and there is a partial function $p \in L_{\tau_{\alpha+1}}[T]$ from ${}^\omega\omega$ to ${}^\omega\omega$ as in (G.2) at Stage $\alpha+1$. Then t satisfies (G.2) at Stage $\alpha+1$ by the choice of e_0, e_1, v_0, v_1. Since $\gamma \le \alpha$, t is the $\le_{L[T]}$-least real which satisfies (G.1) or (G.2) at Stage $\alpha+1$. Thus, by Lemma 3.8, there is $x \in {}^\omega\omega$ such that $\{e_0\}(x, v_0, \chi_{K\oplus G_i}, {}^2E) \not\cong \{e_1\}(x, v_1, \chi_{K\oplus G_j}, {}^2E)$. This is a contradiction and completes the proof of Claim 1.

Claim 2. *For all $\alpha \in \aleph_1$ with $\gamma \le \alpha$ and for all partial functions $p, p' \in L_{\tau_{\alpha+1}}[T]$ from ${}^\omega\omega$ to ${}^\omega\omega$, if $g_\alpha \subseteq p, p'$, $\mathrm{Dom}(p) = \mathrm{Dom}(p')$, $P_k = P'_k$, and there is $\sigma \le \tau_\alpha$ such that $L_\sigma[T]$ is K^{SJ}-admissible, $t \in L_\sigma[T]$, and $\mathrm{Rng}(p - g^\alpha), \mathrm{Rng}(p' - g^\alpha) \subseteq L_\sigma[T]$, then for all $x \in {}^\omega\omega \cap L_{\tau_{\alpha+1}}[T]$, $\{e_0\}(x, v_0, \chi_{K\oplus P_i * \mathbf{0}}, {}^2E) \cong \{e_0\}(x, v_0, \chi_{K\oplus P'_i * \mathbf{0}}, {}^2E)$.*

Proof. Assume $\gamma \le \alpha < \aleph_1$ and Claim 2 does not hold for some partial functions p, p'. Then there is $x \in {}^\omega\omega \cap L_{\tau_{\alpha+1}}[T]$ such that

$$\{e_0\}(x, v_0, \chi_{K\oplus P_i * \mathbf{0}}, {}^2E) \not\cong \{e_0\}(x, v_0, \chi_{K\oplus P'_i * \mathbf{0}}, {}^2E).$$

Since Claim 1 implies that p' is not as in (G.2) at Stage $\alpha+1$,

$$\{e_0\}(x, v_0, \chi_{K\oplus P'_i * \mathbf{0}}, {}^2E) \cong \{e_1\}(x, v_1, \chi_{K\oplus P'_j * \mathbf{0}}, {}^2E).$$

Hence

$$\{e_0\}(x, v_0, \chi_{K\oplus P_i * \mathbf{0}}, {}^2E) \not\cong \{e_1\}(x, v_1, \chi_{K\oplus P'_j * \mathbf{0}}, {}^2E).$$

Thus, by Lemma 3.11, there is a partial function $p'' \in L_{\tau_{\alpha+1}}[T]$ as in (G.2) at Stage $\alpha+1$. This contradicts Claim 1 and completes the proof of Claim 2.

Claim 3. *For all $\alpha \in \aleph_1$ with $\gamma \le \alpha$, set $H_\alpha = G_i^\alpha \cup \{\langle x, y\rangle \in {}^\omega\omega \mid x \notin L_{\tau_\alpha}[T] \wedge \exists a \in {}^\omega\omega(y = f^a(i) \wedge a$ is the $\le_{L[T]}$-least real such that $\langle x, f^a(k)\rangle \in G_k)\}$. Then*:

(1) *H_α is uniformly Δ_1-definable over all T, G_k-admissible sets of which τ_α is an element.*

(2) *For all $x \in {}^\omega\omega \cap L_{\tau_{\alpha+1}}[T]$,*

$$\{e_0\}(x, v_0, \chi_{K\oplus G_i}, {}^2E) \cong \{e_0\}(x, v_0, \chi_{K\oplus H_\alpha}, {}^2E).$$

Proof. (1) It is sufficient to prove that $H_\alpha - G_i^\alpha$ is uniformly Δ_1-definable over all T, G_k-admissible sets of which τ_α is an element. By Lemma 3.4, for all $x \in {}^\omega\omega - L_{\tau_\alpha}[T]$ (notice $\varrho(t) \le \tau_\gamma \le \tau_\alpha \le \mathrm{rk}(x)$), there is $\sigma \le \mathrm{rk}(x)$ such that $L_\sigma[T]$ is K^{SJ}-admissible and $g(x), i, k \in L_\sigma[T]$, and moreover if a is the $\le_{L[T]}$-least real such that $\langle x, f^a(k)\rangle \in G_k$, then since $a \le_{L[T]} g(x)$, we have $a \in L_\sigma[T]$ and so $f^a(k), f^a(i) \in L_\sigma[T]$. Hence, for any T, G_k-admissible set M with $\tau_\alpha \in M$ and for all $x, y \in {}^\omega\omega \cap M$,

$$\begin{aligned}
&\langle x,y\rangle \in H_\alpha - G_i^\alpha\\
&\Leftrightarrow M \models \text{“}x \notin L_{\tau_\alpha}[T] \wedge \exists\sigma \le \mathrm{rk}(x)\exists a \in {}^\omega\omega \cap L_\sigma[T]\\
&\qquad\qquad (L_\sigma[T] \text{ is } K^{\mathrm{SJ}}\text{-admissible} \wedge i,k \in L_\sigma[T]\\
&\qquad\qquad \wedge \exists z \in L_\sigma[T]((z = f^a(k))^{L_\sigma[T]} \wedge \langle x,z\rangle \in G_k)\\
&\qquad\qquad \wedge \forall b,z \in {}^\omega\omega \cap L_\sigma[T]((b <_{L[T]} a \wedge z = f^b(k))^{L_\sigma[T]} \Rightarrow \langle x,z\rangle \notin G_k)\\
&\qquad\qquad \wedge (y = f^a(i))^{L_\sigma[T]})\text{”},
\end{aligned}$$

i.e. the quantifiers in the formula which states "$\langle x,y\rangle \in H_\alpha - G_i^\alpha$" are bounded by $L_\sigma[T]$ and $\mathrm{rk}(x)$.

(2) By definition, there is a partial function $p \in L_{\tau_{\alpha+1}}[T]$ such that $g^\alpha \subseteq p$ and

$$g^{\alpha+1}(x) = \begin{cases} p(x) & \text{if } x \in \mathrm{Dom}(p),\\ \mathbf{0} & \text{if } x \in {}^\omega\omega \cap L_{\tau_{\alpha+1}}[T] - \mathrm{Dom}(p).\end{cases}$$

Then $P_i * \mathbf{0} \cap L_{\tau_{\alpha+1}}[T] = G_i^{\alpha+1} = G_i \cap L_{\tau_{\alpha+1}}[T]$. By Lemma 3.4, there is $\sigma \le \tau_\alpha$ such that $L_\sigma[T]$ is K^{SJ}-admissible, $t \in L_\sigma[T]$, and $\mathrm{Rng}(p - g^\alpha) \subseteq L_\sigma[T]$. We define $p' : \mathrm{Dom}(p) \to {}^\omega\omega$ by

$$p'(x) = \begin{cases} g^\alpha(x) & \text{if } x \in \mathrm{Dom}(g^\alpha),\\ \text{the } \le_{L[T]}\text{-least } a \in {}^\omega\omega & \\ \text{such that } \langle x, f^a(k)\rangle \in G_k & \text{if } x \in \mathrm{Dom}(p) - \mathrm{Dom}(g^\alpha).\end{cases}$$

Then for all $x \in \mathrm{Dom}(p)$ we have $f^{p'(x)}(k) = f^{g(x)}(k) = f^{p(x)}(k)$, and so $P'_k = P_k$. Since $p'(x) \le_{L[T]} g(x)$ for all $x \in \mathrm{Dom}(p)$, it follows that $\mathrm{Rng}(p' - g^\alpha) \subseteq L_\sigma[T]$. Since $L_{\tau_{\alpha+1}}[T]$ is G_k-admissible, similarly to (1), p' is Δ_1 over $L_{\tau_{\alpha+1}}[T]$ and so $p' \in L_{\tau_{\alpha+1}}[T]$ by Δ_1-separation. Moreover, $P'_i * \mathbf{0} \cap L_{\tau_{\alpha+1}}[T] = H_\alpha \cap L_{\tau_{\alpha+1}}[T]$ by definition (notice the assumption that $\mathbf{0}$ is the $\le_{L[T]}$-least real). Thus, by Claim 2, for all $x \in {}^\omega\omega \cap L_{\tau_{\alpha+1}}[T]$, $\{e_0\}(x, v_0, \chi_{K\oplus P_i*\mathbf{0}}, {}^2E) \cong \{e_0\}(x, v_0, \chi_{K\oplus P'_i*\mathbf{0}}, {}^2E)$ and hence

$$\{e_0\}(x, v_0, \chi_{K\oplus G_i}, {}^2E) \cong \{e_0\}(x, v_0, \chi_{K\oplus H_\alpha}, {}^2E).$$

This completes the proof of Claim 3.

Let $x \in {}^\omega\omega - L_{\tau_\gamma}[T]$ and $n \in 2$, and $M = L_{\omega_1^{K\oplus G_k;x}}[K \oplus G_k; x]$. By Lemma 2.2, M is T-admissible, and if $x \in L_{\tau_{\alpha+1}}[T] - L_{\tau_\alpha}[T]$, then $\gamma \le \alpha$ and $\tau_\alpha \le \mathrm{rk}(x) \in \tau_{\alpha+1} \cap M$. Hence by Claim 3,

$$\begin{aligned}
&\chi_X(x) \cong n\\
&\quad \Leftrightarrow \exists\alpha \in \aleph_1(x \in L_{\tau_{\alpha+1}}[T] - L_{\tau_\alpha}[T] \wedge \{e_0\}(x, v_0, \chi_{K\oplus H_\alpha}, {}^2E) \cong n)\\
&\quad \Leftrightarrow M \models \text{“}\exists\alpha \le \mathrm{rk}(x)(\tau_\alpha \le \mathrm{rk}(x)\\
&\qquad\qquad \wedge \neg\exists\tau \le \mathrm{rk}(x)(\tau_\alpha < \tau \wedge \tau \text{ satisfies (T.1)})\\
&\qquad\qquad \wedge \{e_0\}(x, v_0, \chi_{K\oplus H_\alpha}, {}^2E) \cong n)\text{”}.
\end{aligned}$$

Therefore, $X - L_{\tau_\gamma}[T]$ and $({}^\omega\omega - X) - L_{\tau_\gamma}[T]$ are uniformly Σ_1-definable over all $(K \oplus G_k; w)$-admissible sets, where w is a real in WO such that o.t.$(w) = \tau_\gamma$. Since $L_{\tau_\gamma}[T]$ is countable, $X \leq_{\mathcal{K}} K \oplus G_k$. ■

This completes the proof of the Theorem.

Remark. In the Theorem, we may replace "$(\mathcal{L}, \leq_{\mathcal{L}}, \vee_{\mathcal{L}}, \wedge^{\mathcal{L}})$ is Kleene recursive in K^{SJ}" by "$(\mathcal{L}, \leq_{\mathcal{L}}, \vee_{\mathcal{L}}, \wedge^{\mathcal{L}})$ is Kleene recursive in the finite times superjump of K".

Concerning, for example, $(K^{\mathrm{SJ}})^{\mathrm{SJ}}$, for any K-admissible set N, N is closed under $\lambda x.\omega_1^{K;x}$ and $\lambda x.\omega_1^{K^{\mathrm{SJ}};x}$ iff N is $(K^{\mathrm{SJ}})^{\mathrm{SJ}}$-admissible, and the quantifiers in the statement "N is closed under $\lambda x.\omega_1^{K^{\mathrm{SJ}};x}$" are bounded by N as "$\forall x \in {}^\omega\omega \cap N \exists\alpha \in \mathrm{On} \cap N(L_\alpha[K;x]$ is $(K;x)$-admissible$\wedge \forall y \in {}^\omega\omega \cap L_\alpha[K;x] \exists\beta < \alpha(L_\beta[K;y]$ is $(K;y)$-admissible$))^N$". Replacing "$L_\sigma[T]$ is K^{SJ}-admissible" by "$L_\sigma[T]$ is $(K^{\mathrm{SJ}})^{\mathrm{SJ}}$-admissible" in the proof of the Theorem, we can prove the following:

Theorem′ (ZFC+CH). *Let $K_0 \oplus K_1 \leq_{\mathcal{K}} K \subseteq {}^\omega\omega$. For any lattice $\mathcal{L}$, if $\mathcal{L} \subseteq {}^\omega\omega$ and $(\mathcal{L}, \leq_{\mathcal{L}}, \vee_{\mathcal{L}}, \wedge^{\mathcal{L}})$ is Kleene recursive in $(K^{\mathrm{SJ}})^{\mathrm{SJ}}$, then $\mathcal{L}$ can be embedded in $\mathcal{K}[K, K^{\mathrm{SJ}}]$.*

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