Contents of Volume 132, Number 3

STUDIA MATHEMATICA

Executive Editors: **Z. Ciesielski, A. Pełczyński, W. Żelazko**

The journal publishes original papers in English, French, German and Russian, mainly in functional analysis, abstract methods of mathematical analysis and probability theory. Usually 3 issues constitute a volume.

Detailed information for authors is given on the inside back cover. Manuscripts and correspondence concerning editorial work should be addressed to

STUDIA MATHEMATICA

Śniadeckich 8, P.O. Box 137, 00-950 Warszawa, Poland, fax 48-22-6293997
E-mail: studia@impan.gov.pl

Subscription information (1999): Vols. 132–137 (18 issues); $33.50 per issue.

Correspondence concerning subscription, exchange and back numbers should be addressed to

Institute of Mathematics, Polish Academy of Sciences
Publications Department

Śniadeckich 8, P.O. Box 137, 00-950 Warszawa, Poland, fax 48-22-6293997
E-mail: publ@impan.gov.pl

Published by the Institute of Mathematics, Polish Academy of Sciences
Typeset using TEX at the Institute
Printed and bound by

drukarnia
herman & herman
SPÓŁKA CYWILNA

PRINTED IN POLAND

ISSN 0039-3223

Strictly singular operators and the invariant subspace problem

by

C. J. READ (Cambridge)

Abstract. Properties of strictly singular operators have recently become of topical interest because the work of Gowers and Maurey in [GM1] and [GM2] gives (among many other brilliant and surprising results, such as those in [G1] and [G2]) Banach spaces on which every continuous operator is of form $\lambda I + S$, where S is strictly singular. So if strictly singular operators had invariant subspaces, such spaces would have the property that all operators on them had invariant subspaces. However, in this paper we exhibit examples of strictly singular operators without nontrivial closed invariant subspaces. So, though it may be true that operators on the spaces of Gowers and Maurey have invariant subspaces, yet this cannot be because of a general result about strictly singular operators. The general assertion about strictly singular operators is false.

0. Introduction

0.1. The author would like to thank Prof. A. Pełczyński for suggesting this line of investigation to him.

0.2. Operators without invariant subspaces were first found independently by P. Enflo and this author ([E1], [R1]), on an unknown Banach space. They were found on l_1 and c_0 by the present author ([R4], [R7]) and various extensions of the method were found ([R2], [R3], [R5], [R6], [R8]), of which the nearest to the present paper is the construction of a quasinilpotent operator without invariant subspaces on l_1 in [R8]. A general account of the theory of invariant subspaces, written before all these counterexamples were discovered, will be found in [Ra1]. A short account of the basic properties of the James space J will be found in Singer [S1], pp. 273–279. The original article is [J1].

0.3. A continuous linear map $T : E \to F$, where E and F are normed spaces, is *norm increasing* if there is an $\varepsilon > 0$ such that

1991 *Mathematics Subject Classification*: Primary 47A15.
Key words and phrases: operator, invariant subspace, strictly singular.

(0.3.0) $$\|Tx\| \geq \varepsilon\|x\|$$

for all $x \in E$.

0.4. A continuous linear map $T : E \to F$, where E and F are Banach spaces, is said to be *strictly singular* if there is no infinite-dimensional subspace $W \subset E$ such that $T|_W$ is norm increasing.

0.5. The *James p-space* J_p $(1 < p < \infty)$ is the set of all sequences $(a_i)_{i=1}^{\infty} \in c_0$ such that

$$\|a\| = \sup\Big\{ \Big(\sum_{j=2}^{n} |a_{i_j} - a_{i_{j-1}}|^p \Big)^{1/p} : i_1 < \ldots < i_n,\ n \in \mathbb{N} \Big\} < \infty.$$

It is a fact that J_p is nonreflexive, $\dim(J_p^{**}/J_p) = 1$, but that every infinite-dimensional subspace of J_p contains a subspace isomorphic to l_p.

1. Strictly singular noncompact operators. It is well known that on l_p $(1 \leq p < \infty)$ or c_0, any strictly singular operator is compact. On the other hand, the inclusion map $l_p \hookrightarrow J_q$ $(1 < p < q < \infty)$ is strictly singular but not compact. For our purposes we want something like the inclusion map $l_p \hookrightarrow l_q$, but which happens between nonreflexive Banach spaces (and which happens, let it be said, in a manner which has respect for the nonreflexivity, in the sense that there is a sequence of unit vectors in the domain space with no weak-* convergent subsequence, which is mapped to a sequence in the image space which also has no weak-* convergent subsequence).

For such a map we look to the James p-spaces J_p.

LEMMA 1.1. *The natural inclusion $i : J_p \hookrightarrow J_q$ $(1 < p < q < \infty)$ is strictly singular.*

Proof. If not, there is an infinite-dimensional subspace $E \subset J_p$ on which the norms $\|\cdot\|_{J_p}$ and $\|\cdot\|_{J_q}$ are equivalent. Taking a subspace of E as necessary, this tells us that $(E, \|\cdot\|_{J_q})$ is isomorphic to l_p (for every infinite-dimensional subspace of J_p contains a subspace isomorphic to l_p; see 0.5). Taking a further subspace, we find l_q embedded up to isomorphism in l_p, which is nonsense. ■

DEFINITION 1.2. Let us choose, once and for all, a strictly increasing sequence $(p_i)_{i=1}^{\infty}$ of real numbers strictly greater than 2. The Banach space X is defined as the l_2-direct sum

(1.2.0) $$X = \Big(l_2 \bigoplus_{i=1}^{\infty} J_{p_i}\Big)_{l_2}.$$

It is on this Banach space X that we will construct a strictly singular operator without invariant subspaces. We will write $(f_{ij})_{j=0}^{\infty}$ for the unit vector basis of J_{p_i}, and $(f_{0j})_{j=0}^{\infty}$ for the unit vector basis of the space l_2.

An element $\mathbf{x} \in X$ can be regarded as a sequence $(\mathbf{x}_i)_{i=0}^{\infty}$ with $\mathbf{x}_0 \in l_2$, $\mathbf{x}_i \in J_{p_i}$ $(i > 0)$. It can be shown that if $(\delta_i)_{i=0}^{\infty}$ is a sequence of scalars tending to zero, then the "weighted shift" operator

$$W : (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots) \to (\mathbf{0}, \delta_0\mathbf{x}_0, \delta_1\mathbf{x}_1, \delta_2\mathbf{x}_2, \ldots)$$

is strictly singular (see §3.3). We will construct an operator on X without invariant subspaces, which has a good deal in common with a weighted shift W.

The next few definitions follow [R4].

DEFINITION 1.3. Our construction will be built around a strictly increasing sequence $\mathbf{d} = (d_i)_{i=1}^{\infty}$ of positive integers. This sequence will be required to "increase sufficiently rapidly" in the sense of [R1], §1. We will write $a_i = d_{2i-1}$ $(i = 1, 2, \ldots)$ and $b_i = d_{2i}$. Thus, $a_1 < b_1 < a_2 < b_2 < \ldots$ We define $a_0 = 1$, $v_0 = 0$, $v_n = n(a_n + b_n)$ $(n > 0)$. We will use the symbol **pd** to mean, "provided **d** increases sufficiently rapidly", as we did in [R4]. We define $w_n = 1 + \sum_{r=0}^{n-1}(1 + v_r)$, $w_0 = 1$.

DEFINITION 1.4. F will denote the dense subspace of X spanned by the unit vectors $\{f_{ij} : i \geq 0,\ j \geq 0\}$. If $S \subset \mathbb{Z}^+ \times \mathbb{Z}^+$, then $F_S \subset F$ will denote the linear span of the set $\{f_{ij} : (i, j) \in S\}$. Further, π_S will denote the projection $F \to F_S$ such that $\pi_S(f_{ij}) = f_{ij}$ $((i, j) \in S)$ or 0 $((i, j) \notin S)$. This π_S is continuous only for certain choices of S; we shall not be using any S for which it is discontinuous, however. f_{ij}^* will denote the norm-1 linear functional on F such that $f_{ij}^*(f_{kl}) = \delta_{ik}\delta_{jl}$.

DEFINITION 1.5. Let $|p|$ denote the sum of the absolute values of the coefficients of the polynomial p. For a finite set S, let $|S|$ denote the number of elements of S.

2. The main definition. We will now define, in terms of the sequence $\mathbf{d}$ as in 1.3, a sequence $(e_i)_{i=0}^{\infty}$ whose linear span is the dense subspace F of X.

We shall begin by rearranging the fundamental set $(f_{j,k})_{j,k=0}^{\infty}$ into a fundamental sequence $(f_i)_{i=0}^{\infty}$. Each $f_{j,k}$ is equal to $f_{\mathcal{I}(j,k)}$, where $\mathcal{I} : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$ is a suitable bijection (see Definition 2.4). We will write F_n for the linear span $\text{lin}\{f_0, \ldots, f_n\}$—a special case of the subspaces F_S as in Definition 1.4. This particular choice of S will be called S_n, the unique subset $\mathcal{I}^{-1}([0, n])$ of $\mathbb{Z}^+ \times \mathbb{Z}^+$ such that

(2.0.0) $$\text{lin}\{f_0, \ldots, f_n\} = \text{lin}\{f_{i,j} : (i, j) \in S_n\}.$$

We then define linear relationships of general form $f_i = \sum_{j=0}^{i}\lambda_{i,j}e_j$, with $\lambda_{i,i} \neq 0$, for each $i \in \mathbb{Z}^+$ (this is done in Definition 2.5). Because the matrix with entries $\lambda_{i,j}$ is lower triangular with nonzero diagonal entries, this linear map can be inverted providing us with an alternative vector space basis $(e_i)_{i=0}^{\infty}$ of F, given uniquely as linear combinations of general form $e_i = \sum_{j=0}^{i} \varrho_{i,j} f_j$.

So there is a unique linear map $T : F \to F$ that acts as a right shift operator sending each e_i to e_{i+1}. It turns out that **pd**, T extends to a continuous operator $X \to X$ that is strictly singular, and has no nontrivial closed invariant subspaces.

DEFINITION 2.1. Let the sequence **d** be given. Let $\Omega \subset \mathbb{Z}^+$ be the set

$$(2.1.0)\qquad \bigcup_{n=1}^{\infty}\bigcup_{r=1}^{n}[ra_n, ra_n + v_{n-r}] = \bigcup_{s=0}^{\infty}\bigcup_{n=s+1}^{\infty}[(n-s)a_n, (n-s)a_n + v_s].$$

Provided **d** increases sufficiently rapidly, the union (2.1.0) is disjoint, and both Ω and $\mathbb{Z}^+\setminus\Omega$ are infinite sets. If **d** does indeed increase sufficiently fast for this to happen, we make the following definitions:

DEFINITION 2.2. Let γ be the unique increasing bijection $\mathbb{Z}^+\setminus\Omega \to \mathbb{Z}^+$.

DEFINITION 2.3. (a) For each $s \ge 0$, let σ_s be the natural bijection from the set $\bigcup_{n=s+1}^{\infty}[(n-s)a_n, (n-s)a_n + v_s] \subset \mathbb{Z}^+$ to the set $[0, v_s] \times \mathbb{Z}^+ \subset \mathbb{Z}^+ \times\mathbb{Z}^+$ that sends the integer $(n-s)a_n+i$ $(0 \le i \le v_s)$ to the pair $(i, n-s-1)$.

(b) Define maps χ_s, each with the same domain as σ_s, by $\chi_s(j) = \sigma_s(j)+(w_s, 0)$, so that the image of χ_s is equal to $[w_s, w_s+v_s]\times\mathbb{Z}^+ = [w_s, w_{s+1})\times\mathbb{Z}^+$ (for $w_{s+1} = w_s + v_s + 1$, by Definition 1.3).

(c) Let $\chi : \Omega \to \mathbb{Z}^+ \times \mathbb{Z}^+$ be the unique map whose restriction to each subset $\bigcup_{n=s+1}^{\infty}[(n-s)a_n, (n-s)a_n + v_s]$ of Ω is equal to χ_s.

Now χ is a bijection from Ω onto $[w_0, \infty) \times \mathbb{Z}^+$, that is, onto $\mathbb{N} \times \mathbb{Z}^+$. We may obtain a bijection $\mathbb{Z}^+ \to \mathbb{Z}^+ \times \mathbb{Z}^+$ by making sure that $\mathbb{Z}^+ \setminus \Omega$ gets mapped onto $\{0\} \times \mathbb{Z}^+$, thus:

DEFINITION 2.4. Let us extend χ to a map $\mathbb{Z}^+ \to \mathbb{Z}^+ \times \mathbb{Z}^+$ by defining $\chi(i) = (0, \gamma^{-1}(i))$ for each $i \notin \Omega$. Since χ is always a bijection, we may also define the map $\mathcal{I} = \chi^{-1} : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$.

DEFINITION 2.5. Let the sequence **d** be given, and let it increase sufficiently fast that the maps χ and $\mathcal{I}$ are defined. For each i then, we define $f_i = f_{j,k}$, where $(j, k) = \chi(i)$. We shall show that, **pd**, there is a unique sequence $(e_i)_{i=0}^{\infty}$ in F with the following properties. Firstly,

$$(2.5.0)\qquad f_0 = e_0.$$

Secondly, if integers r, n, i satisfy $0 < r \le n$, $i \in [0, v_{n-r}] + ra_n$, then

$$(2.5.1)\qquad f_i = (n-r+1)^{i-ra_n} a_{n-r}((1+n)^{ra_n} e_i - n^{(r-1)a_{n-1}} e_{i-ra_n+(r-1)a_{n-1}}).$$

Thirdly, if $0 < r < n$, $i \in (ra_n + v_{n-r}, (r+1)a_n)$ (respectively, if $1 \le n$, $i \in (v_{n-1}, a_n)$), then

$$(2.5.2)\qquad f_i = (1+n)^i 2^{(h-i)/\sqrt{a_n}} e_i$$

where $h = (r+1/2)a_n$ (respectively, $h = (1/2)a_n$). If integers r, n, i satisfy $0 < r \le n$, $i \in [r(a_n + b_n), na_n + rb_n]$, then

$$(2.5.3)\qquad f_i = (1+n)^i e_i - b_n(1+n)^{i-b_n} e_{i-b_n}.$$

If integers r, n, i satisfy $0 \le r < n$, $i \in (na_n + rb_n, (r+1)(a_n + b_n))$, then

$$(2.5.4)\qquad f_i = (1+n)^i 2^{(h-i)/\sqrt{b_n}} e_i$$

where $h = (r+1/2)b_n$.

LEMMA 2.6. **Pd**, *the sequence* $(e_i)_{i=0}^{\infty}$ *satisfying* (2.5.0)–(2.5.4) *does indeed exist, is unique, and is a vector space basis of* F. *There is a unique linear map* $T : F \to F$ *such that* $Te_i = e_{i+1}$ *for each* i.

Proof. Each definition is of form $f_i = \sum_{l=0}^{i} \lambda_{il} e_l$, with $\lambda_{ii} \ne 0$. The values taken by the index i in formulae (2.5.0)–(2.5.2) include zero, $[0, v_{n-r}] + ra_n$ $(0 < r \le n)$; $(ra_n+v_{n-r}, (r+1)a_n)$ $(0 < r < n)$; and (v_{n-1}, a_n) $(1 \le n)$. **Pd**, this means each value $i = 0$ or $i \in (v_{n-1}, na_n]$ $(n \ge 1)$ is mentioned once and only once.

The remaining values of i are taken care of by (2.5.3), (2.5.4). These cases cover intervals $[r(a_n + b_n), na_n + rb_n]$ $(0 < r \le n)$ and $(na_n + rb_n, (r+1) \times (a_n + b_n))$ $(0 \le r < n)$, whose union is $(na_n, n(a_n + b_n)] = (na_n, v_n]$. As the index n varies, we catch the rest of $\mathbb{Z}^+$.

Pd, then, each f_i $(i \ge 0)$ is defined once and only once, and has the general form $\sum_{l=0}^{i} \lambda_{il} e_l$.

Because $\lambda_{ii} \ne 0$ the linear relationship between the e_i and the f_i is invertible (we have a lower triangular matrix with nonzero entries in the diagonal) so the e_i do exist, are unique, and span F. Note by the way that if $i = \mathcal{I}(j, k)$ then

$$(2.6.0)\qquad f_{jk}^*(\lambda_{ii} e_i) = 1$$

since $f_{jk}^*(f_i) = 1$, and obviously $f_{jk}^*(e_m) = 0$ for $m < i$.

It is then also true that for each n,

$$\operatorname{lin}\{e_0, \ldots, e_n\} = \operatorname{lin}\{f_{j,k} : \mathcal{I}(j, k) = 0, 1, \ldots, n\} = F_{S_n}$$

say, where $S_n = \chi\{0, 1, \ldots, n\}$, $|S_n| = n + 1$. As we remarked at the beginning of §2, we will abbreviate F_{S_n} to F_n. $(e_i)_{i=0}^{\infty}$ is an alternative vector space basis for F, so of course there is a unique map T such that $Te_i = e_{i+1}$ for all i—as yet we say nothing about continuity! ∎

From now on, we will always assume that the given sequence **d** increases sufficiently rapidly for Lemma 2.6 to hold.

Obviously we must now prove that (pd) T is continuous and strictly singular. This is the object of the next section.

3. T is continuous and strictly singular

3.1. *Continuity of T.* The method of achieving this result is to approximate T by an appropriate "weighted shift" operator W, and then estimate the norm of the "error term" $T-W$ by *ad hoc* methods. This also gives us a natural direction to take when proving that T is strictly singular.

DEFINITION 3.2. Let $W_0 : l_2 \to l_2$ be a weighted shift operator with $W_0 f_{0j} = \alpha_j f_{0,j+1}$; we define the weights α_j as follows. Writing $i=\gamma^{-1}(j)$, we know that either i is zero, or it lies in one of the intervals (v_{n-1}, a_n), $(ra_n+v_{n-r}, (r+1)a_n)$, $(na_n+rb_n, (r+1)(a_n+b_n))$ or $[r(a_n+b_n), na_n+rb_n]$ that feature in parts (2.5.2), (2.5.3) and (2.5.4) of Definition 2.5. With an eye on that definition, we set

$$\alpha_j = \begin{cases} \dfrac{2^{1/\sqrt{a_n}}}{1+n} & \text{if } i \in (ra_n + v_{n-r}, (r+1)a_n - 1) \\ & \quad (0<r<n) \text{ or } i \in (v_{n-1},\, a_n - 1)\ (n>0), \\ \dfrac{1}{1+n} & \text{if } i \in [r(a_n+b_n),\, na_n+rb_n)\ (0<r\le n), \\ \dfrac{2^{1/\sqrt{b_n}}}{1+n} & \text{if } i \in (na_n+rb_n, (r+1)(a_n+b_n)-1) \\ & \quad (0\le r<n), \\ 0 & \text{otherwise.} \end{cases} \tag{3.2.0}$$

It is easily checked from Definition 2.5 that in the cases when $\alpha_j \ne 0$, $W_0 f_{0j} = T f_{0j}$. For example, if $j=\gamma(i)$ with $i \in ((na_n+rb_n), (r+1)\times(a_n+b_n)-1)$, then both i and $i+1$ lie in the interval $((na_n+rb_n), (r+1)(a_n+b_n))$, hence for suitable h, we have

$$f_{0,j} = f_i = (1+n)^i 2^{(h-i)/\sqrt{b_n}} e_i$$

and

$$f_{0,j+1} = f_{i+1} = (1+n)^{i+1} 2^{(h-i-1)/\sqrt{b_n}} e_{i+1}.$$

Hence,

$$T f_{0,j} = \frac{1}{n+1} 2^{1/\sqrt{b_n}} f_{0,j+1} = W_0 f_{0,j}.$$

W_0 is a weighted shift operator on l_2, obviously of norm $\frac12 \cdot 2^{1/\sqrt{b_1}}$ (if we assume the interval (a_1, a_1+b_1-1) is nonempty, a rather mild condition of "rapid increase" on the sequence $\mathbf{d}$). Note it is also compact, for the weights tend to zero.

DEFINITION 3.3. Let $W_1 : (\bigoplus_{i=1}^\infty J_{p_i})_{l_2} \to (\bigoplus_{i=1}^\infty J_{p_i})_{l_2}$ be the map such that the sequence $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots)$ with $\mathbf{x}_i \in J_{p_i}$ is sent to the sequence

$(0, \beta_1\mathbf{x}_1, \beta_2\mathbf{x}_2, \beta_3\mathbf{x}_3, \ldots)$ where the coefficients β_i are as follows:

$$\beta_j = \begin{cases} \dfrac{1}{m+1} & \text{if } j \in [w_m,\, w_m+v_m)\ (m\ge 0), \\ 0 & \text{if } j = w_m+v_m\ (m \ge 0). \end{cases} \tag{3.3.0}$$

Once again, in the cases when $\beta_j \ne 0$, the actions of T on J_{p_j} and of W_1 are identical. For if $j \in [w_s, w_s+v_s)$ and $k\ge 0$, let us write $n=k+1+s$ and $i=j-w_s$. Then Definition 2.3 gives us $\chi((n-s)a_n+i) = (w_s+i, n-s-1) = (j,k)$; and likewise $\chi((n-s)a_n+i+1) = (j+1,k)$. Therefore, Definition 2.1 gives us

$$\begin{aligned} f_{j,k} &= f_{(n-s)a_n+i} \\ &= (s+1)^i a_s((n+1)^{(n-s)a_n} e_{(n-s)a_n+i} - n^{(n-s-1)a_{n-1}} e_{(n-s-1)a_{n-1}+i}), \end{aligned}$$

and

$$\begin{aligned} f_{j+1,k} &= f_{(n-s)a_n+i+1} \\ &= (s+1)^{i+1} a_s((n+1)^{(n-s)a_n} e_{(n-s)a_n+i+1} \\ &\quad - n^{(n-s-1)a_{n-1}} e_{(n-s-1)a_{n-1}+i+1}). \end{aligned}$$

Hence

$$T f_{j,k} = \frac{1}{s+1} f_{j+1,k} = \beta_j f_{j+1,k}$$

for all $k \ge 0$; this of course agrees with $W_1 f_{j,k}$.

Since the embedding $J_{p_i} \hookrightarrow J_{p_{i+1}}$ is (strictly singular and) of norm 1, we conclude that $\|W_1\| = 1/2$. Writing W for the unique continuous linear map $X \to X$ which agrees with W_0 on l_2 and W_1 on $\bigoplus_{i=1}^\infty J_{p_i}$, we have $\|W\| = \frac12 \cdot 2^{1/\sqrt{b_1}}$. The "error term" $T-W$ acts as follows. By (2.5.0) and (2.5.1),

$$(T-W) f_{0,0} = e_1 = \frac12 \cdot 2^{(1-a_1/2)/\sqrt{a_1}} f_{0,1} \tag{3.3.1}$$

By (2.5.2), if $i=(r+1)a_n-1$ and $j=\gamma(i)$ $(0\le r<n)$ then

$$(T-W) f_{0,j} = (1+n)^{(r+1)a_n-1} 2^{(1-a_n/2)/\sqrt{a_n}} e_{(r+1)a_n}. \tag{3.3.2}$$

By (2.5.1), if $j=w_m+v_m$ and $k\ge 0$ then writing $r=k+1$, $n=m+k+1$ and $i=ra_n+j-w_m=ra_n+v_{n-r}$, we have

$$\begin{aligned} (T-W) f_{j,k} &= (n-r+1)^{v_{n-r}} a_{n-r}((n+1)^{ra_n} e_{1+ra_n+v_{n-r}} \\ &\quad - n^{(r-1)a_{n-1}} e_{1+(r-1)a_{n-1}+v_{n-r}}). \end{aligned} \tag{3.3.3}$$

By (2.5.3), if $j=\gamma(i)$, $i=na_n+rb_n$, $0<r\le n$, then

$$\begin{aligned} (T-W) f_{0,\gamma(i)} &= (1+n)^i e_{i+1} - b_n(1+n)^{i-b_n} e_{i+1-b_n} \\ &= (1+n)^{na_n+rb_n} e_{1+na_n+rb_n} \\ &\quad - b_n(1+n)^{na_n+(r-1)b_n} e_{1+na_n+(r-1)b_n}. \end{aligned} \tag{3.3.4}$$

Lastly, by (2.5.4), if $j=\gamma(i)$, $i=(r+1)(a_n+b_n)-1$, $0\le r<n$, then

$$(3.3.5)\qquad (T-W)f_{0,j}=(1+n)^{(r+1)(a_n+b_n)-1}\times 2^{(1-b_n/2+(r+1)a_n)/\sqrt{b_n}}e_{(r+1)(a_n+b_n)}.$$

In all other cases, $(T-W)f_{i,j}=0$.

LEMMA 3.4. *For every $\eta>0$ the following is true*: **pd**, $T-W$ *is a nuclear operator of nuclear norm at most* η.

Proof. It is necessary to estimate the sum of the norms of all the vectors in (3.3.1)–(3.3.5), add up the estimates and check that (**pd**) the sum is less than η. These sort of details will be very familiar to readers of [R1]–[R8].

Obviously (3.3.1) contributes $2^{-1+(1-a_1/2)/\sqrt{a_1}}$ to our sum (which is less than $\eta/5$, **pd**, let us say). Now (2.5.1) gives us (for $0<r\le n$)

$$(3.4.0)\qquad \begin{aligned}(1+n)^{ra_n}e_{ra_n}&=a_{n-r}^{-1}f_{w_{n-r},r-1}+n^{(r-1)a_{n-1}}e_{(r-1)a_{n-1}}\\&=a_{n-r}^{-1}(f_{w_{n-r},r-1}+f_{w_{n-r},r-2})\\&\quad+(n-1)^{(r-2)a_{n-2}}e_{(r-2)a_{n-2}}\\&=\dots\\&=a_{n-r}^{-1}\sum_{s=0}^{r-1}f_{w_{n-r},s}+e_0.\end{aligned}$$

Now the J_p spaces have the special property—closely related to their non-reflexivity—that for all r,j we have $\|\sum_{s=0}^{r-1}f_{j,s}\|=1$. So,

$$(3.4.1)\qquad \|(1+n)^{ra_n}e_{ra_n}-e_0\|=1/a_{n-r}$$

and

$$(3.4.2)\qquad \|(1+n)^{ra_n}e_{ra_n}\|=\sqrt{1+a_{n-r}^{-2}}$$

Hence, (3.3.2) contributes to our sum at most

$$(3.4.3)\qquad \sum_{0\le r<n}\frac{\sqrt{1+a_{n-r+1}^{-2}}}{n+1}2^{(1-a_n/2)/\sqrt{a_n}}<\frac{\eta}{5},$$

pd. In view of (2.5.2), if $0<r<n$ then

$$(3.4.4)\qquad \|e_{1+ra_n+v_{n-r}}\|=(1+n)^{-ra_n-1-v_{n-r}}2^{(1+v_{n-r}-a_n/2)/\sqrt{a_n}}.$$

If $r=n>0$, (2.5.4) gives

$$(3.4.5)\qquad \|e_{1+ra_n+v_{n-r}}\|=\|e_{1+na_n}\|=(1+n)^{-na_n-1}2^{(1+na_n-b_n/2)/\sqrt{b_n}}.$$

If $r=0<n$, (2.5.2) gives

$$(3.4.6)\qquad \|e_{1+ra_n+v_{n-r}}\|=\|e_{1+v_n}\|=(2+n)^{-v_n-1}2^{(1+v_n-a_{n+1}/2)/\sqrt{a_{n+1}}}.$$

Hence the contribution made by (3.3.3) to our sum is at most

$$(3.4.7)\qquad \begin{aligned}&\sum_{n=1}^{\infty}\sum_{r=1}^{n}(n-r+1)^{v_{n-r}}((n+1)^{ra_n}\|e_{1+ra_n+v_{n-r}}\|\\&\qquad+n^{(r-1)a_{n-1}}\|e_{1+(r-1)a_{n-1}+v_{n-r}}\|)\\&=\sum_{n=1}^{\infty}\sum_{r=1}^{n-1}(1+n)^{-1-v_{n-r}}2^{(1+v_{n-r}-a_n/2)/\sqrt{a_n}}(n-r+1)^{v_{n-r}}a_{n-r}\\&\quad+\sum_{n=1}^{\infty}\frac{1}{n+1}2^{(1+na_n-b_n/2)/\sqrt{b_n}}\\&\quad+\sum_{n=1}^{\infty}\sum_{r=2}^{n}n^{-1-v_{n-r}}2^{(1+v_{n-r}-a_{n-1}/2)/\sqrt{a_{n-1}}}(n-r+1)^{v_{n-r}}a_{n-r}\\&\quad+\sum_{n=1}^{\infty}2^{(1+v_{n-1}-a_n/2)/\sqrt{a_n}}n^{v_{n-1}}(n+1)^{-1-v_{n-1}}<\frac{\eta}{5},\end{aligned}$$

pd (the first two terms on the right hand side of (3.4.7) are summing appropriate multiples of the norms of the vectors $e_{1+ra_n+v_{n-r}}$ on the left hand side; the last two terms do the same for the vectors $e_{1+(r-1)a_{n-1}+v_{n-r}}$).

Then again, (2.5.4) gives us

$$(3.4.8)\qquad \|e_{1+na_n+rb_n}\|=(1+n)^{-1-na_n-rb_n}2^{(1+na_n-b_n/2)/\sqrt{b_n}}$$

when $0\le r<n$; if $r=n$ we are looking at $\|e_{1+v_n}\|$, which is given by (3.4.6). Hence the contribution to our sum made by (3.3.4) is at most

$$(3.4.9)\qquad \begin{aligned}&\sum_{n=2}^{\infty}\sum_{r=1}^{n-1}(1+n)^{-1}2^{(1+na_n-b_n/2)/\sqrt{b_n}}\\&\qquad+\sum_{n=1}^{\infty}\frac{(1+n)^{v_n}}{(2+n)^{1+v_n}}2^{(1+v_n-a_{n+1}/2)/\sqrt{a_{n+1}}}\\&\qquad+\sum_{n=1}^{\infty}\sum_{r=1}^{n}\frac{b_n}{n+1}2^{(1+na_n-b_n/2)/\sqrt{b_n}}<\frac{\eta}{5},\end{aligned}$$

pd (here the first two terms sum the norms of the vectors $e_{1+na_n+rb_n}$ appearing in (3.3.4), with appropriate weights; and the last term does the same for the vectors $e_{1+na_n+(r-1)b_n}$).

Lastly, (2.5.3) gives us (for each $0<r\le s\le n$)

$$\|(1+n)^{sa_n+rb_n}e_{rb_n+sa_n}\|\le 1+b_n\|(1+n)^{sa_n+(r-1)b_n}e_{(r-1)b_n+sa_n}\|,$$

hence for $0 < r \le n$,

$$(3.4.10)\quad \|(1+n)^{r(a_n+b_n)} e_{r(a_n+b_n)}\| \le 1 + b_n + b_n^2 + s + b_n^r \|e_{ra_n}\| \le 1 + b_n + b_n^2 + s + b_n^r \cdot 2 \quad (\text{by } (3.4.2)) \le 3b_n^r.$$

Therefore the contribution to our sum from (3.3.5) is, **pd**, at most

$$(3.4.11)\qquad \sum_{0\le r<n} \frac{3b_n^r}{1+n} 2^{(1-b_n/2+(r+1)a_n)/\sqrt{b_n}} < \frac{\eta}{5}.$$

Adding up our estimates ((3.4.7), (3.4.3), (3.4.9), (3.4.11) and our remark about Tf_{00}) we find that **pd**,

$$\sum_{i,j} \|(T-W)f_{ij}\| < \eta/5 + \eta/5 + \eta/5 + \eta/5 + \eta/5 = \eta,$$

which gives the result. ∎

COROLLARY 3.5. **Pd**, $\|T\| < 1$.

The proof is obvious.

COROLLARY 3.6. **Pd**, *T is strictly singular.*

Proof. Strict singularity is not affected if an operator is perturbed by an operator in the norm closure of the finite rank operators. Since $T - W$ is nuclear **pd**, it is enough to show that W is strictly singular. Now with slight abuse of notation, we have

$$W = W_0 + W_1$$

where W_0 is a compact operator on l_2. So it is enough to show that W_1 is strictly singular. Now W_1 is the map $(\bigoplus_{i=1}^\infty J_{p_i})_{l_2} \to (\bigoplus_{i=1}^\infty J_{p_i})_{l_2}$ which sends the sequence $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots)$ to $(\mathbf{0}, \beta_1\mathbf{x}_1, \beta_2\mathbf{x}_2, \ldots)$. Furthermore, $\beta_j \to 0$ as $j \to \infty$ (see (3.3.0)). All we need then for our corollary is the easy lemma:

LEMMA 3.7. *If $W_1 : (\bigoplus_{i=1}^\infty J_{p_i})_{l_2} \to (\bigoplus_{i=1}^\infty J_{p_i})_{l_2}$ is the map sending*

$$(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots) \to (\mathbf{0}, \beta_1\mathbf{x}_1, \beta_2\mathbf{x}_2, \ldots)$$

$(\mathbf{x}_i \in J_{p_i})$, then W_1 is strictly singular provided $\beta_i \to 0$ as $i \to \infty$.

Proof. If not, write $X_1 = (\bigoplus_{i=1}^\infty J_{p_i})$ and let $E \subset X_1$ be an infinite-dimensional subspace, and $\varepsilon > 0$, such that for all $\mathbf{x} \in E$,

$$(3.7.0)\qquad \|W_1\mathbf{x}\| \ge \varepsilon\|\mathbf{x}\|.$$

Let P_N denote the natural projection onto $\bigoplus_{i=1}^N J_{p_i}$ sending $(\mathbf{x}_1, \mathbf{x}_2, \ldots)$ to $(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N, \mathbf{0}, \mathbf{0}, \mathbf{0}, \ldots)$. Let $P_0 = 0$. Now $\|W_1P_N - W_1\| \to 0$ as $N \to \infty$

because $\beta_i \to 0$. Therefore there is an N such that for all $\mathbf{x} \in E$,

$$\|W_1P_N\mathbf{x}\| \ge \varepsilon\|\mathbf{x}\|/2.$$

So, W_1 is norm increasing on an infinite-dimensional subspace of P_NX_1 (namely, P_NE). Let $S \subset \mathbb{Z}^+ \times \mathbb{Z}^+ = \{(M,N) : M < N$, and there is an infinite-dimensional subspace of $(P_N - P_M)X_1$ on which W_1 is norm increasing$\}$. We have shown $(O, N) \in S$ for large N. Let $(M,N) \in S$ be such that $N-M$ is minimal, and let E be a subspace of $(P_N - P_M)X_1$ on which W_1 is norm increasing, spanned by vectors $\mathbf{x}^{(i)} = (\mathbf{0}, \ldots, \mathbf{0}, \mathbf{x}_{M+1}^{(i)}, \mathbf{x}_{M+2}^{(i)}, \ldots \mathbf{x}_N^{(i)}, \mathbf{0}, \ldots, \mathbf{0})$ $(i = 1, \ldots, \infty)$. If $N - M = 1$ we find the inclusion map $J_{p_N} \to J_{p_{N+1}}$ is norm increasing on E, contradicting Lemma 1.1. If not, then taking a subspace of E and perturbing slightly as necessary, we can assume that for each j the $\mathbf{x}_j^{(i)}$ are a block basis in J_{p_i} (here we allow a "block basis" to perhaps include some zero vectors). Now the subspace of $J_{p_{M+1}}$ spanned by the $\mathbf{x}_{M+1}^{(i)}$ must be infinite-dimensional, or we can remove the $\mathbf{x}_{M+1}^{(i)}$ (and reduce $N-M$) by passing to a subspace. So taking a subsequence as necessary we may assume the $\mathbf{x}_{M+1}^{(i)}$ to be independent, and likewise we may assume the $\mathbf{x}_N^{(i)}$ are independent. Consider the two norms

$$\|\lambda\|_{M+1} = \Big\|\sum_{i=1}^\infty \lambda_i \mathbf{x}_{M+1}^{(i)}\Big\|_{J_{p_{M+1}}} \quad\text{and}\quad \|\lambda\|_N = \Big\|\sum_{i=1}^\infty \lambda_i \mathbf{x}_N^{(i)}\Big\|_{J_{p_N}}$$

on the finitely nonzero sequences $\lambda \in c_{00}$. If on any infinite-dimensional subspace of c_{00} they are equivalent, we have a subspace of $J_{p_{M+1}}$ isomorphic to one of J_{p_N}; this leads to $l_{p_{M+1}} \subset J_{p_N}$, a contradiction as in Lemma 1.1. If not, there is a normalized block basis $\mathbf{y}^{(i)}$ of the $\mathbf{x}^{(j)}$ such that writing $\mathbf{y}^{(i)} = (\mathbf{0}, \ldots, \mathbf{0}, \mathbf{y}_{M+1}^{(i)}, \ldots, \mathbf{y}_N^{(i)}, \mathbf{0}, \ldots, \mathbf{0})$ we either have $\|\mathbf{y}_{M+1}^{(i)}\| \to 0$ or $\|\mathbf{y}_N^{(i)}\| \to 0$. Perturbing a subsequence of this block basis very slightly, we can obtain an increasing sequence (n_i), and vectors $\mathbf{z}^{(i)}$ close to $\mathbf{y}^{(n_i)}$, spanning a norm increasing subspace of X_1, for which $\mathbf{z}_N^{(i)} = \mathbf{0}$ (or $\mathbf{z}_{M+1}^{(i)} = \mathbf{0}$). Hence, either $(M, N-1) \in S$ or $(M+1, N) \in S$; so $N-M$ was not minimal. This contradiction shows that W_1 is strictly singular.

4. Estimates concerning $\|T^{a_n+b_n}\|$. This of course is essentially a repeat of arguments given in [R1]–[R8].

LEMMA 4.1 (cf. [R7], §3.4). *Recall that $F_n = \mathrm{lin}(e_0, \ldots, e_n)$. On F_n, one may consider the two norms*

$$\Big|\sum_{i=0}^n \lambda_i e_i\Big| = \sum_{i=0}^n |\lambda_i| \quad\textit{and}\quad \Big\|\sum_{i=0}^n \lambda_i e_i\Big\| = \Big\|\sum_{i=0}^n \lambda_i e_i\Big\|_X.$$

For suitable functions $N_1 : \mathbb{N}^2 \to \mathbb{N}$ *and* $N_2 : \mathbb{N}^2 \to \mathbb{N}$, *we have*

$$\text{(4.1.0)} \qquad \frac{1}{N_1(n_1, a_n)}|x| \le \|x\| \le N_1(n_1, a_n)|x|$$

for all $\mathbf{x} \in F_{na_n}$, *and*

$$\text{(4.1.1)} \qquad \frac{1}{N_2(n_1, b_n)}|x| \le \|x\| \le N_2(n_1, b_n)|x|$$

for all $\mathbf{x} \in F_{v_n}$, *provided* $\mathbf{d}$ *increases sufficiently rapidly that Definition* 2.5 *is meaningful.*

Proof. For F_{na_n} is $F_{S_{na_n}} = \operatorname{lin}\{f_{ij} : (i,j) \in S_{na_n}\}$ where S_n is the set mentioned in (2.0.0). Furthermore, the matrix of the map on F_{na_n} sending e_i to f_{jk} (where $i = \mathcal{I}(j,k)$, as in 2.6) is determined by the values $a_1, b_1, \ldots, a_n$ as used in Definition 2.5. If we take the relevant f_{jk} as our basis for F_{na_n} then of course the norm is between the c_0 norm and the l_1 norm. If we make the change of basis to e_i ($i = 1, \ldots, na_n$) then for a suitable function $M_1(n, a_1, b_1, \ldots, a_n)$ the inequality (4.1.0) must hold. Because $\mathbf{d}$ is an increasing sequence, we can write $M_1(n, a_1, b_1, \ldots, a_n) \le N_1(n, a_n)$ for a suitable function N_1. Similarly, a suitable function N_2 exists such that (4.1.1) holds. ∎

Definition 4.2. Recall that we write f_i for the vector $f_{j,k}$, where $i = \mathcal{I}(j,k)$. Let $Q_m^0 : F \to F_{ma_m}$ be the projection such that

$$Q_m^0(f_j) = \begin{cases} f_j, & 0 \le j \le ma_m, \\ -(m-r+1)^{j-ra_m}(m+1)^{(r-1)a_m} a_{m-r} e_{j-ra_{m+1}+(r-1)a_m}, & \\ \qquad j \in [0, v_{m+1-r}] + ra_{m+1},\ 1 < r \le m, & \\ 0 & \text{otherwise.} \end{cases}$$

We shall establish the following lemma (cf. [R7], Lemma 4.3).

Lemma 4.3. $\mathbf{Pd}$, *for all* n *we have*

$$\|T^{a_n+b_n} \circ (I - Q_n^0)\| \le 2(n+1)^{-(a_n+b_n)}.$$

Later on, we will establish that, for an arbitrary norm-1 vector $\mathbf{x} \in l_1$, and for any $\varepsilon > 0$, there is a polynomial q and an integer n such that

$$\|q(T)T^{a_n+b_n}Q_n^0 x - e_0\| < \varepsilon/2 \quad \text{and} \quad \|q(T)T^{a_n+b_n} \circ (I - Q_n^0)x\| < \varepsilon/2.$$

This will show that $e_0 \in \operatorname{lim}\{T_n^r \mathbf{x} : r \ge 0\}$ and hence that $\mathbf{x}$ is cyclic, since e_0 obviously is. So T is strictly singular, and has no invariant subspaces.

Proof of Lemma 4.3. As with proving T continuous, we split the operator involved (in this case, $T^{a_n+b_n} \circ (I - Q_n^0)$) into a part that looks like a weighted shift operator, and a nuclear operator. In certain cases, we now find that $T^{a_n+b_n} \circ (I - Q_n^0) f_i$ is of form $\varepsilon_i f_{i+a_n+b_n}$. These cases are as follows:

Case 4.3.1. If $i \in [0, na_n]$ then $Q_n^0 f_i = f_i$ so of course

$$\text{(4.3.0)} \qquad T^{a_n+b_n} \circ (I - Q_n^0) f_i = 0.$$

Case 4.3.2. If $i \in [r(a_n + b_n), (n-1)a_n + rb_n]$ with $0 < r < n$, then by two applications of (2.5.3) we find that

$$\text{(4.3.1)} \qquad T^{a_n+b_n} \circ (I - Q_n^0) f_i = T^{a_n+b_n} f_i = (1+n)^{-a_n-b_n} f_{i+a_n+b_n}.$$

Case 4.3.3. If $i \in (na_n + rb_n, (r+1)(a_n + b_n))$ with $0 \le r < n-1$ then two applications of (2.5.4) likewise give us

$$\text{(4.3.2)} \qquad T^{a_n+b_n} \circ (I - Q_n^0) f_i = T^{a_n+b_n} f_i = (1+n)^{-a_n-b_n} 2^{a_n/\sqrt{b_n}} f_{i+a_n+b_n}.$$

Case 4.3.4. If $i \in [0, v_{m-r} - a_n - b_n] + ra_m$ with $0 < r \le m - n$ then (2.5.1) gives

$$\text{(4.3.3)} \qquad T^{a_n+b_n} \circ (I - Q_n^0) f_i = T^{a_n+b_n} f_i = (m-r+1)^{-a_n-b_n} f_{i+a_n+b_n}.$$

Case 4.3.5. If $i \in (ra_m + v_{m-r}, (r+1)a_m - a_n - b_n)$ with $0 < r < m$, $m > n$, or if $i \in (v_{m-1}, a_m - a_n - b_n)$, $m > n$, then by (2.5.1),

$$\text{(4.3.4)} \qquad \begin{aligned} T^{a_n+b_n} \circ (I - Q_n^0) f_i &= T^{a_n+b_n} f_i \\ &= (1+m)^{-a_n-b_n} 2^{(a_n+b_n)/\sqrt{a_m}} f_{i+a_n+b_m}. \end{aligned}$$

Case 4.3.6. If $i \in e[r(a_m + b_m), ma_m + rb_m - a_n - b_n]$ with $0 < r \le m$, $m > n$, then (2.5.3) gives

$$\text{(4.3.5)} \qquad T^{a_n+b_n} \circ (I - Q_n^0) f_i = T^{a_n+b_n} f_i = (1 + m^{-a_n-b_n}) f_{i+a_n+b_n}.$$

Case 4.3.7. Finally if $i \in (ma_m + rb_m, (r+1)(a_m + b_m) - a_n - b_n))$ with $0 \le r < m$, $m > n$, then (2.5.4) gives

$$\text{(4.3.6)} \qquad T^{a_n+b_n} \circ (I - Q_n^0) f_i = (1+m)^{-a_n-b_n} 2^{(a_n+b_n)/\sqrt{b_m}} f_{i+a_n+b_n}. \ ∎$$

Lemma 4.4. $\mathbf{Pd}$, *the following is true: The operator* $W = W^{(n)} : F \to F$ *such that* Wf_i *is as in* (4.3.0)–(4.3.6) *if the integer* i *is mentioned in these cases, and* $Wf_i = 0$ *otherwise, has norm*

$$(1+n)^{-a_n-b_n} 2^{a_n/\sqrt{b_n}} < \tfrac{3}{2}(1+n)^{-a_n-b_n}.$$

Proof. As earlier, we split up W into an operator W_0 acting on l_2, and W_1 acting on $\bigoplus_{i=1}^{\infty} J_{p_i}$. The operator W_0 covers all cases except Case 4.3.4, and it acts as

$$W_0(f_i) = \varepsilon_i f_{0,\gamma(a_n+b_n+\gamma^{-1}(i))}$$

with $\varepsilon_i \le (1+n)^{-a_n-b_n} 2^{a_n/\sqrt{b_n}}$, $\mathbf{pd}$, and equality achieved for certain values i such that $\gamma^{-1}(i)$ is covered by Case 4.3.3. The operator W_1 deals with Case 4.3.4. It sends (for $i \in [0, v_{m-r} - a_n - b_n] + ra_m$, $0 < r \le m - n$) f_i to $(m-r+1)^{-a_n-b_n} f_{i+a_n+b_n}$; that is, writing $j = i - ra_m \in [0, v_{m-r} - a_n - b_n]$,

$$f_{j+w_{m-r}, r-1} \to (m-r+1)^{-a_n-b_n} f_{j+w_{m-r}+a_n+b_n, r-1}.$$

Writing $m-r=k\geq n$ we find that W_1 sends $f_{j+w_k,\,s}$ to

$$(k+1)^{-a_n-b_n}f_{j+w_k+a_n+b_n,\,s}$$

for all $s\geq 0$ and $j\in[0,v_k-a_n-b_n]$.

Otherwise, $W_1f_{l,\,s}=0$. So W_1 acts on each J_{p_i} space as a multiple $\beta_i\times\varepsilon_i^{(n)}$, where $\beta_i\times\varepsilon_i^{(n)}$ is the inclusion $J_{p_i}\to J_{p_{i+a_n+b_n}}$, and β_i does not exceed $(n+1)^{-a_n-b_n}$. So, $\|W_1\|=(n+1)^{-a_n-b_n}$. Hence,

$$\|W\|=\max\{\|W_0\|,\|W_1\|\}=(n+1)^{-a_n-b_n}2^{a_n/\sqrt{b_n}}. \ \blacksquare$$

LEMMA 4.5. *For each $\eta>0$, the following is true*: **pd**, *for every $n>0$ the operator $T^{a_n+b_n}\circ(I-Q_n^0)-W^{(n)}$ is nuclear, of nuclear norm at most $\eta(n+1)^{-a_n-b_n}$.*

Proof. We must consider the error terms $T^{a_n+b_n}\circ(I-Q_n^0)f_i-W^{(n)}f_i$, sum all their norms, and obtain at most $\eta(n+1)^{-a_n-b_n}$. This is not in fact difficult to do, there are roughly six cases, corresponding to values i which were "missed out" of Cases 4.3.2–4.3.7 above.

CASE 4.5.1. If $i\in((n-1)a_n+rb_n,na_n+rb_n]$ with $0<r<n$ (these are some of the "missing values" from Case 4.3.2 above), then (2.5.3) gives us

$$f_i=(1+n)^ie_i-b_n(1+n)^{i-b_n}e_{i-b_n}$$

and hence, writing $j=i+a_n+b_n-na_n-(r+1)b_n-1\geq 0$, we have

$$\begin{aligned}(4.5.0)\quad & T^{a_n+b_n}(I-Q_n^0)f_i\\ &=T^{a_n+b_n}f_i=T^j\{(1+n)^ie_{1+na_n+(r+1)b_n}-b_n(1+n)^{i-b_n}e_{1+na_n+rb_n}\}\end{aligned}$$

Now (3.4.8) and (3.4.6) together give us, for all $0\leq r\leq n$,

$$(4.5.1)\qquad \|e_{1+na_n+rb_n}\|\leq(1+n)^{-1-na_n-rb_n}2^{(1+na_n-b_n/2)/\sqrt{b_n}}$$

(for, pd, (4.3.6) is less than (3.4.9) for the same value of n). Applying this twice, we find that (4.5.0) is, **pd**, at most

$$(4.5.2)\qquad (1+m)^{i-na_n-(r+1)b_n-1}2^{(1+na_n-b_n/2)/\sqrt{b_n}}(1+b_n)\leq 2^{-\sqrt{b_n}/3}.$$

CASE 4.5.2. If $i=v_n$ (the final value "missed out" of Case 4.3.2 above), then (2.5.3) and (3.4.6), together with the ever-useful fact that $\|T\|<1$, give us

$$\begin{aligned}(4.5.3)\quad \|T^{a_n+b_n}(I-Q_n^0)f_i\|&=\|T^{a_n+b_n}f_i\|\\ &\leq(1+n)^{v_n}\|T^{a_n+b_n}e_{v_n}\|\\ &\quad+b_n(1+n)^{na_n+(n-1)b_n}\|T^{a_n+b_n}e_{na_n+(n-1)b_n}\|\\ &\leq((1+n)^{v_n}+b_n(1+n)^{na_n+(n-1)b_n})\|e_{1+v_n}\|\\ &\leq((1+n)^{v_n}+b_n(1+n)^{na_n+(n-1)b_n})(2+n)^{-(1+v_n)}\\ &\quad\times 2^{(1+v_n-a_{n+1}/2)/\sqrt{a_{n+1}}}\\ &\leq 2^{-\sqrt{a_{n+1}}/3}\end{aligned}$$

for all n, **pd**.

CASE 4.5.3. If $i\in(na_n+(n-1)b_n,n(a_n+b_n))$ (not covered by Case 4.3.3 above), then by (2.5.4), f_i is a multiple λ_ie_i with λ_i (crudely) at most $(1+n)^{v_n-1}$. Therefore, $T^{a_n+b_n}f_i$ is $T^k(\lambda_ie_{1+v_n})$ for some $k\geq 0$. Since $\|T\|\leq 1$, it follows that, **pd**,

$$\begin{aligned}(4.5.5)\quad \|T^{a_n+b_n}f_i\|&\leq|\lambda_i|\|e_{1+v_n}\|\\ &\leq(1+n)^{v_n-1}(2+n)^{-1-v_n}2^{(1+v_n-a_{n+1}/2)/\sqrt{a_{n+1}}}\quad\text{(by (3.4.6))}\\ (4.5.6)\quad &\leq 2^{-\sqrt{a_{n+1}}/3}.\end{aligned}$$

CASE 4.5.4. If $i\in(v_{m-r}-a_n-b_n,v_{m-r}]+ra_m$ with $0<r\leq m-n$ (not covered by Case 4.3.4 above), or if $i\in[0,v_{m-r}]+ra_m$ with $0<r\leq m>n$, $m-r<n$, then we see by (2.5.1) that

$$\begin{aligned}(4.5.7)\quad f_i=&((1+m)^{ra_m}e_i-m^{(r-1)a_{m-1}}e_{i-ra_m+(r-1)a_{m-1}})\\ &\times(m-r+1)^{i-ra_m}a_{m-r}\end{aligned}$$

and in either case, $i+a_n+b_n>ra_m+v_{m-r}$. (In the second case, $i+a_n+b_n\geq ra_m+a_n+b_n>ra_m+v_{m-r}$ **pd**, since $m-r<n$). Writing

$$(4.5.8)\qquad j=i+a_n+b_n-v_{m-r}-ra_m-1$$

we have

$$\begin{aligned}(4.5.9)\quad \|T^{a_n+b_n}f_i\|&=(m-r+1)^{i-ra_m}a_{m-r}\|T^j((1+m)^{ra_m}e_{1+ra_m+v_{m-r}}\\ &\quad-m^{(r-1)a_{m-1}}e_{(r-1)a_{m-1}+v_{m-r}+1})\|\\ &\leq(m-r+1)^{i-ra_m}a_{m-r}((1+m)^{ra_m}\|e_{1+ra_m+v_{m-r}}\|\\ &\quad+m^{(r-1)a_{m-1}}\|e_{(r-1)a_{m-1}+v_{m-r}+1}\|).\end{aligned}$$

Now (3.4.4), (3.4.5) tell us that **pd**, for all $0\leq r\leq n$ we have

$$(4.5.10)\qquad \|e_{1+ra_n+v_{n-r}}\|\leq(1+n)^{-ra_n-1-v_{n-r}}2^{(1-a_n/2)/\sqrt{a_n}}.$$

So, (4.5.9) is at most

$$\begin{aligned}(4.5.11)\quad &(m-r+1)^{i-ra_m}a_{m-r}((1+m)^{-1-v_{m-r}}2^{(1-a_m/2)/\sqrt{a_m}}\\ (4.5.12)\quad &\qquad+m^{-1-v_{m-r}}2^{(1-a_{m-1}/2)/\sqrt{a_{m-1}}})\\ (4.5.13)\quad &\leq 2^{(1-a_{m-1}/2)/\sqrt{a_{m-1}}}\,\frac{2}{m}a_{m-1}\end{aligned}$$

(the worst case in this estimate is when $r=1$). Now if we have $m>n+1$, then $(I-Q_n^0)f_i$ is just f_i (Definition 4.2), and (4.5.13) above is the upper bound we need; thus,

$$\|T^{a_n+b_n}\circ(I-Q_n^0)f_i\|\le 2^{(1-a_{m-1}/2)/\sqrt{a_{m-1}}}\frac{2}{m}a_{m-1}. \tag{4.5.14}$$

If, on the other hand, $m=n+1$, then $(I-Q_n^0)f_i$ is in fact not f_i but just $(1+m)^{ra_m}(m-r+1)^{i-ra_m}e_i$ (see Definition 4.2 and (2.5.3)). Our argument then gives, pd, the better estimate

$$\begin{aligned}\|T^{a_n+b_n}\circ(I-Q_n^0)f_i\| &\le (1+m)^{ra_m} \\ &\quad\times(m-r+1)^{i-ra_m}\|e_{1+ra_m+v_{m-r}}\| \qquad (4.5.15)\\ &\le 2^{(1-a_{n+1}/2)/\sqrt{a_{n+1}}}\frac{1}{n+2}a_n. \qquad (4.5.16)\end{aligned}$$

CASE 4.5.5. If $i\in[(r+1)a_m-a_n-b_n,(r+1)a_m)$ with $0\le r<m$, $m>n$ (these are the "missing values" from Case 4.3.5 above), then (2.5.2) gives us

$$\begin{aligned}T^{a_n+b_n}\circ(I-Q_n^0)f_i=T^{a_n+b_n}f_i&=(1+m)^i2^{((r+1/2)a_m-i)/\sqrt{a_m}}e_{i+a_n+b_n}\\ &=(1+m)^i2^{((r+1/2)a_m-i)/\sqrt{a_m}}T^je_{(r+1)a_m}\end{aligned}$$

for some $j\ge 0$. As we have remarked in (3.4.2), $\|(1+n)^{ra_n}e_{ra_n}\|=\sqrt{1+a_{n-r}^{-2}}$, so since $\|T\|\le 1$ this is at most

$$\begin{aligned}&2(1+m)^{i-(r+1)a_m}2^{((r+1/2)a_m-i)/\sqrt{a_m}} \qquad (4.5.17)\\ &\le\frac{2}{1+m}2^{(-a_m/2+a_n+b_n)/\sqrt{a_m}}\qquad(\text{since } i\ge(r+1)a_m-a_n-b_n)\\ &\le 2^{-\sqrt{a_m}/3}\end{aligned}$$

for all $0\le r<m>n$, pd.

CASE 4.5.6. If $i\in(ma_m+rb_m-a_n-b_n,ma_m+rb_m]$ with $0<r\le m>n$, then we have one of the "missing values" from Case 4.3.6 above. Formula (2.5.3) gives us

$$f_i=(1+m)^ie_i-b_m(1+i)^{i-b_m}e_{i-b_m}$$

and hence, writing $j=i+a_n+b_n-ma_m-rb_m-1\ge0$, we have

$$\begin{aligned}T^{a_n+b_n}f_i=T^j\{&(1+m)^ie_{1+ma_m+rb_m}\\ &-b_m(1+i)^{i-b_m}e_{1+ma_m+(r-1)b_m}\}.\end{aligned} \tag{4.5.18}$$

Now (3.4.8) and (3.4.6) together give us, for all $0\le r\le n$,

$$\|e_{1+na_n+rb_n}\|\le(1+n)^{-1-na_n-rb_n}2^{(1+na_n-b_n/2)/\sqrt{b_n}} \tag{4.5.19}$$

(for, pd, (3.4.6) is less than (3.4.9) for the same value of n). Applying this twice, we find that, pd, (4.5.18) is at most

$$(1+m)^{i-ma_m-rb_m-1}2^{(1+ma_m-b_m/2)/\sqrt{b_m}}(1+b_m)\le 2^{-\sqrt{b_m}/3}. \tag{4.5.20}$$

CASE 4.5.7. Finally, if $i\in[(r+1)(a_m+b_m)-a_n-b_n,(r+1)(a_m+b_m))$ $(0\le r<m>n)$ then we are among the "missing values" from Case 3.4.7. So, (2.5.4) gives us

$$f_i=(1+m)^i2^{((r+1/2)b_m-i)/\sqrt{b_m}}e_i$$

and hence, for $j=i+a_n+b_n-(r+1)(a_m+b_m)\ge0$,

$$\begin{aligned}T^{a_n+b_n}f_i&=T^{a_n+b_n}\circ Q_n^0f_i\\ &=(1+m)^i2^{((r+1/2)b_m-i)/\sqrt{b_m}}T^je_{(r+1)(a_m+b_m)}.\end{aligned}\tag{4.5.21}$$

As in (3.4.10),

$$\|e_{(r+1)(a_m+b_m)}\|\le 3b_m^{r+1}(1+m)^{-(r+1)(a_m+b_m)} \tag{4.5.22}$$

so

$$\begin{aligned}\|T^{a_n+b_n}f_i\|&\le3(1+m)^{i-(r+1)(a_m+b_m)}2^{((r+1/2)b_m-i)/\sqrt{b_m}}b_m^{r+1}\\ &\le\frac{3}{1+m}b_m^{r+1}2^{(-b_m/2-a_m+a_n+b_n)/\sqrt{b_m}}\\ &\le 2^{-\sqrt{b_m}/3}\end{aligned}\tag{4.5.23}$$

in all cases, pd. We now add up all our estimates (4.5.2), (4.5.4), (4.5.6), (4.5.14), (4.5.16), (4.5.17), (4.5.20), and (4.5.23), counting according to the multiplicity of values i that are involved. We obtain this estimate of the nuclear norm:

$$\begin{aligned}&\sum_i\|(T^{a_n+b_n}\circ(I-Q_n^0)-W^{(n)})f_i\|\\ &\le(n-1)a_n2^{-\sqrt{b_n}/3}+2^{-\sqrt{a_{n+1}}/3}\\ &\quad+b_n2^{-\sqrt{a_{n+1}}/3}+(a_n+b_n)\sum_{m=n+2}^{\infty}\sum_{r=1}^{m-n}\frac{2}{m}a_{m-1}2^{(1-a_{m-1}/2)/\sqrt{a_{m-1}}}\\ &\quad+(a_n+b_n)\frac{1}{n+2}a_n2^{(1-a_{n+1}/2)/\sqrt{a_{n+1}}}\\ &\quad+\sum_{\substack{m>n\\0\le r<m}}(a_n+b_n)2^{-\sqrt{a_m}/3}+\sum_{\substack{m>n\\0\le r<m}}(a_n+b_n)2^{-\sqrt{b_m}/3}\\ &\quad+\sum_{\substack{m>n\\0\le r<m}}(a_n+b_n)2^{-\sqrt{b_m}/3}.\end{aligned}$$

The reader will observe that as a function of n and $\mathbf{d}$, this sum is, **pd**, at most $\eta(n+1)^{-a_n-b_n}$. Thus 4.5 is proved, and 4.3 follows immediately. ∎

5. Further estimates concerning T

DEFINITION 5.1. Let Q_m $(m \geq 1)$ be the projection $F \to F_{ma_m}$ such that

$$Q_m(f_j) = \begin{cases} f_j, & 0 \leq j \leq ma_m, \\ 0, & \text{otherwise}. \end{cases} \tag{5.1.0}$$

Note that in terms of what happens to the e_j, this amounts to much the same as (6.1) of [R8], though it does not look the same.

LEMMA 5.2. *$\|Q_m\| = 1$ for all m.*

Proof. We claim that for each i, the vectors f_{ik} $(k = 0, 1, \ldots)$ appear as a subsequence $(f_{j_k})_{k=0}^{\infty}$ of the f_j in their proper order $(j_0 < j_1 < j_2 < \ldots)$. This is true because γ is an increasing function (Definition 2.1) and $f_{0j} = f_{\gamma(j)}$, and because for $i > 0$, say $i \in [w_m, w_m + v_m]$, we have $f_{ik} = f_{(k+1)a_m+k+1+i-w_m}$ (see 2.3). Hence, for each i there is a k such that

$$Q_m(f_{ij}) = \begin{cases} f_{ij}, & j \leq k, \\ 0, & \text{otherwise}. \end{cases} \tag{5.2.0}$$

The norm of the projection that thus "truncates" a sequence is 1 on l_2 (of course), and also on any J_{p_i}. Hence, $\|Q_m\| = 1$. ∎

DEFINITION 5.3. Let $P_{n,m}$ $(m > n \geq 1)$ be the operator $\tau_{nm} \circ Q_m : F_{ma_m} \to F_{ma_m}$, where

$$\tau_{nm}(e_j) = \begin{cases} e_j, & 0 \leq j < (m-n)a_m, \\ 0, & \text{otherwise}. \end{cases} \tag{5.3.0}$$

LEMMA 5.4. *$\|Q_m^0\| \leq a_m$ for all m,* **pd**.

Proof. We know $\|Q_m\| = 1$; and 4.2 tells us that $(Q_m - Q_m^0)f_j$ is zero unless $j \in [0, v_{m+1-r}] + ra_{m+1}$, $1 < r \leq m+1$. In this case, it is

$$e_{j-ra_{m+1}+(r-1)a_m}(m-r+2)^{j-ra_n}(m+1)^{(r-1)a_m}a_{m+1-r}.$$

Hence, crudely,

$$\begin{aligned} \|Q_m - Q_m^0\| &\leq \sum_j \|(Q_m - Q_m^0)f_j\| \\ &\leq \sum_{r=2}^{m+1}(1+v_{m+1-r})(m+1)^{(r-1)a_m}a_{m+1-r} \\ &\quad \times \max\{\|e_{j-ra_{m+1}+(r-1)a_m}\| : j \in [0, v_{m+1-r}] + ra_{m+1}\} \\ &\leq \sum_{r=2}^{m+1}(1+v_{m+1-r})(m+1)^{(r-1)a_m}\|e_{(r-1)a_m}\| \\ &\leq \sum_{r=2}^{m+1}(1+v_{m+1-r}) \cdot 2 \qquad \text{(by (3.4.2))} \\ &\leq 2m(1+v_{m-1}) < a_m \end{aligned} \tag{5.4.0}$$

for all m, **pd**. ∎

LEMMA 5.5. *$\|P_{nm}\| \leq a_{n+1}$ for all $n < m$,* **pd**.

Proof. $\|Q_m\| = 1$ so $\|P_{nm}\| = \|\tau_{nm}\|$. Examining (5.3) and (2.5) we find

$$\tau_{nm}f_i = \begin{cases} f_i, & 0 \leq i < (m-n)a_m, \\ -e_{i-ra_m+(r-1)a_{m-1}}(m-r+1)^{i-ra_n}m^{(r-1)a_{m-1}}a_{m-r}, & \\ \qquad i \in [0, v_{m-r}] + ra_m,\ m-n \leq r \leq n, & \\ 0, & \text{otherwise}. \end{cases} \tag{5.5.0}$$

Now the projection τ' such that

$$\tau' f_i = \begin{cases} f_i, & 0 \leq i < (m-n)a_m, \\ 0, & \text{otherwise}, \end{cases} \tag{5.5.1}$$

has norm 1, for the same reasons as in 5.2. Therefore

$$\begin{aligned} \|\tau_{nm}\| &\leq 1 + \sum_{\substack{i \in [0, v_{m-r}] + ra_m \\ m-n \leq r \leq m}} \|e_{i-ra_m+(r-1)a_{m-1}}\| \\ &\qquad \times (m-r+1)^{i-ra_n}m^{(r-1)a_{m-1}}a_{m-r} \\ &\leq 1 + \sum_{r=m-n}^{m}(1+v_{m-r})\|e_{(r-1)a_{m-1}}\| \\ &\qquad \times (m-r+1)^{i-ra_n}m^{(r-1)a_{m-1}}a_{m-r} \end{aligned} \tag{5.5.2}$$

since $\|T\| < 1$. Recall from (3.4.2) that $\|(1+n)^{ra_n}e_{ra_n}\| = \sqrt{1+a_{n-r}^{-2}}$. Substituting into (5.5.2) we have

$$\begin{aligned} \|\tau_{nm}\| &\leq 1 + \sum_{r=m-n}^{m}(1+v_{m-r}) \\ &\qquad \times \sqrt{1+a_{n-r}^{-2}}(m-r+1)^{v_{m-r}}a_{m-r} \\ &= 1 + \sum_{u=1}^{n}(1+v_u)\sqrt{1+a_u^{-2}}(u+1)^{v_u}a_u \\ &< a_{n+1} \end{aligned} \tag{5.5.3}$$

for all n, **pd**. ∎

6. T has no invariant subspaces

DEFINITION 6.1. For each $1 < n \le m$, let $K_{n,m} \subset F_{ma_m}$ be the set of vectors such that $\|\mathbf{x}\| \le a_m$ and $\|\tau_{n,m}\mathbf{x}\| \ge 1/a_m$. Let $T_m : F_{ma_m} \to F_{ma_m}$ be the "truncated" version of T, i.e. $T_m(e_i) = e_{i+1}$ ($i < ma_m$) or zero ($i = ma_m$).

LEMMA 6.2. *There is a function $N_3 : \mathbb{N}^2 \to \mathbb{N}$ with the following property:* **pd**, *for all $1 < n < m$ and $\mathbf{x} \in K_{n,m}$, there is a polynomial p such that $|p| < N_3(m, a_m)$, $p(t)$ is of form $\sum_{i=a_m}^{ma_m} \lambda_i t^i$, and*

$$\|p(T_m)\mathbf{x} - e_0\| < 1/a_m + 1/a_{n-1}. \tag{6.2.0}$$

Proof. For any $\mathbf{y} \in K_{n,m}$ we can write $\mathbf{y} = \sum_{i=1}^{\alpha} \lambda_i e_i$ where $\lambda_\alpha \ne 0$. Then

$$\operatorname{lin}\{T_m^r \mathbf{y} : a_m \le r \le ma_m\} = \operatorname{lin}\{e_{\alpha+a_m}, e_{\alpha+a_m+1}, \ldots, e_{ma_m}\}. \tag{6.2.1}$$

Since $\tau_{nm}\mathbf{y} \ne 0$ we know $\alpha < (m-n)a_m$ so certainly $e_{(m-n+1)a_m} \in \operatorname{lin}\{T_m^r e_i : a_m \le r \le ma_m\}$. Since K_{nm} is compact, there are a finite number of polynomials $p_1, \ldots, p_k$ of form $p_j(t) = \sum_{i=a_m}^{ma_m} \lambda_{ji} t^i$ such that for all $\mathbf{x} \in K_{nm}$ there is a j such that

$$\|p_j(T_m)\mathbf{x} - (m+1)^{(m-n+1)a_m} e_{(m-n+1)a_m}\| < 1/a_m. \tag{6.2.2}$$

Writing $N = \max_j |p_j|$, note that N depends only on elements of the underlying sequence $\mathbf{d}$ up to and including a_m; so $N < N_3(m, a_m)$ for a suitable function $N_3 : \mathbb{N}^2 \to \mathbb{N}$. Since in view of (3.4.1) we have

$$\|(m+1)^{(m-n+1)a_m} e_{(m-n+1)a_m} - e_0\| = 1/a_{n-1}, \tag{6.2.3}$$

this concludes the proof. ∎

We now extend the previous lemma as follows.

LEMMA 6.3. *With the notation of (6.2), the polynomial*

$$q(t) = t^{b_m}(m+1)^{b_m}/b_m \cdot p(t)$$

satisfies $t^{a_m+b_m} \mid q(t)$, $\deg q \le b_m + ma_m$, $|q| \le N_3(m, a_m)(m+1)^{b_m}/b_m$ and

$$\|q(T)\mathbf{x} - e_0\| \le 2/a_{n-1} + 3/a_m. \tag{6.3.0}$$

Proof. Given $\mathbf{x} \in K_{n,m}$ let p be the polynomial as in 6.2, and write $q(t) = t^{b_m}(m+1)^{b_m}/b_m \cdot p(t)$. Let us consider the vector $q(T)\mathbf{x}$. For all $i \in [a_m + b_m, ma_m + b_m]$ we have $f_i = (m+1)^i e_i - b_m(m+1)^{i-b_m} e_{i-b_m}$; so if we write $p(T_m)\mathbf{x} = \sum_{i=a_m}^{ma_m} \lambda_i e_i$ then, **pd**,

$$\begin{aligned}
&\left\|\frac{(m+1)^{b_m}}{b_m} T^{b_m} p(T_m)\mathbf{x} - p(T_m)\mathbf{x}\right\| \qquad (6.3.1)\\
&= \left\|\sum_{i=a_m}^{ma_m} \lambda_i \left\{\frac{(m+1)^{b_m}}{b_m} e_{i+b_m} - e_i\right\}\right\|\\
&= \left\|\sum_{i=a_m}^{ma_m} \lambda_i \frac{(m+1)^{-i}}{b_m} f_{i+b_m}\right\| \le \frac{1}{b_m}\sum_{i=a_m}^{ma_m} |\lambda_i| = \frac{1}{b_m}|p(T_m)\mathbf{x}|\\
&\le \frac{1}{b_m}|p||\mathbf{x}| \le \frac{1}{b_m} N_3(m, a_m)\|\mathbf{x}\| N_1(m, a_m)\\
&\le \frac{1}{b_m} N_3(m, a_m) a_m N_1(m, a_m) \quad (\text{since } \mathbf{x} \in K_{nm} \text{ so } \|\mathbf{x}\| \le a_m)\\
&\le 1/a_m.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
T^{b_m}(p(T) - p(T_m))\mathbf{x} &\in T^{b_m} \operatorname{lin}\{e_j : ma_m < j \le 2ma_m\} && (6.3.2)\\
&= \operatorname{lin}\{e_j : b_m + ma_m < j \le b_m + 2ma_m\}. && (6.3.3)
\end{aligned}$$

Since $\|T\| \le 1$, **pd**, we deduce that

$$\begin{aligned}
&\frac{(m+1)^{b_m}}{b_m}\|T^{b_m}(p(T) - p(T_m))\mathbf{x}\|\\
&\le |p||\mathbf{x}|\|e_{1+b_m+ma_m}\| \frac{(m+1)^{b_m}}{b_m} && (6.3.4)\\
&\le \frac{(m+1)^{b_m}}{b_m} N_3(m, a_m) N_1(m, a_m) a_m \|e_{1+b_m+ma_m}\| \quad (\text{as above}) && (6.3.5)\\
&\le \frac{(m+1)^{b_m}}{b_m} N_3(m, a_m) N_1(m, a_m) && (6.3.6)\\
&\quad \times a_m(1+m)^{-1-ma_m-b_m} 2^{(1+ma_m-b_m/2)/\sqrt{b_m}} \quad (\text{by } (3.4.9))\\
&\le \frac{1}{a_m} && (6.3.7)
\end{aligned}$$

for all m, **pd**. Adding up (6.3.7) and (6.3.1) we have

$$\|q(T)\mathbf{x} - p(T_m)\mathbf{x}\| \le 2/a_m. \tag{6.3.8}$$

Using 6.2 we have our result. ∎

We now have the following very convenient lemma (it corresponds to Lemma 5.17 of [R1]):

LEMMA 6.4. *For all $j \in [0, v_{k-r}]$, $1 \le r < k - n$ and $s \ge r$, we have*

$$f^*_{j+w_{k-r},s-1} \circ \tau_{n,k+s-r} \circ Q^0_{k+s-r} = -f^*_{j+w_{k-r},s} + f^*_{j+w_{k-r},s-1}. \tag{6.4.0}$$

Proof. The vector $f_{j+w_{k-r},s-1} = f_{j+sa_{k+s-r}}$ (Definition 2.3) is in the image of the projection $\tau_{n,k+s-r} \circ Q^0_{k+s-r}$ and is fixed by it. The vector $f_{j+w_{k-r},s} = f_{j+(s+1)a_{k+s-r+1}}$ is mapped (by 4.2, 5.3) to

$$u = -e_{j+sa_{k+s-r}}(k+s-r+1)^{(s-1)a_{k+s-r}}(k-r+1)^j, \tag{6.4.1}$$

which by (2.6.0), (2.5.1) satisfies $f^*_{j+w_{k-r},s-1}(u) = -1$. It is easily seen that for all other vectors f_m, $\tau_{n,k+s-r} \circ Q^0_{k+s-r}(f_m)$ is either f_m, or zero, or another vector similar to u above, being therefore of form $f_{j',s'} + h$ with the pair (j', s') not equal to $(j + w_{k-r}, s)$, and with $h \in F_{v_{k+s-r-1}}$. In all such cases $f^*_{j+w_{k-r},s}(\tau_{n,k+s-r}Q^0_{k+s-r}f_m) = 0$, hence the result. ∎

THEOREM 6.5. Pd, *T has no invariant subspace.*

Proof. Let $\mathbf{x} \in X$, $\|\mathbf{x}\| = 1$ and $n > 0$. Since e_0 is cyclic for T, it is enough to show that for all such $\mathbf{x}$ and n there is a polynomial q such that $\|q(T)\mathbf{x} - e_0\| \le 2/a_{n-1}$. We claim there is an $m > n$ such that

$$\|\tau_{nm} \circ Q^0_m \mathbf{x}\| \ge 1/a_m. \tag{6.5.0}$$

Now $\|P_{nk}\| \le a_{n+1}$ for all k, and certainly for all $x \in F$ we have $P_{nk}\mathbf{x} = \mathbf{x}$ for all but finitely many k. Therefore, $P_{nk}\mathbf{x} \to \mathbf{x}$ as $k \to \infty$ for any vector $\mathbf{x} \in X$.

Choose, then, a k so large that $\|P_{nk}\mathbf{x}\| = \|\tau_{nk} \circ Q_k\mathbf{x}\| > 1/2$. If $\|\tau_{nk} \circ Q^0_k\mathbf{x}\| > 1/4$ our assertion is proved; if not then

$$\|\tau_{nk} \circ (Q_k - Q^0_k)\mathbf{x}\| > 1/4. \tag{6.5.1}$$

For all $j > 0$ we either have

$$(Q_k - Q^0_k)f_j = e_{j-ra_{k+1}-(r-1)a_k}(k-r+1)^{j-ra_k}(k+1)^{(r-1)a_k} \tag{6.5.2}$$

if $j \in [0, v_{k-r}] + ra_{k+1}$, $1 < r \le n$, or else $(Q_k - Q^0_k)f_j = 0$. Hence, $\tau_{nk} \circ (Q_k - Q^0_k)f_j$ is either

$$e_{j-ra_{k+1}-(r-1)a_k}(k-r+1)^{j-ra_k}(k+1)^{(r-1)a_k} \tag{6.5.3}$$

if $j \in [0, v_{k-r}] + ra_{k+1}$, $1 < r \le k - n$, or else zero. Thus, $\tau_{nk} \circ (Q_k - Q^0_k) = \tau_{nk} \circ (Q_k - Q^0_k) \circ \pi_S$ where S is the finite set

$$S = \bigcup_{1 \le r < k-n} [0, v_{k-r}] + (r+1)a_{k+1} = S_{nk}, \tag{6.5.4}$$

say. Crudely, then, we may say that there is an $i \in S_{nk}$ such that

$$|f^*_i(\mathbf{x})| \ge \frac{1}{\|\tau_{nk}(Q_k - Q^0_k)\|\,|S_{nk}|}. \tag{6.5.5}$$

Now if $i = j + (r+1)a_{k+1}$, $j \in [0, v_{k-r}]$, $1 \le r < k - n$, then we know by (2.5.1) that $f_i = f_{j+w_{k-r},r}$. Because any $\mathbf{x} \in J_{p_{j+w_{k-r}}}$ is necessarily in c_0,

we know that as $s \to \infty$, $|f^*_{j+w_{k-r},s}(\mathbf{x})| \to 0$. Therefore there is an $s > r$ such that

$$|f^*_{j+w_{k-r},s}(\mathbf{x}) - f^*_{j+w_{k-r},s-1}(\mathbf{x})| \ge \frac{2^{r-s-1}}{\|\tau_{nk}(Q_k - Q^0_k)\|\,|S_{nk}|}. \tag{6.5.6}$$

If (6.5.6) holds then we may deduce from (6.4) that, pd,

$$\|\tau_{n,k+s-r}Q^0_{k+s-r}\mathbf{x}\| \ge |f^*_{j+w_{k-r},s}(\mathbf{x}) - f^*_{j+w_{k-r},s-1}(\mathbf{x})|$$

$$\ge \frac{2^{r-s-1}}{\|\tau_{nk} \circ (Q_k - Q^0_k)\|\,|S_{nk}|} \ge \frac{2^{r-s-1}}{4a_{n+1}(1+a_k)w_{k-1}} \tag{6.5.7}$$

(because $\tau_{nk} \circ (Q_k - Q^0_k) = \tau_{nk} \circ Q_k \circ (Q_k - Q^0_k) = P_{nk} \circ (Q_k - Q^0_k)$, and $|S_{nk}| = w_{k-1}$)

$$> \frac{1}{a_{k+s-r}}. \tag{6.5.8}$$

This proves our assertion that there is indeed an $m > n$ such that (6.5.0) holds. Pick such an m, and write $\mathbf{y} = Q^0_m\mathbf{x}$. We know that $\|\mathbf{y}\| \le \|Q^0_m\| \le a_m$, pd. But $\|\tau_{nm}\mathbf{y}\| \ge 1/a_m$ so $\mathbf{y} \in K_{nm}$. Therefore by 6.3 there is a polynomial q such that $\|q(T)\mathbf{y} - e_0\| \le 1/a_{n-1} + 3/a_m$, $t^{a_m+b_m} \mid q(t)$, $\deg q \le b_m + ma_m$, and $|q| \le (m+1)^{b_m}/b_m$. Using our estimate on $\|T^{a_m+b_m} \circ (I - Q^0_m)\|$ and the fact that $\|T\| < 1$, we find that

$$\|q(T)(I - Q^0_m)\| < \frac{2N_3(m, a_m)(m+1)^{-a_m}}{b_m}. \tag{6.5.9}$$

Therefore, pd,

$$\|q(T)\mathbf{x} - e_0\| \le \|q(T)\mathbf{y} - e_0\| + \|q(T)(\mathbf{y} - \mathbf{x})\| \tag{6.5.10}$$

$$= \|q(T)\mathbf{y} - e_0\| + \|q(T)(I - Q^0_m)\mathbf{x}\| \tag{6.5.11}$$

$$\le \frac{1}{a_{n-1}} + \frac{3}{a_m} + \frac{2N_3(m, a_m)(m+1)^{-a_m}}{b_m} \le \frac{2}{a_{n-1}}. \tag{6.5.12}$$

This inequality (which can be repeated with different values of n by choosing suitable alternative q) shows that in fact $\mathbf{x}$ is cyclic; and so we conclude the proof. ∎

7. Conclusion. The author would like to emphasize that in spite of this negative result, there is some hope that the spaces of Gowers and Maurey may indeed have no invariant subspace free operators. Though the general result about strictly singular operators is false, yet operators on the spaces of Gowers and Maurey have more properties than just being of form $\lambda I+$ (strictly singular). These spaces do not seem to support many operators like weighted shift operators, whose presence is so important for the construction of operators without invariant subspaces, by our methods at any rate. So there is some chance that operators on these very special spaces may indeed have invariant subspaces.

References

[E1] P. Enflo, *On the invariant subspace problem for Banach spaces*, Acta Math. 158 (1987), 212–313.

[G1] W. T. Gowers, *A solution to Banach's hyperplane problem*, Bull. London Math. Soc. 26 (1994), 523–530.

[G2] —, *A solution to the Schroeder–Bernstein problem for Banach spaces*, ibid. 28 (1996), 297–304.

[GM1] W. T. Gowers and B. Maurey, *The unconditional basic sequence problem*, J. Amer. Math. Soc. 16 (1993), 851–874.

[GM2] —, —, *Banach spaces with small spaces of operators*, Math. Ann. 307 (1997), 543–568.

[J1] R. C. James, *A non-reflexive Banach space isometric with its second conjugate space*, Proc. Nat. Acad. Sci. U.S.A. 37 (1951), 174–177.

[R1] C. J. Read, *A solution to the invariant subspace problem*, Bull. London Math. Soc. 16 (1984), 337–401.

[R2] —, *A short proof concerning the invariant subspace problem*, J. London Math. Soc. (2) 33 (1986), 335–348.

[R3] —, *The invariant subspace problem, a description with further applications of a combinatorial proof*, in: Advances in Invariant Subspaces and Other Results of Operator Theory, Birkhäuser, 1986, 275–300.

[R4] —, *A solution to the invariant subspace problem on the space l_1*, Bull. London Math. Soc. 17 (1985), 305–317.

[R5] —, *The invariant subspace problem on a class of nonreflexive Banach spaces*, in: Geometric Aspects of Functional Analysis, Israel Seminar (GAFA) 1986-7, Lecture Notes in Math. 1317, Springer, 1988, 1–20.

[R6] —, *The invariant subspace problem for a class of Banach spaces*, 2: *Hypercyclic operators*, Israel J. Math. 63 (1988), 1–40.

[R7] —, *The invariant subspace problem on some Banach spaces with separable dual*, Proc. London Math. Soc. (3) 58 (1989), 583–607.

[R8] —, *Quasinilpotent operators and the invariant subspace problem*, J. London Math. Soc. (2) 56 (1997), 595–606.

[Ra1] H. Radjavi and P. Rosenthal, *Invariant Subspaces*, Springer, 1973.

[S1] I. Singer, *Bases in Banach Spaces I*, Springer, 1970.

Trinity College
Cambridge
CB2 1TQ, England
E-mail: cr25@cam.ac.uk

Received March 7, 1996
Revised version June 25, 1998 (3627)

STUDIA MATHEMATICA 132 (3) (1999)

A general geometric construction for affine surface area

by

ELISABETH WERNER (Cleveland, Ohio, and Lille)

Abstract. Let K be a convex body in $\mathbb{R}^n$ and B be the Euclidean unit ball in $\mathbb{R}^n$. We show that
$$\lim_{t\to 0}\frac{|K|-|K_t|}{|B|-|B_t|}=\frac{\operatorname{as}(K)}{\operatorname{as}(B)},$$
where $\operatorname{as}(K)$ respectively $\operatorname{as}(B)$ is the affine surface area of K respectively B and $\{K_t\}_{t\ge0}$, $\{B_t\}_{t\ge0}$ are general families of convex bodies constructed from K, B satisfying certain conditions. As a corollary we get results obtained in [M-W], [Schm], [S-W] and [W].

The affine surface area $\operatorname{as}(K)$ was introduced by Blaschke [B] for convex bodies in $\mathbb{R}^3$ with sufficiently smooth boundary and by Leichtweiss [L1] for convex bodies in $\mathbb{R}^n$ with sufficiently smooth boundary as follows:
$$\operatorname{as}(K)=\int\limits_{\partial K}\kappa(x)^{1/(n+1)}\,d\mu(x),$$
where $\kappa(x)$ is the Gaussian curvature at $x\in\partial K$ and μ is the surface measure on ∂K. As it occurs naturally in many important questions, for example in the approximation of convex bodies by polytopes (see the survey article of Gruber [Gr] and the paper by Schütt [S]) or in a priori estimates for PDEs [Lu-O], one wanted to have extensions of the affine surface area to arbitrary convex bodies in $\mathbb{R}^n$ without any smoothness assumptions on the boundary.

Such extensions were given in recent years by Leichtweiss [L2], Lutwak [Lu], Meyer and Werner [M-W], Schmuckenschläger [Schm], Schütt and Werner [S-W] and Werner [W].

The extensions of affine surface area to an arbitrary convex body K in $\mathbb{R}^n$ in [L2], [M-W], [Schm], [S-W] and [W] have a common feature. First a specific family $\{K_t\}_{t\ge0}$ of convex bodies is constructed. This family is different in each of the cited extensions but of course related to the given convex body K. Typically the families $\{K_t\}_{t\ge0}$ are obtained from K through a "geometric" construction. In [L2] respectively [S-W] this construction gives

1991 *Mathematics Subject Classification*: Primary 52A20.
Partially supported by a grant from the National Science Foundation.