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Received May 15, 1997
Revised version August 24, 1998
(3885)

STUDIA MATHEMATICA 133 (3) (1999)

On strongly asymptotically developable functions and the Borel–Ritt theorem

by

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Abstract. We show that the holomorphic functions on polysectors whose derivatives remain bounded on proper subpolysectors are precisely those strongly asymptotically developable in the sense of Majima. This fact allows us to solve two Borel–Ritt type interpolation problems from a functional-analytic viewpoint.

Introduction. It is well known that, for a function f holomorphic on a sector S in the complex plane with vertex at 0, the existence of asymptotic expansion as the variable tends to 0 amounts to the boundedness of the derivatives of f on bounded proper subsectors of S. The Borel–Ritt theorem assures the existence of holomorphic functions on a given sector S admitting a prescribed asymptotic expansion at 0 in S. There are several classical proofs of this result in the literature (see, e.g., [Ol, Chapter 1, §9, p. 22], [Wa, Chapter III, §9.2, p. 43]). One of them (based on the ideas of [Ol, Chapter 4, §1.1, p. 106]; see Theorem 5.1 in this paper) has the particular feature that the derivatives of the solution are in fact bounded on unbounded proper subsectors of S. So, the Borel–Ritt interpolation problem is solvable in a different setting.

The aim of this paper is to transfer this characterization and results to the case of strongly asymptotically developable holomorphic functions of several complex variables, as defined by Majima [Ma]. To this end, Section 3 is devoted to the study of the space $\mathcal{A}(S)$ of holomorphic functions on a polysector S of $\mathbb{C}^n$ whose derivatives remain bounded in bounded proper subpolysectors of S; we give $\mathcal{A}(S)$ a natural Fréchet space topology, and prove that it is precisely the space of holomorphic functions on S strongly asymptotically developable at the origin. This equivalence allows us to obtain many properties of these functions in an elementary way. The main ideas in this section first appeared, for the Gevrey case, in the paper of Haraoka [Ha]; the results, in the present terms, come from the work of Hernández [He].

1991 *Mathematics Subject Classification*: 34E05, 41A60, 41A.

In Section 4, the corresponding Borel–Ritt problem in this context is stated: given a coherent family $\mathcal{F}$ (see Section 3 for the definition), does there exist $f \in \mathcal{A}(S)$ such that $\mathrm{TA}(f) = \mathcal{F}$?

Majima [Ma2, Part I, Theorem 3.1, p. 35] gives a partial solution: for such an $\mathcal{F}$ and for a bounded proper subpolysector T of S, there exists $f \in \mathcal{A}(T)$ such that $\mathrm{TA}(f) = \mathcal{F}$. Hernández [He] solves the problem, as initially stated, by a constructive method which strongly depends on the boundedness of the subpolysectors imposed in the definition of $\mathcal{A}(S)$. He considers the Fréchet space $\mathcal{A}(S, E)$ of holomorphic functions on a polysector S, with values in a Fréchet space E, and whose derivatives remain bounded on proper bounded subpolysectors of S. After obtaining a solution in series form when S is a sector, he studies its properties in the particular case in which E is of the type $\mathcal{A}(U, E)$, U being a polysector; this, together with the fact that $\mathcal{A}(S, \mathcal{A}(U, E))$ and $\mathcal{A}(S \times U, E)$ are isomorphic, allows applying an induction argument on the number of variables to conclude.

The solution in this paper is completely different, due to the following reasons. Section 5, of mainly theoretical interest, is devoted to obtaining a Borel–Ritt type theorem in the framework of the space $\mathcal{B}(S)$ of holomorphic functions on an unbounded polysector S of $\mathbb{C}^n$ whose derivatives are bounded on unbounded proper subpolysectors of S. This result corresponds, in the several variables case, to Theorem 5.1, and it was the motivation for the present work. We were able neither to obtain a solution in series form in the one-dimensional case, nor to make a suitable study of the solution obtained in Theorem 5.1 (which remains valid when we consider the space $\mathcal{B}(S, E)$, E being a Fréchet space, instead of $\mathcal{B}(S)$) so that we might apply induction. So, a different approach was necessary. Functional-analysis techniques turned out to be fruitful not only in this situation, but also in a similar treatment for $\mathcal{A}(S)$.

2. Notation. For $n \in \mathbb{N}$, $n \geq 1$, put $N = \{1, \ldots, n\}$. Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$, $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$ be two multiindices, m a natural number, and $\boldsymbol{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$. We set

$$\boldsymbol{\alpha} + \boldsymbol{\beta} = (\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n), \quad m\boldsymbol{\alpha} = (m\alpha_1, \ldots, m\alpha_n),$$
$$|\boldsymbol{\alpha}| = \alpha_1 + \ldots + \alpha_n, \quad \boldsymbol{\alpha}! = \alpha_1! \ldots \alpha_n!,$$
$$\boldsymbol{\alpha} \leq \boldsymbol{\beta} \Leftrightarrow \alpha_j \leq \beta_j,\ j \in N, \quad \boldsymbol{\alpha} < \boldsymbol{\beta} \Leftrightarrow \alpha_j < \beta_j,\ j \in N,$$
$$\mathbf{1} = (1, \ldots, 1), \quad \boldsymbol{e}_j = (\delta_{ij})_{i=1}^n,$$
$$\boldsymbol{z}^{\boldsymbol{\alpha}} = z_1^{\alpha_1} \ldots z_n^{\alpha_n}, \quad |\boldsymbol{z}^{\boldsymbol{\alpha}}| = |\boldsymbol{z}|^{\boldsymbol{\alpha}} = |z_1|^{\alpha_1} \ldots |z_n|^{\alpha_n},$$
$$D^{\boldsymbol{\alpha}} = \frac{\partial^{\boldsymbol{\alpha}}}{\partial \boldsymbol{z}^{\boldsymbol{\alpha}}} = \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial z_1^{\alpha_1} \ldots \partial z_n^{\alpha_n}}.$$

If J is a nonempty subset of N, the number of elements of J will be $\#J$.

Let n be a natural number, $n \geq 1$, and consider, for $j = 1, \ldots, n$, an open sector in $\mathbb{C}$ with vertex at the origin,

$$S_j = \{z \in \mathbb{C} : \theta_{1j} < \arg(z) < \theta_{2j}\}, \quad 0 < \theta_{2j} - \theta_{1j} \leq 2\pi.$$

Any cartesian product of open sectors in $\mathbb{C}$ with vertex at 0, $S = \prod_{j=1}^n S_j \subset \mathbb{C}^n$, will be called an (unbounded open) *polysector* in $\mathbb{C}^n$ with vertex at $\mathbf{0}$.

We say a polysector T in $\mathbb{C}^n$ (with vertex at the origin) is a *proper subpolysector* of S if $T = \prod_{j=1}^n T_j$ with $\overline{T}_j \subset S_j \cup \{0\}$, $j = 1, \ldots, n$. If

$$T_j = \{z \in \mathbb{C} : \varphi_{1j} < \arg(z) < \varphi_{2j},\ 0 < |z| < r_j\},$$

we say T is a *bounded* proper subpolysector of S.

If $J = \{j_1 < \ldots < j_k\}$ is a nonempty subset of N and $\boldsymbol{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$, we put $\boldsymbol{z}_J = (z_{j_1}, \ldots, z_{j_k})$. Let J and L be nonempty disjoint subsets of N. For $\boldsymbol{z}_J \in \mathbb{C}^J$ and $\boldsymbol{z}_L \in \mathbb{C}^L$, $(\boldsymbol{z}_J, \boldsymbol{z}_L)$ represents the element of $\mathbb{C}^{J \cup L}$ satisfying $(\boldsymbol{z}_J, \boldsymbol{z}_L)_J = \boldsymbol{z}_J$ and $(\boldsymbol{z}_J, \boldsymbol{z}_L)_L = \boldsymbol{z}_L$; we also write $J^c = N - J$, and for $j \in N$ we use j^c instead of $\{j\}^c$. In particular, we shall use these conventions for multiindices.

Finally, if $S = \prod_{j=1}^n S_j$ is a polysector of $\mathbb{C}^n$, then $S_J = \prod_{j \in J} S_j \subset \mathbb{C}^J$.

3. Characterization of strongly asymptotically developable functions. Denote by $\mathcal{A}(S)$ the complex vector space consisting of the complex functions f defined and holomorphic in S, such that for each bounded proper subpolysector T of S and each $\boldsymbol{\alpha} \in \mathbb{N}^n$,

$$Q_{T,\boldsymbol{\alpha}}(f) = \sup\{|D^{\boldsymbol{\alpha}} f(\boldsymbol{z})| : \boldsymbol{z} \in T\} < \infty.$$

Clearly, $\mathcal{A}(S)$ is closed under product and differentiation. We equip $\mathcal{A}(S)$ with the topology generated by the family $\{Q_{T,\boldsymbol{\alpha}}\}$ of seminorms; this makes $\mathcal{A}(S)$ a Fréchet space.

Let $f \in \mathcal{A}(S)$. Since all its derivatives are bounded on bounded proper subpolysectors of S, Barrow's formula implies that they are also lipschitzian. Hence, if $\emptyset \neq J \subset N$ and $\boldsymbol{\alpha}_J \in \mathbb{N}^J$, we can define a function from S_{J^c} to $\mathbb{C}$ by

$$f_{\boldsymbol{\alpha}_J}(\boldsymbol{z}_{J^c}) = \lim_{\substack{\boldsymbol{z}_J \to \mathbf{0} \\ \boldsymbol{z}_J \in T_J}} \frac{D^{(\boldsymbol{\alpha}_J, \mathbf{0}_{J^c})} f(\boldsymbol{z})}{\boldsymbol{\alpha}_J!}, \quad \boldsymbol{z}_{J^c} \in S_{J^c}, \tag{1}$$

for any subpolysector T_J of S_J; the limit is uniform on bounded proper subpolysectors of S_{J^c} (whenever $J \neq N$), which implies that $f_{\boldsymbol{\alpha}_J} \in \mathcal{A}(S_{J^c})$ (setting $\mathcal{A}(S_{N^c}) = \mathbb{C}$). Also, the map from $\mathcal{A}(S)$ to $\mathcal{A}(S_{J^c})$ sending f to $f_{\boldsymbol{\alpha}_J}$ is continuous.

Accordingly, we may associate with f a family

$$\mathcal{F}(f) = \{f_{\boldsymbol{\alpha}_J} \in \mathcal{A}(S_{J^c}) : \emptyset \neq J \subset N,\ \boldsymbol{\alpha}_J \in \mathbb{N}^J\},$$

which we call the *derived family* for f. The aforementioned uniformity in the limits defining its elements entails

PROPOSITION 3.1 (Coherence conditions). *Let $f \in \mathcal{A}(S)$, and let $\mathcal{F}(f)$ be its derived family. Then:*

(i) *For any disjoint nonempty subsets J and L of N with $J \cup L \neq N$, for every $\boldsymbol{\alpha}_J \in \mathbb{N}^J$ and $\boldsymbol{\alpha}_L \in \mathbb{N}^L$, and for every proper subpolysector T_L of S_L,*

$$\lim_{\substack{\boldsymbol{z}_L \to \boldsymbol{0} \\ \boldsymbol{z}_L \in T_L}} \frac{D^{(\boldsymbol{\alpha}_L, \boldsymbol{0}_{(J\cup L)^c})} f_{\boldsymbol{\alpha}_J}(\boldsymbol{z}_{J^c})}{\boldsymbol{\alpha}_L!} = f_{(\boldsymbol{\alpha}_J, \boldsymbol{\alpha}_L)}(\boldsymbol{z}_{(J\cup L)^c}),$$

uniformly on bounded proper subpolysectors of $S_{(J\cup L)^c}$.

(ii) *For each nonempty subset J of N, for each multiindex $\boldsymbol{\alpha} \in \mathbb{N}^n$ and for each proper subpolysector T_{J^c} of S_{J^c},*

$$\lim_{\substack{\boldsymbol{z}_{J^c} \to \boldsymbol{0} \\ \boldsymbol{z}_{J^c} \in T_{J^c}}} \frac{D^{\boldsymbol{\alpha}_{J^c}} f_{\boldsymbol{\alpha}_J}(\boldsymbol{z}_{J^c})}{\boldsymbol{\alpha}_{J^c}!} = f_{\boldsymbol{\alpha}} \in \mathbb{C}.$$

Hereafter, we will say that a family

$$\mathcal{F} = \{f_{\boldsymbol{\alpha}_J} \in \mathcal{A}(S_{J^c}) : \emptyset \neq J \subset N,\ \boldsymbol{\alpha}_J \in \mathbb{N}^J\},$$

or briefly $\mathcal{F} = \{f_{\boldsymbol{\alpha}_J}\}$, is *coherent* if it satisfies (i) and (ii).

The concept of strong asymptotic development was introduced by Majima (see [Ma]) in order to study solutions for integrable connections with irregular singular points. Let f be a complex function defined and holomorphic in a polysector S of $\mathbb{C}^n$ with vertex at $\boldsymbol{0}$. We say that f is *strongly asymptotically developable* at $\boldsymbol{0}$ if there exists a family

$$\mathcal{F} = \{f_{\boldsymbol{\alpha}_J} : \emptyset \neq J \subset N,\ \boldsymbol{\alpha}_J \in \mathbb{N}^J\},$$

where $f_{\boldsymbol{\alpha}_J}$ is a holomorphic function from S_{J^c} to $\mathbb{C}$ when $J \neq N$, and $f_{\boldsymbol{\alpha}_J} \in \mathbb{C}$ when $J = N$, satisfying the following bounds: if we define

$$\mathrm{App}_{\boldsymbol{\alpha}}(\mathcal{F})(\boldsymbol{z}) = \sum_{\emptyset \neq J \subset N} (-1)^{\#J+1} \sum_{\substack{\boldsymbol{\beta}_J \in \mathbb{N}^J \\ \boldsymbol{\beta}_J \leq \boldsymbol{\alpha}_J - \boldsymbol{1}_J}} f_{\boldsymbol{\beta}_J}(\boldsymbol{z}_{J^c}) \boldsymbol{z}_J^{\boldsymbol{\beta}_J}, \quad \boldsymbol{\alpha} \in \mathbb{N}^n,\ \boldsymbol{z} \in S,$$

then for every bounded proper subpolysector T of S and for every $\boldsymbol{\alpha} \in \mathbb{N}^n$,

$$\sup\left\{\left|\frac{f(\boldsymbol{z}) - \mathrm{App}_{\boldsymbol{\alpha}}(\mathcal{F})(\boldsymbol{z})}{\boldsymbol{z}^{\boldsymbol{\alpha}}}\right| : \boldsymbol{z} \in T\right\} < \infty.$$

Under these conditions, $\mathcal{F}$ will be called the *total family* of strongly asymptotic expansion associated with f, and will be denoted by $\mathrm{TA}(f)$. For $\boldsymbol{\alpha} \in \mathbb{N}^n$, the function $\mathrm{App}_{\boldsymbol{\alpha}}(\mathcal{F})$, defined and holomorphic from S to $\mathbb{C}$, is called the *approximate function* of order $\boldsymbol{\alpha}$ corresponding to the family $\mathcal{F}$.

THEOREM 3.2. *Let $f : S \to \mathbb{C}$ be holomorphic. Then f is strongly asymptotically developable at $\boldsymbol{0}$ in S if and only if $f \in \mathcal{A}(S)$. If this is the case, then $\mathcal{F}(f) = \mathrm{TA}(f)$.*

Proof. Assume f is strongly asymptotically developable. Consider a proper bounded subpolysector T of S and $\boldsymbol{\alpha} \in \mathbb{N}^n$. We may take a new proper bounded subpolysector T_1 of S such that T is proper in T_1, which allows us to find $r > 0$ such that for every $\boldsymbol{z} \in T$, the closed polydisc centered at $\boldsymbol{z}$ with polyradius $\boldsymbol{r}(\boldsymbol{z}) = (r|z_1|, \ldots, r|z_n|) \in (0,\infty)^n$, denoted by $\overline{D}_{\boldsymbol{r}(\boldsymbol{z})}(\boldsymbol{z})$, is contained in T_1. If $\boldsymbol{\omega}$ belongs to the distinguished boundary $\partial_0 \overline{D}_{\boldsymbol{r}(\boldsymbol{z})}(\boldsymbol{z})$, then $|\boldsymbol{\omega}|^{\boldsymbol{\alpha}} \leq (1+r)^{|\boldsymbol{\alpha}|} |\boldsymbol{z}|^{\boldsymbol{\alpha}}$ and $|\boldsymbol{\omega} - \boldsymbol{z}|^{\boldsymbol{\alpha}+\boldsymbol{1}} = r^{|\boldsymbol{\alpha}|+n} |\boldsymbol{z}|^{\boldsymbol{\alpha}+\boldsymbol{1}}$. As f admits a strongly asymptotic expansion at $\boldsymbol{0}$, there exists $C_{T_1,\boldsymbol{\alpha}} > 0$ such that

$$|f(\boldsymbol{z}) - \mathrm{App}_{\boldsymbol{\alpha}}(\mathrm{TA}(f))(\boldsymbol{z})| \leq C_{T_1,\boldsymbol{\alpha}} |\boldsymbol{z}|^{\boldsymbol{\alpha}}, \quad \boldsymbol{z} \in T_1.$$

Since $D^{\boldsymbol{\alpha}} \mathrm{App}_{\boldsymbol{\alpha}}(\mathrm{TA}(f)) \equiv 0$ on S, we can apply Cauchy's integral formula to find that, for every $\boldsymbol{z} \in T$,

$$|D^{\boldsymbol{\alpha}} f(\boldsymbol{z})| = \left| \frac{\boldsymbol{\alpha}!}{(2\pi i)^n} \int_{\partial_0 \overline{D}_{\boldsymbol{r}(\boldsymbol{z})}(\boldsymbol{z})} \frac{f(\boldsymbol{\omega}) - \mathrm{App}_{\boldsymbol{\alpha}}(\mathrm{TA}(f))(\boldsymbol{\omega})}{(\boldsymbol{\omega} - \boldsymbol{z})^{\boldsymbol{\alpha}+\boldsymbol{1}}}\, d\boldsymbol{\omega} \right|$$

$$\leq \boldsymbol{\alpha}!\, C_{T_1,\boldsymbol{\alpha}} \left(\frac{1+r}{r}\right)^{|\boldsymbol{\alpha}|} < \infty,$$

and we deduce that $f \in \mathcal{A}(S)$.

Conversely, let $f \in \mathcal{A}(S)$ and $\mathcal{F}(f)$ be its derived family. The error formula

$$f(\boldsymbol{z}) - \mathrm{App}_{\boldsymbol{\alpha}}(\mathcal{F}(f))(\boldsymbol{z})$$
$$= \prod_{\substack{j=1 \\ \alpha_j \neq 0}}^{n} \Big(\int_0^{z_j} dt_{j,1} \int_0^{t_{j,1}} dt_{j,2} \ldots \int_0^{t_{j,\alpha_j - 1}} dt_{j,\alpha_j} \Big) D^{\boldsymbol{\alpha}} f(t_{1,\alpha_1}, \ldots, t_{n,\alpha_n})$$

was given by Haraoka (cf. [Ha]; a proof valid in our setting can be found in a paper by Zurro [Zu]). Consider a proper bounded subpolysector T of S. Since $f \in \mathcal{A}(S)$, for every $\boldsymbol{\alpha} \in \mathbb{N}^n$ we have $\sup_{\boldsymbol{z} \in T} |D^{\boldsymbol{\alpha}} f(\boldsymbol{z})| = C_{T,\boldsymbol{\alpha}} < \infty$. Hence if $\boldsymbol{z} \in T$, then

$$|f(\boldsymbol{z}) - \mathrm{App}_{\boldsymbol{\alpha}}(\mathcal{F}(f))(\boldsymbol{z})| \leq \frac{|\boldsymbol{z}|^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!} C_{T,\boldsymbol{\alpha}},$$

so f admits a strongly asymptotic expansion at $\boldsymbol{0}$, and $\mathrm{TA}(f) = \mathcal{F}(f)$. ∎

Some remarks are in order. The uniqueness of $\mathrm{TA}(f)$ follows from the expressions given in (1). So, the approximate functions will be denoted henceforth as $\mathrm{App}_{\boldsymbol{\alpha}}(f)$, $\boldsymbol{\alpha} \in \mathbb{N}^n$. For $\emptyset \neq J \subset N$ and $\boldsymbol{\alpha}_J \in \mathbb{N}^J$, $f_{\boldsymbol{\alpha}_J} \in \mathcal{A}(S_{J^c})$; thus, the elements of the total family are strongly asymptotically developable, and from the coherence conditions it becomes obvious that

$$\mathrm{TA}(f_{\boldsymbol{\alpha}_J}) = \{f_{(\boldsymbol{\alpha}_J,\boldsymbol{\beta}_L)} : \emptyset \neq L \subset J^c,\ \boldsymbol{\beta}_L \in \mathbb{N}^L\}.$$

We also note that the notion of consistent family given by Majima (see [Ma2, Part I, p. 25]) is now seen to be equivalent to that of coherent family. Finally, it is evident that the family $\{P_{T,\boldsymbol{\alpha}}\}$ of seminorms, defined on $\mathcal{A}(S)$ for every bounded proper subpolysector T of S and $\boldsymbol{\alpha} \in \mathbb{N}^n$ as

$$P_{T,\boldsymbol{\alpha}}(f) = \sup\left\{\left|\frac{f(z) - \mathrm{App}_{\boldsymbol{\alpha}}(f)(z)}{z^{\boldsymbol{\alpha}}}\right| : z \in T\right\},$$

generates in $\mathcal{A}(S)$ the original topology.

4. Interpolation problem of Borel–Ritt type in $\mathcal{A}(S)$. If $f \in \mathcal{A}(S)$, then $\mathrm{TA}(f)$ is coherent. Thus, the following Borel–Ritt type problem arises:

Given a coherent family $\mathcal{F} = \{f_{\boldsymbol{\alpha}_J}\}$, does there exist $f \in \mathcal{A}(S)$ such that $\mathrm{TA}(f) = \mathcal{F}$?

In order to solve this problem we now give another two equivalent settings, obtained by changing the initial interpolation data. This will also let us go deeper into the relations linking the different elements in the concept of strongly asymptotic expansion.

First approach. If $f \in \mathcal{A}(S)$, then the *first order family associated with* f is

$$\mathrm{TA}'(f) = \{f_{m_{\{j\}}} \in \mathcal{A}(S_{j^c}) : j \in N,\ m \in \mathbb{N}\},$$

i.e., the subfamily of $\mathrm{TA}(f)$ consisting of those elements in $n-1$ variables. For convenience, we write f_{jm} instead of $f_{m_{\{j\}}}$. $\mathrm{TA}'(f)$ satisfies "first order" coherence conditions:

(a) Let $\boldsymbol{\alpha} \in \mathbb{N}^n$ and $j, l \in N$. For each bounded proper subpolysector T of S,

$$\lim_{\substack{z_{j^c} \to 0 \\ z_{j^c} \in T_{j^c}}} \frac{D^{\boldsymbol{\alpha}_{j^c}} f_{j\alpha_j}(z_{j^c})}{\boldsymbol{\alpha}_{j^c}!} = \lim_{\substack{z_{l^c} \to 0 \\ z_{l^c} \in T_{l^c}}} \frac{D^{\boldsymbol{\alpha}_{l^c}} f_{l\alpha_l}(z_{l^c})}{\boldsymbol{\alpha}_{l^c}!}.$$

Moreover, for $n \geq 3$, we have:

(b) Let L be a proper subset of N consisting of at least two elements, $\boldsymbol{\alpha}_L \in \mathbb{N}^L$ and T_L a bounded proper subpolysector of S_L. For every $j, l \in L$,

$$\lim_{\substack{z_{L-\{j\}} \to 0 \\ z_{L-\{j\}} \in T_{L-\{j\}}}} \frac{D^{(\boldsymbol{\alpha}_{L-\{j\}}, \mathbf{0}_{L^c})} f_{j\alpha_j}(z_{j^c})}{\boldsymbol{\alpha}_{L-\{j\}}!} = \lim_{\substack{z_{L-\{l\}} \to 0 \\ z_{L-\{l\}} \in T_{L-\{l\}}}} \frac{D^{(\boldsymbol{\alpha}_{L-\{l\}}, \mathbf{0}_{L^c})} f_{l\alpha_l}(z_{l^c})}{\boldsymbol{\alpha}_{L-\{l\}}!},$$

uniformly on bounded proper subpolysectors of S_{L^c}.

It turns out that $\mathrm{TA}'(f)$, under these first order conditions, determines $\mathrm{TA}(f)$ uniquely. Indeed, the case $n = 2$ is obvious; otherwise, let J be a subset of N consisting of at least two elements, and let $\boldsymbol{\alpha}_J \in \mathbb{N}^J$. Choose $j \in J$; then $f_{\boldsymbol{\alpha}_J}$ can be recovered as

$$f_{\boldsymbol{\alpha}_J}(z_{J^c}) = \lim_{\substack{z_{J-\{j\}} \to 0 \\ z_{J-\{j\}} \in T_{J-\{j\}}}} \frac{D^{(\boldsymbol{\alpha}_{J-\{j\}}, \mathbf{0}_{J^c})} f_{j\alpha_j}(z_{j^c})}{\boldsymbol{\alpha}_{J-\{j\}}!}, \quad z_{J^c} \in S_{J^c}, \tag{2}$$

$T_{J-\{j\}}$ being a bounded proper subpolysector of $S_{J-\{j\}}$. In fact, if we consider a family

$$\mathcal{F}' = \{f_{jm} \in \mathcal{A}(S_{j^c}) : j \in N,\ m \in \mathbb{N}\}$$

under the previous first order coherence conditions (henceforth, we will say that $\mathcal{F}' = \{f_{jm}\}$ is a *coherent first order family*), the relations in (2) define, with no ambiguity, a function $f_{\boldsymbol{\alpha}_J} \in \mathcal{A}(S_{J^c})$, and we may construct a family $\mathcal{F} = \{f_{\boldsymbol{\alpha}_J}\}$ that, by Proposition 3.1 applied to the functions f_{jm}, is seen to be coherent. So, the Borel–Ritt problem may be rewritten as follows:

Given a coherent first order family $\mathcal{F}'$, prove the existence of a function $f \in \mathcal{A}(S)$ such that $\mathrm{TA}'(f) = \mathcal{F}'$.

Second approach. Consider the family of approximate functions for $f \in \mathcal{A}(S)$, $\mathrm{App}(f) = (\mathrm{App}_{\boldsymbol{\alpha}}(f))_{\boldsymbol{\alpha} \in \mathbb{N}^n}$. Of course, the knowledge of $\mathrm{TA}(f)$ entails that of $\mathrm{App}(f)$; the converse is also true, since, for $\emptyset \neq J \subset N$ and $\boldsymbol{\alpha}_J \in \mathbb{N}^J$ we have, from the coherence conditions,

$$f_{\boldsymbol{\alpha}_J}(z_{J^c}) = \lim_{\substack{z_J \to 0 \\ z_J \in T_J}} \frac{D^{(\boldsymbol{\alpha}_J, \mathbf{0}_{J^c})} \mathrm{App}_{(\boldsymbol{\alpha}_J + \mathbf{1}_J, \mathbf{0}_{J^c})}(f)(z_J, z_{J^c})}{\boldsymbol{\alpha}_J!},$$

T_J being a bounded proper subpolysector of S_J. Next, observe that, for every $\boldsymbol{\alpha} \in \mathbb{N}^n$, we have $\mathrm{App}_{\boldsymbol{\alpha}}(f) \in \mathcal{A}_{\boldsymbol{\alpha}}$, where

$$\mathcal{A}_{\boldsymbol{\alpha}} = \{g \in \mathcal{A}(S) : \text{there exists } h \in \mathcal{A}(S) \text{ with } g = \mathrm{App}_{\boldsymbol{\alpha}}(h)\}.$$

Moreover, if $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^n$ and $\boldsymbol{\beta} \leq \boldsymbol{\alpha}$, a straightforward calculation gives

$$\mathrm{App}_{\boldsymbol{\beta}}(\mathrm{App}_{\boldsymbol{\alpha}}(f)) = \mathrm{App}_{\boldsymbol{\beta}}(f),$$

and so, if we set $\mathcal{P} = \prod_{\boldsymbol{\alpha} \in \mathbb{N}^n} \mathcal{A}_{\boldsymbol{\alpha}}$, then $\mathrm{App}(f) \in \mathcal{D}$, where

$$\mathcal{D} = \{(g_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha} \in \mathbb{N}^n} \in \mathcal{P} : \text{if } \boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{N}^n \text{ and } \boldsymbol{\beta} \leq \boldsymbol{\gamma}, \text{ then } \mathrm{App}_{\boldsymbol{\beta}}(g_{\boldsymbol{\gamma}}) = g_{\boldsymbol{\beta}}\}.$$

Conversely, if we begin with $(g_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha} \in \mathbb{N}^n} \in \mathcal{D}$, we may construct a family $\mathcal{F} = \{f_{\boldsymbol{\alpha}_J} \in \mathcal{A}(S_{J^c}) : \emptyset \neq J \subset N,\ \boldsymbol{\alpha}_J \in \mathbb{N}^J\}$ by defining

$$f_{\boldsymbol{\alpha}_J}(z_{J^c}) = \lim_{\substack{z_J \to 0 \\ z_J \in T_J}} \frac{D^{(\boldsymbol{\alpha}_J, \mathbf{0}_{J^c})} g_{(\boldsymbol{\alpha}_J + \mathbf{1}_J, \mathbf{0}_{J^c})}(z_J, z_{J^c})}{\boldsymbol{\alpha}_J!},$$

and it may be easily proved that $\mathcal{F}$ is coherent. Hence, our problem may also be expressed in these terms:

Given a family $\mathcal{I} \in \mathcal{D}$, does there exist a function $f \in \mathcal{A}(S)$ such that $\mathrm{App}(f) = \mathcal{I}$*?*

We shall answer the problem in the affirmative in this second approach, while the first one will come into play in the last section.

Define $\psi : \mathcal{A}(S) \to \mathcal{D}$ by $\psi(f) = \mathrm{App}(f)$. Solving the problem amounts to proving the surjectivity of ψ.

For each $\boldsymbol{\alpha} \in \mathbb{N}^n$, equip $\mathcal{A}_{\boldsymbol{\alpha}} \subset \mathcal{A}(S)$ with the subspace topology. Then $\mathcal{A}_{\boldsymbol{\alpha}}$ is a Fréchet space. $\mathcal{P} = \prod_{\boldsymbol{\alpha}\in\mathbb{N}^n} \mathcal{A}_{\boldsymbol{\alpha}}$ is given the product topology, and $\mathcal{D} \subset \mathcal{P}$ the subspace topology. Again, $\mathcal{D}$ is a Fréchet space.

The map $\psi : \mathcal{A}(S) \to \mathcal{D}$ is continuous. Indeed, it suffices to prove that for every $\boldsymbol{\alpha} \in \mathbb{N}^n$, the map $\psi_{\boldsymbol{\alpha}} : \mathcal{A}(S) \to \mathcal{A}_{\boldsymbol{\alpha}}$ defined by

$$\psi_{\boldsymbol{\alpha}}(f) = \mathrm{App}_{\boldsymbol{\alpha}}(f), \qquad f \in \mathcal{A}(S),$$

is continuous. Suppose a sequence $\{f_l\}_{l=1}^{\infty} \subset \mathcal{A}(S)$ converges to $g \in \mathcal{A}(S)$ and $\{\mathrm{App}_{\boldsymbol{\alpha}}(f_l)\}_{l=1}^{\infty}$ converges to $h \in \mathcal{A}_{\boldsymbol{\alpha}}$; then the continuity of the map from $\mathcal{A}(S)$ to $\mathcal{A}(S_{J^c})$ sending f to $f_{\boldsymbol{\alpha}_J}$ implies that

$$\mathrm{App}_{\boldsymbol{\alpha}}(g)(z) = \lim_{l\to\infty} \mathrm{App}_{\boldsymbol{\alpha}}(f_l)(z), \quad z \in S.$$

Since the convergence in $\mathcal{A}_{\boldsymbol{\alpha}}$ assures the pointwise convergence, this last limit equals $h(z)$, and we conclude with the Closed Graph Theorem.

We may then apply the following result (see [Ho, Chapter 3, §13, Proposition 3 and its Corollary, pp. 263–264]): ψ is surjective if and only if its transpose ${}^t\psi : \mathcal{D}' \to \mathcal{A}(S)'$ is injective and ${}^t\psi(\mathcal{D}')$ is $\sigma(\mathcal{A}(S)', \mathcal{A}(S))$-closed in $\mathcal{A}(S)'$. Also, ${}^t\psi$ is injective if and only if $\psi(\mathcal{A}(S))$ is dense in $\mathcal{D}$.

To prove the density, let $\mathcal{I} = (f_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in\mathbb{N}^n} \in \mathcal{D}$. A neighbourhood V of $\mathcal{I}$ is the product of neighbourhoods $V_{\boldsymbol{\alpha}}$ of $f_{\boldsymbol{\alpha}}$ for each $\boldsymbol{\alpha} \in \mathbb{N}^n$, where $V_{\boldsymbol{\alpha}} = \mathcal{A}_{\boldsymbol{\alpha}}$ except for finitely many multiindices, $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_m$. Consider the multiindex

$$\boldsymbol{\alpha}_0 = \boldsymbol{\alpha}_1 + \ldots + \boldsymbol{\alpha}_m \in \mathbb{N}^n,$$

and the function $f_{\boldsymbol{\alpha}_0} \in \mathcal{A}_{\boldsymbol{\alpha}_0} \subset \mathcal{A}(S)$. Since $\boldsymbol{\alpha}_l \le \boldsymbol{\alpha}_0$ for $l = 1, \ldots, m$, we have $\mathrm{App}_{\boldsymbol{\alpha}_l}(f_{\boldsymbol{\alpha}_0}) = f_{\boldsymbol{\alpha}_l}$, and so $\psi(f_{\boldsymbol{\alpha}_0}) = (\mathrm{App}_{\boldsymbol{\alpha}}(f_{\boldsymbol{\alpha}_0}))_{\boldsymbol{\alpha}\in\mathbb{N}^n} \in V$. Hence, ${}^t\psi$ is injective.

To prove that ${}^t\psi(\mathcal{D}')$ is weakly closed in $\mathcal{A}(S)'$, the following three results are needed.

PROPOSITION 4.1. *Let $L \in \mathcal{A}(S)'$ belong to the weak closure of ${}^t\psi(\mathcal{D}')$. Then*

$$\bigcap_{\boldsymbol{\alpha}\in\mathbb{N}^n} \mathrm{Ker}(\psi_{\boldsymbol{\alpha}}) \subset \mathrm{Ker}(L),$$

i.e., if $f \in \mathcal{A}(S)$ is such that $\mathrm{App}_{\boldsymbol{\alpha}}(f) = 0$ for all $\boldsymbol{\alpha} \in \mathbb{N}^n$, then $L(f) = 0$.

Proof. Let $f \in \bigcap_{\boldsymbol{\alpha}\in\mathbb{N}^n} \mathrm{Ker}(\psi_{\boldsymbol{\alpha}})$. For $p \in \mathbb{N}$, $p \ge 1$, consider the weak neighbourhood V_p of L,

$$V_p = \{M \in \mathcal{A}(S)' : |M(f) - L(f)| < 1/p\}.$$

Since $V_p \cap {}^t\psi(\mathcal{D}') \neq \emptyset$, there exists $\xi_p \in \mathcal{D}'$ such that $|{}^t\psi(\xi_p)(f) - L(f)| < 1/p$. As $\mathcal{D}$ is provided with the subspace topology of $\mathcal{P} = \prod_{\boldsymbol{\alpha}\in\mathbb{N}^n} \mathcal{A}_{\boldsymbol{\alpha}}$, we have $\mathcal{P}' = \bigoplus_{\boldsymbol{\alpha}\in\mathbb{N}^n} \mathcal{A}'_{\boldsymbol{\alpha}} \subset \mathcal{D}'$, and it makes sense to define $\overline{\psi} : \bigoplus_{\boldsymbol{\alpha}\in\mathbb{N}^n} \mathcal{A}'_{\boldsymbol{\alpha}} \to \mathcal{A}(S)'$ by

$$\overline{\psi}(\xi) = {}^t\psi(\xi|_{\mathcal{D}}) = {}^t\psi \circ \mu(\xi), \qquad \xi \in \bigoplus_{\boldsymbol{\alpha}\in\mathbb{N}^n} \mathcal{A}'_{\boldsymbol{\alpha}},$$

where μ represents restriction to $\mathcal{D}$ of continuous linear functionals defined on $\mathcal{P}$. By the Hahn–Banach theorem, μ is surjective. Thus, there exists $\eta_p \in \mathcal{P}' = \bigoplus_{\boldsymbol{\alpha}\in\mathbb{N}^n} \mathcal{A}'_{\boldsymbol{\alpha}}$ such that $\mu(\eta_p) = \xi_p$. Then

$$\overline{\psi}(\eta_p) = {}^t\psi \circ \mu(\eta_p) = {}^t\psi(\xi_p).$$

We may write, for a suitable $\boldsymbol{\alpha}_0 \in \mathbb{N}^n$, $\eta_p = \sum_{\boldsymbol{\alpha}\in\mathbb{N}^n, \boldsymbol{\alpha}\le\boldsymbol{\alpha}_0} \eta_{\boldsymbol{\alpha}}$, $\eta_{\boldsymbol{\alpha}} \in \mathcal{A}'_{\boldsymbol{\alpha}}$. For $g \in \mathcal{A}(S)$,

$$\begin{aligned}\overline{\psi}(\eta_{\boldsymbol{\alpha}})(g) &= {}^t\psi \circ \mu(\eta_{\boldsymbol{\alpha}})(g) = {}^t\psi(\eta_{\boldsymbol{\alpha}}|_{\mathcal{D}})(g) = \eta_{\boldsymbol{\alpha}}|_{\mathcal{D}} \circ \psi(g)\\ &= \eta_{\boldsymbol{\alpha}}|_{\mathcal{D}}((\mathrm{App}_{\boldsymbol{\alpha}}(g))_{\boldsymbol{\alpha}\in\mathbb{N}^n}) = \eta_{\boldsymbol{\alpha}}(\mathrm{App}_{\boldsymbol{\alpha}}(g)) = \eta_{\boldsymbol{\alpha}} \circ \psi_{\boldsymbol{\alpha}}(g).\end{aligned}$$

Since $\psi_{\boldsymbol{\alpha}}(f) = 0$ for all $\boldsymbol{\alpha}$, we see that for each $p \in \mathbb{N}$, $p \ge 1$,

$$\begin{aligned}|L(f)| &= \Big|L(f) - \sum_{\boldsymbol{\alpha}\le\boldsymbol{\alpha}_0} \eta_{\boldsymbol{\alpha}} \circ \psi_{\boldsymbol{\alpha}}(f)\Big| = |L(f) - \overline{\psi}(\eta_p)(f)|\\ &= |L(f) - {}^t\psi(\xi_p)(f)| < 1/p,\end{aligned}$$

and hence, $L(f) = 0$. ■

PROPOSITION 4.2. *Let $L \in \mathcal{A}(S)'$ have the following property: If $f \in \mathcal{A}(S)$ and $\mathrm{App}_{\boldsymbol{\alpha}}(f) = 0$ for all $\boldsymbol{\alpha} \in \mathbb{N}^n$, then $L(f) = 0$. Then there exists $r \in \mathbb{N}$, $r \ge 1$, such that if $f \in \mathcal{A}(S)$ and $\mathrm{App}_{\boldsymbol{\alpha}}(f) = 0$ for all $\boldsymbol{\alpha} \in \mathbb{N}^n$ with $|\boldsymbol{\alpha}| \le r + 1$, then $L(f) = 0$.*

Proof. For $j \in N$, consider a sequence of bounded proper subsectors of S_j, $\{T_{jr}\}_{r\in\mathbb{N}}$, such that:

(i) T_{jr} is a bounded proper subpolysector of $T_{j,r+1}$, $r \in \mathbb{N}$.

(ii) If K is a compact subset of S_j, then there exists $r \in \mathbb{N}$ such that $K \subset T_{jr}$.

(iii) T_{j1} has nonempty interior.

For $r \in \mathbb{N}$, define $T_r = \prod_{j\in N} T_{jr}$, and let p_r be the seminorm on $\mathcal{A}(S)$ given by

$$p_r(f) = \sup_{\boldsymbol{\alpha}\in\mathbb{N}^n,\, 0\le|\boldsymbol{\alpha}|\le r}\ \sup_{z\in T_r} \left|\frac{f(z) - \mathrm{App}_{\boldsymbol{\alpha}}(f)(z)}{z^{\boldsymbol{\alpha}}}\right|.$$

The increasing sequence $\{p_r\}_{r\in\mathbb{N}}$ of seminorms defines the topology of $\mathcal{A}(S)$. Hence, the sets $V_{\delta,r}=\{f\in\mathcal{A}(S):p_r(f)<\delta\}$, $r\in\mathbb{N}$, $\delta>0$, form a fundamental system of neighbourhoods of the origin in $\mathcal{A}(S)$, and given $\varepsilon=1$, there exist $\delta>0$ and $r\in\mathbb{N}$ such that, if $f\in V_{\delta,r}$, then $|L(f)|<1$. Notice that $V_{\delta,r'}\subset V_{\delta,r}$ if $r'>r$, so that we can assume, without loss of generality, that $r\geq 1$.

Let $g\in\mathcal{A}(S)$ be such that $\mathrm{App}_{\boldsymbol\alpha}(g)=0$ for all $\boldsymbol\alpha\in\mathbb{N}^n$ with $|\boldsymbol\alpha|\leq r+1$. We will prove that $L(g)=0$ by constructing a sequence $\{g_k\}_{k=1}^{\infty}\subset\mathcal{A}(S)$ such that $L(g_k)=L(g)$ for all k, and $\lim_{k\to\infty}L(g_k)=0$.

For $\boldsymbol\alpha$ with $|\boldsymbol\alpha|\leq r+1$, there exists $C_{\boldsymbol\alpha,r}>0$ such that

$$\sup_{\boldsymbol z\in T_r}\left|\frac{g(\boldsymbol z)}{\boldsymbol z^{\boldsymbol\alpha}}\right|\leq C_{\boldsymbol\alpha,r}. \tag{3}$$

The following fact can be easily proved (cf. [Co]): given a sector U in $\mathbb{C}$ with vertex at 0, there exists a holomorphic function $\beta:U\to\mathbb{C}$ satisfying:

(i) $\sup_{z\in U}|\beta(z)|<\infty$;

(ii) for each $j\in\mathbb{N}$,

$$\lim_{\substack{z\to0\\ z\in U}}\frac{|\beta(z)-1|}{|z|^j}=0\quad\text{and}\quad\sup_{z\in U}\frac{|\beta(z)-1|}{|z|^j}<\infty;$$

(iii) there exist $H,R>0$ such that for $z\in U$ with $|z|\geq R$, we have

$$|\beta(z)|\leq H/|z|^{r+1}.$$

Without loss of generality, we may assume that the sectors S_j, $j=1,\ldots,n$, have the positive real semiaxis as their symmetry semiaxis. Then we can consider a function β as above, holomorphic in a sector U of $\mathbb{C}$, such that $S_j\subset U$, $j\in N$. Define $\gamma:S\to\mathbb{C}$ by

$$\gamma(\boldsymbol z)=1-(-1)^n\prod_{j\in N}(\beta(z_j)-1)=\sum_{\emptyset\neq J\subset N}(-1)^{\#J+1}\prod_{j\in J}\beta(z_j)$$

for $\boldsymbol z=(z_1,\ldots,z_n)\in S$. Then γ is obviously holomorphic in S, and it satisfies:

(i′) $\sup_{\boldsymbol z\in S}|\gamma(\boldsymbol z)|\leq 1+\prod_{j\in N}(1+\sup_{z_j\in S_j}|\beta(z_j)|)<\infty$;

(ii′) for every $\boldsymbol\alpha\in\mathbb{N}^n$,

$$\lim_{\substack{\boldsymbol z\to0\\ \boldsymbol z\in S}}\left|\frac{\gamma(\boldsymbol z)-1}{\boldsymbol z^{\boldsymbol\alpha}}\right|=0\quad\text{and}\quad\sup_{\boldsymbol z\in T_l}\left|\frac{\gamma(\boldsymbol z)-1}{\boldsymbol z^{\boldsymbol\alpha}}\right|<\infty,$$

for $l\in\mathbb{N}$; hence $\gamma\in\mathcal{A}(S)$.

For $k\in\mathbb{N}$, $k\geq1$, we define the functions

$$\gamma_k(\boldsymbol z)=\gamma(k\boldsymbol z),\qquad g_k(\boldsymbol z)=\gamma_k(\boldsymbol z)g(\boldsymbol z),\qquad \boldsymbol z\in S.$$

If $\boldsymbol\alpha\in\mathbb{N}^n$ and $l\in\mathbb{N}$, we have

$$\begin{aligned}\sup_{\boldsymbol z\in T_l}\left|\frac{g_k(\boldsymbol z)-\mathrm{App}_{\boldsymbol\alpha}(g)(\boldsymbol z)}{\boldsymbol z^{\boldsymbol\alpha}}\right|&=\sup_{\boldsymbol z\in T_l}\left|\frac{\gamma_k(\boldsymbol z)g(\boldsymbol z)-g(\boldsymbol z)+g(\boldsymbol z)-\mathrm{App}_{\boldsymbol\alpha}(g)(\boldsymbol z)}{\boldsymbol z^{\boldsymbol\alpha}}\right|\\&\leq\sup_{\boldsymbol z\in T_l}|g(\boldsymbol z)|\sup_{\boldsymbol z\in T_l}\left|\frac{\gamma_k(\boldsymbol z)-1}{\boldsymbol z^{\boldsymbol\alpha}}\right|+\sup_{\boldsymbol z\in T_l}\left|\frac{g(\boldsymbol z)-\mathrm{App}_{\boldsymbol\alpha}(g)(\boldsymbol z)}{\boldsymbol z^{\boldsymbol\alpha}}\right|<\infty,\end{aligned}$$

where we have used (ii′). So, $g_k\in\mathcal{A}(S)$, $k\geq1$, and $\mathrm{App}_{\boldsymbol\alpha}(g_k)=\mathrm{App}_{\boldsymbol\alpha}(g)$, $\boldsymbol\alpha\in\mathbb{N}^n$, which leads to $L(g_k)=L(g)$, $k\geq1$.

Next we prove that for every $\boldsymbol\alpha\in\mathbb{N}^n$ with $|\boldsymbol\alpha|\leq r$ there exists a constant $C_0(\boldsymbol\alpha)>0$ such that

$$\sup_{\boldsymbol z\in T_r}\left|\frac{g_k(\boldsymbol z)}{\boldsymbol z^{\boldsymbol\alpha}}\right|<\frac{C_0(\boldsymbol\alpha)}{\sqrt{k}}$$

whenever $k>R^2$ (R being the constant introduced in (iii)).

Fix $\boldsymbol\alpha\in\mathbb{N}^n$ with $|\boldsymbol\alpha|\leq r$, and let $\overline{\boldsymbol\alpha}=\boldsymbol\alpha+\boldsymbol e_j$, $j\in N$. As $|\overline{\boldsymbol\alpha}|=|\boldsymbol\alpha|+1\leq r+1$, by (3) and (i′) we see that for every $\boldsymbol z\in T_r$,

$$\left|\frac{g_k(\boldsymbol z)}{\boldsymbol z^{\boldsymbol\alpha}}\right|=\left|\frac{\gamma_k(\boldsymbol z)g(\boldsymbol z)}{\boldsymbol z^{\boldsymbol\alpha}}\right|\leq C_\gamma\left|\frac{g(\boldsymbol z)}{\boldsymbol z^{\overline{\boldsymbol\alpha}}}\right||\boldsymbol z^{\boldsymbol e_j}|\leq C_\gamma C_{\overline{\boldsymbol\alpha},r}|z_j|.$$

If $|z_j|\leq1/\sqrt{k}$ for any $j\in N$, then

$$\left|\frac{g_k(\boldsymbol z)}{\boldsymbol z^{\boldsymbol\alpha}}\right|\leq C_\gamma C_{\boldsymbol\alpha+\boldsymbol e_j,r}\frac{1}{\sqrt{k}}\leq C_\gamma(\max_{j\in N}C_{\boldsymbol\alpha+\boldsymbol e_j,r})\frac{1}{\sqrt{k}}\leq\frac{C_1(\boldsymbol\alpha)}{\sqrt{k}}.$$

If $|z_j|>1/\sqrt{k}$ for all $j\in N$, suppose that $k>R^2$. We have $|kz_j|>\sqrt{k}>R$, and so

$$|\beta(kz_j)|\leq H/|kz_j|^{r+1}.$$

On the other hand, $|\boldsymbol z^{\boldsymbol\alpha}|=\prod_{j\in N}|z_j|^{\alpha_j}>(k^{-1/2})^{\alpha_1+\ldots+\alpha_n}=k^{-|\boldsymbol\alpha|/2}$. Applying again (3) and the definition of γ, we may write

$$\begin{aligned}\left|\frac{g_k(\boldsymbol z)}{\boldsymbol z^{\boldsymbol\alpha}}\right|&=\frac{1}{|\boldsymbol z^{\boldsymbol\alpha}|}|\gamma_k(\boldsymbol z)g(\boldsymbol z)|\leq C_{0,r}\frac{1}{|\boldsymbol z^{\boldsymbol\alpha}|}|\gamma(k\boldsymbol z)|\\&\leq\frac{C_{0,r}}{|\boldsymbol z^{\boldsymbol\alpha}|}\sum_{\substack{J\subset N\\ J\neq\emptyset}}\Big(\prod_{j\in J}|\beta(kz_j)|\Big)\leq\frac{C_{0,r}}{|\boldsymbol z^{\boldsymbol\alpha}|}\sum_{\substack{J\subset N\\ J\neq\emptyset}}\Big(\prod_{j\in J}\frac{H}{|kz_j|^{r+1}}\Big)\\&\leq C_{0,r}\sum_{J\subset N,J\neq\emptyset}\frac{H^{\#J}}{k^{-|\boldsymbol\alpha|/2}k^{(r+1)(\#J)/2}}\leq\frac{(2^n-1)H_1C_{0,r}}{\sqrt{k}}=\frac{C_2(\boldsymbol\alpha)}{\sqrt{k}},\end{aligned}$$

because $(r+1)(\#J)-|\boldsymbol\alpha|\geq r+1-|\boldsymbol\alpha|\geq1$, with $H_1=\max_{\emptyset\neq J\subset N}H^{\#J}$. Therefore, there exists $C_0(\boldsymbol\alpha)=\max(C_1(\boldsymbol\alpha),C_2(\boldsymbol\alpha))>0$ such that, if $k>R^2$, then

$$\sup_{\boldsymbol z\in T_r}\left|\frac{g_k(\boldsymbol z)}{\boldsymbol z^{\boldsymbol\alpha}}\right|<\frac{C_0(\boldsymbol\alpha)}{\sqrt{k}},$$

and thus,

$$p_r(g_k) = \sup_{0\le|\alpha|\le r}\ \sup_{z\in T_r}\left|\frac{g_k(z)}{z^{\alpha}}\right| < \frac{\max_{0\le|\alpha|\le r} C_0(\alpha)}{\sqrt{k}} = \frac{C_0}{\sqrt{k}},$$

or, in another way, $p_r(k^{1/4}g_k) < C_0/k^{1/4}$, $k > R^2$. There exists $k_1 \in \mathbb{N}$ with $k_1 > R^2$ such that if $k \ge k_1$, then $p_r(k^{1/4}g_k) < \delta$, therefore $|L(k^{1/4}g_k)| < 1$, and $L(g) = \lim_{k\to\infty} L(g_k) = 0$. ∎

The existence of the natural number r in Proposition 4.2 ensures the existence of a multiindex $\beta \in \mathbb{N}^n$ (e.g., $\beta = (r+1, r+1, \ldots, r+1)$) such that if $f \in \mathcal{A}(S)$ and $\mathrm{App}_{\alpha}(f) = 0$ for all $\alpha \in \mathbb{N}^n$ with $\alpha \le \beta$, then $L(f) = 0$.

PROPOSITION 4.3. *Let $L \in \mathcal{A}(S)'$ be such that there exists $\beta \in \mathbb{N}^n$ with the following property: $L(f) = 0$ for every function $f \in \mathcal{A}(S)$ such that $\mathrm{App}_{\alpha}(f) = 0$ for all $\alpha \in \mathbb{N}^n$ with $\alpha \le \beta$. Then there exists a functional $H \in \mathcal{D}'$ such that $L = H \circ \psi$.*

Proof. Consider an arbitrary function $h \in \mathcal{A}(S)$. The function $\widetilde{h}$ given as

$$\widetilde{h}(z) = h(z) - \mathrm{App}_{\beta}(h)(z), \quad z \in S,$$

is in $\mathcal{A}(S)$, and for all $\gamma \in \mathbb{N}^n$ with $\gamma \le \beta$ we have

$$\mathrm{App}_{\gamma}(\widetilde{h}) = \mathrm{App}_{\gamma}(h) - \mathrm{App}_{\gamma}(\mathrm{App}_{\beta}(h)) = 0,$$

so that $L(\widetilde{h}) = 0$, i.e., $L(h) = L(\mathrm{App}_{\beta}(h))$ for every $h \in \mathcal{A}(S)$. Define $H : \mathcal{D} \to \mathbb{C}$ given by

$$H((f_{\alpha})_{\alpha\in\mathbb{N}^n}) = L(f_{\beta}), \quad (f_{\alpha})_{\alpha\in\mathbb{N}^n} \in \mathcal{D}.$$

The natural injection from $\mathcal{D}$ into $\mathcal{P}$, the projection from $\mathcal{P}$ to $\mathcal{A}_{\beta}$, and the natural injection from $\mathcal{A}_{\beta}$ into $\mathcal{A}(S)$ are continuous mappings; hence $H \in \mathcal{D}'$. On the other hand, for $f \in \mathcal{A}(S)$,

$$(H \circ \psi)(f) = H((\mathrm{App}_{\alpha})_{\alpha\in\mathbb{N}^n}) = L(\mathrm{App}_{\beta}(f)) = L(f),$$

as desired. ∎

The last three propositions allow us to deduce that ${}^t\psi(\mathcal{D}')$ is weakly closed in $\mathcal{A}(S)'$, and hence, the surjectivity of ψ.

5. A new Borel–Ritt interpolation problem. In this section we prove the corresponding Borel–Ritt theorem in the space $\mathcal{B}(S)$ of holomorphic functions on an unbounded polysector S of $\mathbb{C}^n$ whose derivatives are bounded on unbounded proper subpolysectors of S. Section 3 may be repeated word by word for $\mathcal{B}(S)$, just changing bounded proper subpolysectors of S to unbounded ones. So, elements of $\mathcal{B}(S)$ are strongly asymptotically

developable at $\mathbf{0}$ in S in a somewhat different sense; having this in mind, we use the same phrasing in this section.

The Borel–Ritt interpolation problem is stated as follows:

Given a coherent family $\mathcal{F} = \{f_{\alpha_J} \in \mathcal{B}(S_{J^c}) : \emptyset \ne J \subset N,\ \alpha_J \in \mathbb{N}^J\}$, does there exist $f \in \mathcal{B}(S)$ such that $\mathrm{TA}(f) = \mathcal{F}$?

The main difference between the settings in $\mathcal{A}(S)$ and $\mathcal{B}(S)$ is that, whereas in the former the approximate functions for an element remain in the space considered, in the latter this need not to be so (see the situation in dimension one, in which approximate functions are polynomials). This makes it impossible to use the second approach (see Section 4) to solve the present problem. Instead, we adopt the first one, based on coherent first order families.

For each $j \in N$ and $m \in \mathbb{N}$, define $\mathcal{B}_{jm} = \mathcal{B}(S_{j^c})$, provided with the natural topology. Then $\mathcal{B}_{jm}$ is a Fréchet space. Denote by $\mathcal{P}$ the product space $\prod_{j\in N,\, m\in\mathbb{N}} \mathcal{B}_{jm}$, endowed with the product topology, and let $\mathcal{D}$ be the subspace of $\mathcal{P}$ consisting of coherent first order families $\mathcal{F}' = \{f_{jm}\}$. The subspace topology makes $\mathcal{D}$ a Fréchet space. The map ψ which sends a function $f \in \mathcal{B}(S)$ to its first order family $\mathrm{TA}'(f) \in \mathcal{D}$ is continuous, since so are the maps ψ_{jm}, $j \in N$, $m \in \mathbb{N}$, sending $f \in \mathcal{B}(S)$ to $f_{jm} \in \mathcal{B}_{jm}$.

Our aim is to prove that ψ is surjective; we apply the same argument as in the previous section. Due to the way $\mathcal{D}$ and its topology have been defined, $\psi(\mathcal{B}(S))$ is dense in $\mathcal{D}$ if we prove the possibility of interpolating finitely many elements of a family in $\mathcal{D}$ (in fact, in a "continuous" way, which will be decisive later on). This will be done in Proposition 5.3, after two auxiliary results. The first one is the one-dimensional Borel–Ritt theorem in this context.

THEOREM 5.1. *Let S be an unbounded sector of $\mathbb{C}$ with vertex at the origin. For any sequence $\{a_n\}_{n=0}^{\infty}$ of complex numbers, there exists a holomorphic function $f : S \to \mathbb{C}$ such that for every proper subsector T of S,*

$$\sup_{z\in T}\left|\frac{f(z) - \sum_{j=0}^{m-1} a_j z^j}{z^m}\right| < \infty, \quad m \in \mathbb{N}.$$

In this situation, we write $f \sim_S \sum_{n=0}^{\infty} a_n z^n$.

Proof. It is easy to reduce the problem to the sector $S = \{z : |\arg(z)| < \pi/4\}$. By the classical Borel theorem, we find $q \in \mathcal{C}^{\infty}(\mathbb{R})$ with compact support such that $q^{n)}(0) = a_n$, $n \in \mathbb{N}$. Then the function

$$F(z) = \int_0^{\infty} e^{-tz} q(t)\, dt, \quad z \in S,$$

is holomorphic in S, and $f(z) = z^{-1}F(z^{-1})$ solves the problem in S, as can be seen by taking $\delta = \pi/4$ and $\sigma = 0$ in [Ol, Chapter 4, §1.1, p. 106]. ∎

LEMMA 5.2. *Let $n \in \mathbb{N}$, $n > 1$, and $N = \{1, \ldots, n\}$. Let J be a nonempty proper subset of N, $f \in \mathcal{B}(S_{J^c})$ and $g \in \mathcal{B}(S_J)$ with $\mathrm{TA}'(f) = \{f_{jm}\}$ and $\mathrm{TA}'(g) = \{g_{jm}\}$. Then $fg \in \mathcal{B}(S)$, and if $\mathrm{TA}'(fg) = \{h_{jm}\}$ then*

$$h_{jm}(z_{j^c}) = \begin{cases} f_{jm}(z_{(J\cup\{j\})^c})g(z_J) & \text{if } j \in J^c, \\ g_{jm}(z_{J-\{j\}})f(z_{J^c}) & \text{if } j \in J. \end{cases}$$

PROPOSITION 5.3. *Let S be a polysector of $\mathbb{C}^n$ with vertex at $\mathbf{0}$. Then:*

(i) *Given $\mathcal{F}' = \{f_{jm}\} \in \mathcal{D}$ and $p \in \mathbb{N}$, there exists $F \in \mathcal{B}(S)$ such that if $\mathrm{TA}'(F) = \{F_{jm}\}$, then $F_{jm} = f_{jm}$, $j = 1, \ldots, n$, $m = 0, 1, \ldots, p$.*

(ii) *Let $\{\mathcal{F}'_k\}_{k\in\mathbb{N}} \subset \mathcal{D}$ converge to 0, where $\mathcal{F}'_k = \{f_{jm,k}\}$, $k \in \mathbb{N}$. Given $p \in \mathbb{N}$, there exists a sequence $\{F_k\}_{k\in\mathbb{N}} \subset \mathcal{B}(S)$ converging to 0 such that if $\mathrm{TA}'(F_k) = \{F_{jm,k}\}$, $k \in \mathbb{N}$, then $F_{jm,k} = f_{jm,k}$, $j = 1, \ldots, n$, $m = 0, 1, \ldots, p$.*

For a better understanding of the procedure, and to get rid of cumbersome notation, we limit ourselves to the case $n = 2$. The proof for an arbitrary dimension is analogous.

In the two-dimensional case the statement is as follows: Let $S = S_1 \times S_2$ be a polysector of $\mathbb{C}^2$ with vertex at the origin.

(i) Consider a coherent first order family $\mathcal{F}' = \{f_n \in \mathcal{B}(S_1),\ g_m \in \mathcal{B}(S_2) : n, m \in \mathbb{N}\}$, i.e., for every $m, n \in \mathbb{N}$, and any proper subsectors T_1 of S_1 and T_2 of S_2, we have

$$\lim_{\substack{z\to 0\\ z\in T_1}} \frac{f_n^{m)}(z)}{m!} = \lim_{\substack{\omega\to 0\\ \omega\in T_2}} \frac{g_m^{n)}(\omega)}{n!}. \tag{4}$$

Then, given $p \in \mathbb{N}$, there exists $F \in \mathcal{B}(S)$ with

$$\mathrm{TA}'(F) = \{h_n \in \mathcal{B}(S_1),\ l_m \in \mathcal{B}(S_2) : n, m \in \mathbb{N}\},$$

satisfying

$$h_n = f_n, \quad l_m = g_m, \quad n, m = 0, 1, \ldots, p.$$

(ii) For $k \in \mathbb{N}$, consider coherent first order families

$$\mathcal{F}'_k = \{f_{n,k} \in \mathcal{B}(S_1),\ g_{m,k} \in \mathcal{B}(S_2) : n, m \in \mathbb{N}\}$$

such that $\{\mathcal{F}'_k\}_{k\in\mathbb{N}}$ converges to 0 in $\mathcal{D}$. Then, given $p \in \mathbb{N}$, there exists a sequence $\{F_k\}_{k\in\mathbb{N}} \subset \mathcal{B}(S)$ converging to 0 such that if

$$\mathrm{TA}'(F_k) = \{h_{n,k} \in \mathcal{B}(S_1),\ l_{m,k} \in \mathcal{B}(S_2) : n, m \in \mathbb{N}\}, \quad k \in \mathbb{N},$$

we have $h_{n,k} = f_{n,k}$ and $l_{m,k} = g_{m,k}$, $n, m = 0, 1, \ldots, p$, $k \in \mathbb{N}$.

For brevity, we write $\{f_n, g_m\}$ instead of $\{f_n \in \mathcal{B}(S_1),\ g_m \in \mathcal{B}(S_2) : n, m \in \mathbb{N}\}$. The proof of (i) is carried out in two steps. In the following, $T = T_1 \times T_2$ denotes an arbitrary proper subpolysector of S.

STEP 1. We will obtain a function $F \in \mathcal{B}(S)$ such that if $\mathrm{TA}'(F) = \{h_n, l_m\}$, then $h_n = f_n$, $n = 0, 1, \ldots, p$, and $l_0 = g_0$. Indeed, by Theorem 5.1, we can consider, for $j = 0, 1, \ldots, p$, a function $\alpha_j \in \mathcal{B}(S_2)$ such that $\alpha_j(\omega) \sim_{S_2} \omega^j$. Define $G_j : S \to \mathbb{C}$ by $G_j(z, \omega) = f_j(z)\alpha_j(\omega)$, $(z, \omega) \in S$. According to Lemma 5.2, $G_j \in \mathcal{B}(S)$. Let $\mathrm{TA}'(G_j) = \{h_{n,j}, l_{m,j}\}$. By the choice of α_j, for $z \in S_1$, we have

$$h_{n,j}(z) = \lim_{\substack{\omega\to 0\\ \omega\in T_2}} \frac{D^{(0,n)}G_j(z,\omega)}{n!} = f_j(z)\delta_{n,j},$$

where $\delta_{n,j}$ equals 1 if $n = j$ and 0 if $n \neq j$. Hence, $h_{n,j} = 0$ if $n \neq j$, and $h_{j,j} = f_j$. The function $G = \sum_{i=1}^p G_i$ belongs to $\mathcal{B}(S)$, and if $\mathrm{TA}'(G) = \{H_n, L_m\}$, it is clear that $H_n = 0$ for $n > p$, and $H_n = f_n$ for $n \le p$.

Let $\beta_0 \in \mathcal{B}(S_1)$ be such that $\beta_0(z) \sim_{S_1} 1$, and define $M : S \to \mathbb{C}$ by

$$M(z, \omega) = \beta_0(z)(g_0(\omega) - L_0(\omega)), \quad (z, \omega) \in S.$$

By Lemma 5.2, $M \in \mathcal{B}(S)$. If $\mathrm{TA}'(M) = \{\widetilde{H}_n, \widetilde{L}_m\}$, then, according to (4) and the coherence conditions for the family $\mathrm{TA}'(G)$, for $n \le p$ and $z \in S_1$ we have

$$\begin{aligned}\widetilde{H}_n(z) &= \lim_{\substack{\omega\to 0\\ \omega\in T_2}} \frac{D^{(0,n)}M(z,\omega)}{n!} = \beta_0(z) \lim_{\substack{\omega\to 0\\ \omega\in T_2}} \frac{g_0^{n)}(\omega) - L_0^{n)}(\omega)}{n!} \\ &= \beta_0(z) \lim_{\substack{z\to 0\\ z\in T_1}} (f_n(z) - H_n(z)) = 0;\end{aligned}$$

on the other hand, by the choice of β_0, for every $\omega \in S_2$ we have

$$\widetilde{L}_0(\omega) = \lim_{\substack{z\to 0\\ z\in T_1}} M(z, \omega) = (g_0(\omega) - L_0(\omega)) \lim_{\substack{z\to 0\\ z\in T_1}} \beta_0(z) = g_0(\omega) - L_0(\omega).$$

The additivity of first order families allows us to conclude that the function $F = G + M \in \mathcal{B}(S)$ is a solution for the first step.

STEP 2. To get the result mentioned, we use recurrence, assuming that there exists $G \in \mathcal{B}(S)$ such that if $\mathrm{TA}'(G) = \{H_n, L_m\}$, then $H_n = f_n$ if $n \le p$ and $L_m = g_m$ if $m \le p - 1$.

Consider $\beta_p \in \mathcal{B}(S_1)$ such that $\beta_p(z) \sim_{S_1} z^p$, and define $M : S \to \mathbb{C}$ as

$$M(z, \omega) = \beta_p(z)(g_p(\omega) - L_p(\omega)), \quad (z, \omega) \in S.$$

Lemma 5.2 again yields $M \in \mathcal{B}(S)$; say $\mathrm{TA}'(M) = \{\widetilde{H}_n, \widetilde{L}_m\}$. Because of (4) and of the coherence conditions for $\mathrm{TA}'(G)$, for $n \le p$ and $z \in S_1$ we have

$$\begin{aligned}\widetilde{H}_n(z) &= \lim_{\substack{\omega\to 0\\ \omega\in T_2}} \frac{D^{(0,n)}M(z,\omega)}{n!} = \beta_p(z) \lim_{\substack{\omega\to 0\\ \omega\in T_2}} \frac{g_0^{n)}(\omega) - L_0^{n)}(\omega)}{n!} \\ &= \beta_p(z) \lim_{\substack{z\to 0\\ z\in T_1}} (f_n(z) - H_n(z)) = 0,\end{aligned}$$

whereas, from the choice of β_p, for $m \le p$ and $\omega \in S_2$ we have

$$\widetilde{L}_m(\omega) = \lim_{\substack{z\to 0\\ z\in T_1}} \frac{D^{(m,0)}M(z,\omega)}{m!}$$
$$= (g_p(\omega) - L_p(\omega)) \lim_{\substack{z\to 0\\ z\in T_1}} \frac{\beta_p^{(m)}(z)}{m!} = (g_p(\omega) - L_p(\omega))\delta_{mp}.$$

The function $F = G + M \in \mathcal{B}(S)$ is the solution we were looking for.

(ii) We again divide the proof in two steps.

STEP 1. Our aim is to show the existence of a sequence $\{F_k\}_{k\in\mathbb{N}} \subset \mathcal{B}(S)$ converging to 0 such that if $\mathrm{TA}'(F_k) = \{h_{n,k}, l_{m,k}\}$, $k \in \mathbb{N}$, then $h_{n,k} = f_{n,k}$, $n = 0, 1, \ldots, p$, and $l_{0,k} = g_{0,k}$.

For $j = 0, 1, \ldots, p$, consider $\alpha_j \in \mathcal{B}(S_2)$ such that $\alpha_j(\omega) \sim_{S_2} \omega^j$, and define $G_{j,k} \in \mathcal{B}(S)$ by $G_{j,k}(z,\omega) = f_{j,k}(z)\alpha_j(\omega)$, $(z,\omega) \in S$. We claim that $\{G_{j,k}\}_{k\in\mathbb{N}}$ converges to 0 in $\mathcal{B}(S)$ for $j = 0, 1, \ldots, p$: for every proper subpolysector $T = T_1 \times T_2$ of S and every multiindex $\gamma = (\gamma_1, \gamma_2) \in \mathbb{N}^2$,

$$Q_{T,\gamma}(G_{j,k}) = \sup_{(z,\omega)\in T} |D^\gamma G_{j,k}(z,\omega)| \le \sup_{z\in T_1} |f_{j,k}^{(\gamma_1)}(z)| \sup_{\omega\in T_2} |\alpha_j^{(\gamma_2)}(\omega)|$$
$$= C(T_2, j, \gamma_2) Q_{T_1,\gamma_1}(f_{j,k}).$$

As $\{f_{j,k}\}_{k\in\mathbb{N}}$ converges to 0 in $\mathcal{B}(S_1)$, the conclusion is immediate.

For $k \in \mathbb{N}$, $G_k = \sum_{i=1}^p G_{i,k} \in \mathcal{B}(S)$; say $\mathrm{TA}'(G_k) = \{H_{n,k}, L_{m,k}\}$. The sequence $\{G_k\}_{k\in\mathbb{N}}$ obviously converges to 0 in $\mathcal{B}(S)$. The continuity of the map that sends each element of $\mathcal{B}(S)$ to the corresponding element of its first order family now yields the convergence of $\{L_{0,k}\}_{k\in\mathbb{N}}$ to 0 in $\mathcal{B}(S_2)$.

Let $\beta_0 \in \mathcal{B}(S_1)$ have $\beta_0(z) \sim_{S_1} 1$, and define $M_k \in \mathcal{B}(S)$, $k \in \mathbb{N}$, by

$$M_k(z,\omega) = \beta_0(z)(g_{0,k}(\omega) - L_{0,k}(\omega)), \quad (z,\omega) \in S.$$

We have

$$Q_{T,\gamma}(M_k) = \sup_{(z,\omega)\in T} |D^\gamma M_k(z,\omega)| \le \sup_{z\in T_1} |\beta_0^{(\gamma_1)}(z)| \sup_{\omega\in T_2} |(g_{0,k} - L_{0,k})^{(\gamma_2)}(\omega)|$$
$$\le C(T_1, \gamma_1)(Q_{T_2,\gamma_2}(g_{0,k}) + Q_{T_2,\gamma_2}(L_{0,k})).$$

As $\{g_{0,k}\}_{k\in\mathbb{N}}$ and $\{L_{0,k}\}_{k\in\mathbb{N}}$ converge to 0 in $\mathcal{B}(S_2)$, also $\{M_k\}_{k\in\mathbb{N}}$ converges to 0 in $\mathcal{B}(S)$. So, the sequence $\{F_k\}_{k\in\mathbb{N}}$ defined as $F_k = G_k + M_k \in \mathcal{B}(S)$, $k \in \mathbb{N}$, converges to 0 in $\mathcal{B}(S)$. The proof of the first step of (i) shows that the rest of the statement holds.

STEP 2. We use an induction argument. Assume therefore that there exists a sequence $\{G_k\}_{k\in\mathbb{N}}$ of elements of $\mathcal{B}(S)$ that converges to 0 and, if $\mathrm{TA}'(G_k) = \{H_{n,k}, L_{m,k}\}$ for $k \in \mathbb{N}$, then $H_{n,k} = f_{n,k}$ if $n \le p$ and $L_{m,k} = g_{m,k}$ if $m \le p-1$. Notice that $\{L_{p,k}\}_{k\in\mathbb{N}}$ converges to 0 in $\mathcal{B}(S_2)$.

Take $\beta_p \in \mathcal{B}(S_1)$ with $\beta_p(z) \sim_{S_1} z^p$, and define $M_k \in \mathcal{B}(S)$, $k \in \mathbb{N}$, by

$$M_k(z,\omega) = \beta_p(z)(g_{p,k}(\omega) - L_{p,k}(\omega)), \quad (z,\omega) \in S.$$

We have

$$Q_{T,\gamma}(M_k) = \sup_{(z,\omega)\in T} |D^\gamma M_k(z,\omega)| \le \sup_{z\in T_1} |\beta_p^{(\gamma_1)}(z)| \sup_{\omega\in T_2} |(g_{p,k} - L_{p,k})^{(\gamma_2)}(\omega)|$$
$$\le C(T_1, \gamma_1)(Q_{T_2,\gamma_2}(g_{p,k}) + Q_{T_2,\gamma_2}(L_{p,k})).$$

Since $\{g_{p,k}\}_{k\in\mathbb{N}}$ and $\{L_{p,k}\}_{k\in\mathbb{N}}$ converge to 0 in $\mathcal{B}(S_2)$, we deduce that $\{M_k\}_{k\in\mathbb{N}}$ converges to 0 in $\mathcal{B}(S)$. Hence the sequence $\{F_k\}_{k\in\mathbb{N}}$, defined by $F_k = G_k + M_k \in \mathcal{B}(S)$, $k \in \mathbb{N}$, converges to 0 in $\mathcal{B}(S)$, and the proof of the second step in (i) leads to the desired result. ∎

Our last task consists in proving that ${}^t\psi(\mathcal{D}')$ is weakly closed in $\mathcal{B}(S)'$. We need the following three propositions; the proof of the first one resembles that of Proposition 4.1, with $\mathcal{A}(S)$ replaced by $\mathcal{B}(S)$, and $\mathcal{A}_{\boldsymbol{\alpha}}$ by $\mathcal{B}_{jm}$.

PROPOSITION 5.4. *Let L be a continuous functional on $\mathcal{B}(S)$ that belongs to the weak closure of ${}^t\psi(\mathcal{D}')$. Then*

$$\bigcap_{j\in N,\, m\in\mathbb{N}} \mathrm{Ker}(\psi_{jm}) \subset \mathrm{Ker}(L),$$

i.e., $L(f) = 0$ for every $f \in \mathcal{B}(S)$ such that $\psi_{jm}(f) = f_{jm} = 0$ for all $j \in N$ and $m \in \mathbb{N}$.

The next lemma can be deduced from the coherence conditions for the total family associated with an element of $\mathcal{B}(S)$.

LEMMA 5.5. *Let $f \in \mathcal{B}(S)$ and $\boldsymbol{\alpha} \in \mathbb{N}^n$. Then $\mathrm{App}_{\boldsymbol{\alpha}}(f) = 0$ if and only if $f_{jm} = 0$ for every $j \in N$ and $m \in \mathbb{N}$ such that $\alpha_j \ne 0$ and $m \le \alpha_j - 1$.*

PROPOSITION 5.6. *Let $L \in \mathcal{B}(S)'$ have the following property: If $f \in \mathcal{B}(S)$ is such that $f_{jm} = 0$ for all $j \in N$ and $m \in \mathbb{N}$, then $L(f) = 0$. Then there exists $r \in \mathbb{N}$, $r \ge 1$, such that $L(f) = 0$ for every $f \in \mathcal{B}(S)$ satisfying $f_{jm} = 0$ for $j \in N$ and $m \in \mathbb{N}$ with $m \le r$.*

Proof. Let $g \in \mathcal{B}(S)$ be such that $g_{jm} = 0$ for $j \in N$ and $m \in \mathbb{N}$ with $m \le r$. By the previous lemma, $\mathrm{App}_{\boldsymbol{\alpha}}(g) = 0$ for all $\boldsymbol{\alpha} \in \mathbb{N}^n$ with $0 \le |\boldsymbol{\alpha}| \le r + 1$. Therefore, the assertion follows as in Proposition 4.2. ∎

PROPOSITION 5.7. *Let $L \in \mathcal{B}(S)'$ such that there exists $r \in \mathbb{N}$, $r \ge 1$, with the following property: If $f \in \mathcal{B}(S)$ and $f_{jm} = 0$ for every $j \in N$ and every $m \in \mathbb{N}$ with $m \le r$, then $L(f) = 0$. Then there exists a functional $H \in \mathcal{D}'$ such that $L = H \circ \psi$.*

Proof. According to the proof of Proposition 5.3, for all $\mathcal{G}' = \{g_{jm}\} \in \mathcal{D}$ there exists a function $G \in \mathcal{B}(S)$ with $\mathrm{TA}'(G) = \{G_{jm}\}$ such that $G_{jm} = g_{jm}$ for $j \in N$ and $m \in \mathbb{N}$, $m \le r$.

Define $H : \mathcal{D} \to \mathbb{C}$ by $H(\mathcal{G}') = L(G)$, $\mathcal{G}' \in \mathcal{D}$. H is well defined, as can be easily deduced from the hypothesis imposed on L. For the same reason, it is clear that $L = H \circ \psi$.

To prove the continuity of H, consider a sequence $\{\mathcal{G}'_k\}_{k\in\mathbb{N}} \subset \mathcal{D}$ converging to 0, where $\mathcal{G}'_k = \{g_{jm,k}\}$, $k \in \mathbb{N}$. This implies the convergence of $\{g_{jm,k}\}_{k\in\mathbb{N}}$ to 0 in the corresponding spaces $\mathcal{B}_{jm}$. As shown in Proposition 5.3, there exists a sequence $\{G_k\}_{k\in\mathbb{N}} \subset \mathcal{B}(S)$ converging to 0 such that if $\mathrm{TA}'(G_k) = \{G_{jm,k}\}$, $k \in \mathbb{N}$, then $G_{jm,k} = g_{jm,k}$ for $j \in N$ and $m \in \mathbb{N}$ with $m \le r$. So, $H(\mathcal{G}'_k) = L(G_k)$, $k \in \mathbb{N}$. Now, the continuity of L implies $\lim_{k\to\infty} L(G_k) = 0$, and so H is continuous. Its linearity results immediately. ∎

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Received July 21, 1997 (3928)
Revised version February 5, 1998

STUDIA MATHEMATICA 133 (3) (1999)

Simple systems are disjoint from Gaussian systems

by

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Abstract. We prove the theorem promised in the title. Gaussians can be distinguished from simple maps by their property of divisibility. Roughly speaking, a system is divisible if it has a rich supply of direct product splittings. Gaussians are divisible and weakly mixing simple maps have no splittings at all so they cannot be isomorphic. The proof that they are disjoint consists of an elaboration of this idea, which involves, among other things, the notion of virtual divisibility, which is, more or less, divisibility up to distal extensions. The theory of Kronecker Gaussians also plays a crucial role.

1. Main result and overview of the proof. We deal throughout with (dynamical) systems $\mathbf{X} = (X, \mathcal{B}, \mu, T)$ and $\mathbf{Y} = (Y, \mathcal{C}, \nu, S)$, by which we understand that $(X, \mathcal{B}, \mu)$ is a Lebesgue probability space and $T : X \to X$ is a measurable invertible μ-preserving map. The purpose of this paper is to prove:

THEOREM 1. *If* $\mathbf{X}$ *is simple and* $\mathbf{Y}$ *is Gaussian then* $\mathbf{X}$ *and* $\mathbf{Y}$ *are disjoint.*

In the special case when $\mathbf{Y}$ has minimal self-joinings Theorem 1 was established by Thouvenot in [Th1]. After learning of our result Thouvenot has recently proved that the assumption that Y is Gaussian can be weakened to the assumption that Y is the time one map in a flow which is infinitely divisible (see §3 for the notion of divisibility).

Thouvenot [Th1] has shown that every horocycle flow is a factor of a simple flow, so Theorem 1 has the following corollary.

COROLLARY 2. *The time one map of any horocycle flow is disjoint from any Gaussian system.*

By a result of [J,R], Theorem 1 is equivalent to showing:

1991 *Mathematics Subject Classification*: Primary 28D15.
Research of A. del Junco partially supported by NSERC operating grant.
Research of M. Lemańczyk partially supported by KBN grant KBN 2P3 0103107.