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Complexifications of real Banach spaces, polynomials and multilinear maps

by

GUSTAVO A. MUÑOZ (Madrid),
YANNIS SARANTOPOULOS (Athens)
and ANDREW TONGE (Kent, Ohio)

Abstract. We give a unified treatment of procedures for complexifying real Banach spaces. These include several approaches used in the past. We obtain best possible results for comparison of the norms of real polynomials and multilinear mappings with the norms of their complex extensions. These estimates provide generalizations and show sharpness of previously obtained inequalities.

1. Introduction and notation. Many of the classical Banach function spaces exist in real- or complex-valued versions. This is the case, for example, with the $L_p(\mu)$-spaces and the $C(K)$-spaces. These spaces are actually Banach lattices, and this extra structure makes it easy to construct the complex version from the real version. If E is a real Banach lattice, the product $E\times E$ can be made into a complex Banach space in a natural way. Addition is defined by

$$(x,y)+(u,v)=(x+u,y+v)\quad \forall x,y,u,v\in E,$$

scalar multiplication is given by

$$(a+ib)(x,y)=(ax-by,bx+ay)\quad \forall x,y\in E,\ \forall a,b\in\mathbb{R},$$

and, thanks to the functional calculus, the norm can be specified by

$$\|(x,y)\|=\|(|x|^2+|y|^2)^{1/2}\|\quad \forall x,y\in E.$$

For more details, consult [11, p. 326].

It is straightforward to verify that, if E is a real-valued $L_p(\mu)$-space or $C(K)$-space, this complexification procedure yields the corresponding

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complex-valued space. It is also clear that the procedure is totally dependent on having a lattice structure available. Now, it is sometimes desirable to be able to construct a complexification of a general real Banach space. One situation where this is useful is in the investigation of holomorphic mappings, polynomials and multilinear mappings. Many results depend on techniques special to the complex numbers, but if it is possible to find a way to extend polynomials or multilinear mappings on a real Banach space to a complexification without losing control of norm, then results for complex Banach spaces can be interpreted in a real setting. Techniques such as these have been used by several authors; see A. Alexiewicz and W. Orlicz [1], C. Benítez, Y. Sarantopoulos and A. Tonge [6], J. Bochnak [8], J. Bochnak and J. Siciak [9], and A. E. Taylor [31].

It is interesting to note that several different complexification procedures were used in these papers, and that these in turn are different from the procedures adopted by J. Lindenstrauss and L. Tzafriri [18] and J. Wenzel [35].

In this paper we give a unified treatment of complexification procedures. The topic is trivial on the algebraic level, but it turns out that there is no completely satisfactory analytic theory. We show, for instance, that although a judicious choice of complexification procedure allows the extension of continuous multilinear mappings on real Banach spaces to their complexifications without change in norm, no such extension is generally possible for homogeneous polynomials of degree greater than 3.

We obtain optimal results on the comparison of the norms of homogeneous polynomials on real Banach spaces and the norms of their complexifications. In this way we are able (in Proposition 16) to generalize a classical polynomial inequality due to V. Markov [22]; see also C. Visser [34] and H.-J. Rack [25], [26]. We also show how the classical results can be used to make progress on a conjecture made by L. Harris in his commentary to problem 74 in The Scottish Book [20]. This deals with optimal constants in another extension of Markov's inequality; see Proposition 17.

We note that some of the results in this paper were obtained independently by Pádraig Kirwan in his Ph.D. thesis entitled "Complexification of multilinear and polynomial mappings on normed spaces" (National University of Ireland, Galway, 1997). However, Kirwan's focus was different from ours.

For convenience we recall the basic definitions needed to discuss polynomials from E into F, where E and F are real or complex Banach spaces. A map $P : E \to F$ is a (continuous) *n-homogeneous polynomial* if there is a (continuous) symmetric n-linear mapping $L : E^n \to F$ for which $P(x) = L(x, \ldots, x)$ for all $x \in E$. In this case it is convenient to write $P = \widehat{L}$. We define

$$\|P\| = \sup\{\|P(x)\| : \|x\| \le 1\}.$$

If $L : E^n \to F$ is a continuous n-linear mapping we define

$$\|L\| = \sup\{\|L(x_1, \ldots, x_n)\| : \|x_1\| \le 1, \ldots, \|x_n\| \le 1\}.$$

We let $\mathcal{P}(^nE; F)$, $\mathcal{L}(^nE; F)$ and $\mathcal{L}^s(^nE; F)$ denote respectively the Banach spaces of continuous n-homogeneous polynomials from E into F, the continuous n-linear mappings from E into F and the continuous symmetric n-linear mappings from E into F. If $\mathbb{K}$ is the real or complex field we use the notations $\mathcal{P}(^nE)$, $\mathcal{L}(^nE)$ and $\mathcal{L}^s(^nE)$ in place of $\mathcal{P}(^nE; \mathbb{K})$, $\mathcal{L}(^nE; \mathbb{K})$ and $\mathcal{L}^s(^nE; \mathbb{K})$ respectively. More generally, a map $P : E \to F$ is a *continuous polynomial of degree* $\le n$ if

$$P = P_0 + P_1 + \ldots + P_n$$

where $P_k \in \mathcal{P}(^kE; F)$ $(1 \le k \le n)$, and $P_0 : E \to F$ is a constant function.

2. General results on complexifications of real Banach spaces. To be able to build a coherent framework within which we can discuss complexifications of real Banach spaces, we need to be very precise about what we mean by a complexification. To start, we work at the algebraic level.

Definition. A complex vector space $\widetilde{E}$ is a *complexification* of a real vector space E if the following two conditions hold:

(i) there is a one-to-one real-linear map $j_E : E \to \widetilde{E}$, and
(ii) complex-span$(j_E(E)) = \widetilde{E}$.

When there is no possibility of confusion we write j instead of j_E. It is easy to see that, up to complex isomorphism, a real vector space has just one complexification. There are, however, various alternative concrete descriptions, and we will focus on three of these. The first is modeled on the usual construction of the complex numbers from the reals.

Ordered pair description of a complexification. If E is a real vector space, we can make $E \times E$ into a complex vector space by defining

$$\begin{aligned} (x, y) + (u, v) &:= (x + u, y + v) && \forall x, y, u, v \in E, \\ (\alpha + i\beta)(x, y) &:= (\alpha x - \beta y, \beta x + \alpha y) && \forall x, y \in E,\ \forall \alpha, \beta \in \mathbb{R}. \end{aligned}$$

The map $j : E \to E \times E$, $x \mapsto (x, 0)$, clearly satisfies conditions (i) and (ii) above, and so this complex vector space is a complexification of E. It is convenient to denote it by

$$\widetilde{E} = E \oplus iE$$

and to suppress reference to j by writing $z = x + iy$ for the element $z = (x, y) = j(x) + ij(y)$. It is natural to write $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$.

Tensor product description of a complexification. Let $\{e_1, e_2\}$ be the natural basis for $\mathbb{R}^2$. If E is a real vector space, a typical element of $E\otimes\mathbb{R}^2$ can be written in the form $x\otimes e_1 + y\otimes e_2$ with $x, y \in E$. We can make $E\otimes\mathbb{R}^2$ into a complex vector space by defining

$$(x\otimes e_1 + y\otimes e_2) + (u\otimes e_1 + v\otimes e_2) := (x+u)\otimes e_1 + (y+v)\otimes e_2 \quad \forall x,y,u,v\in E,$$
$$(\alpha+i\beta)(x\otimes e_1 + y\otimes e_2) := (\alpha x - \beta y)\otimes e_1 + (\beta x + \alpha y)\otimes e_2 \quad \forall x,y\in E,\ \forall\alpha,\beta\in\mathbb{R}.$$

The map $j : E\to E\otimes\mathbb{R}^2$, $x\mapsto x\otimes e_1$, clearly satisfies the complexification conditions (i) and (ii), and so $E\otimes\mathbb{R}^2$ can be viewed as a complexification of E, which we also denote by $\widetilde{E}$.

It will often be convenient to write $z = x+iy$ instead of $z = x\otimes e_1 + y\otimes e_2 = j(x)+ij(y)$. Naturally, this prompts the notation $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$.

Linear operator description of a complexification. There is a natural identification between $E\otimes\mathbb{R}^2$ and $\mathcal{L}(\mathbb{R}^2; E)$, the real vector space of linear operators from $\mathbb{R}^2$ to E. This gives us one more way of looking at complexifications, which we now describe explicitly. Again, let $\{e_1, e_2\}$ be the natural basis for $\mathbb{R}^2$. For each $x, y\in E$ we define $T_{x,y}\in\mathcal{L}(\mathbb{R}^2; E)$ by

$$T_{x,y}(e_1) = x \quad\text{and}\quad T_{x,y}(e_2) = y.$$

Notice that all elements of $\mathcal{L}(\mathbb{R}^2; E)$ arise in this way.

Now $\mathcal{L}(\mathbb{R}^2; E)$ can be viewed as a complex vector space by defining

$$T_{x,y} + T_{u,v} = T_{x+u,y+v} \qquad \forall x,y,u,v\in E,$$
$$(\alpha+i\beta)T_{x,y} = T_{\alpha x-\beta y,\beta x+\alpha y} \qquad \forall x,y\in E,\ \forall\alpha,\beta\in\mathbb{R}.$$

The map $j : E\to\mathcal{L}(\mathbb{R}^2; E)$, $x\mapsto T_{x,0}$, clearly satisfies the complexification conditions (i) and (ii), and so we have an explicit representation of $\mathcal{L}(\mathbb{R}^2; E)$ as a complexification of E.

When convenient we write $z = x+iy$ instead of $z = T_{x,0} + iT_{y,0} = j(x)+ij(y)$. Naturally, the notation $x = \operatorname{Re} z$, $y = \operatorname{Im} z$ will also be used.

Natural complexifications of real Banach spaces. The plot thickens when we turn to the question of the complexification of real Banach spaces. Even when we impose some natural conditions on the norm, there are infinitely many possibilities.

DEFINITION. Let E be a real Banach space. We say that a norm on the complexification $\widetilde{E}$ is *reasonable* if

(iii) $\|j(x)\| = \|x\|$ $\forall x\in E$,
(iv) $\|x+iy\| = \|x-iy\|$ $\forall x,y\in E$.

When $\widetilde{E}$ is equipped with such a norm we call it a *reasonable complexification* of E.

Conditions (iii) and (iv) are modeled on basic properties of complex numbers, namely that for all real numbers x we have $|x| = |x+i0|$, and that, if z is a complex number, its complex conjugate $\overline{z}$ satisfies $|\overline{z}| = |z|$. Other simple properties of complex numbers reappear in this new set-up.

PROPOSITION 1. *Let $\widetilde{E}$ be a reasonable complexification of the real Banach space E. For any $x, y\in E$ we have*

$$\|x\|_E \le \|x+iy\|_{\widetilde{E}} \quad\textit{and}\quad \|y\|_E \le \|x+iy\|_{\widetilde{E}}.$$

Proof. By property (iii),

$$2\|x\|_E = \|(x+iy)+(x-iy)\|_{\widetilde{E}} \le \|x+iy\|_{\widetilde{E}} + \|x-iy\|_{\widetilde{E}}.$$

An application of property (iv) gives $\|x\|_E \le \|x+iy\|_{\widetilde{E}}$. The other inequality is just as easy to prove. ■

PROPOSITION 2. *Let $\widetilde{E}$ be a reasonable complexification of the real Banach space E. For any $x, y\in E$ we have*

$$\sup_{0\le t\le 2\pi}\|x\cos t - y\sin t\|_E \le \|x+iy\|_{\widetilde{E}} \le \inf_{0\le t\le 2\pi}(\|x\cos t - y\sin t\|_E + \|x\sin t + y\cos t\|_E).$$

Proof. For each $0\le t\le 2\pi$,

$$\|x+iy\|_{\widetilde{E}} = \|e^{it}(x+iy)\|_{\widetilde{E}} = \|(x\cos t - y\sin t) + i(x\sin t + y\cos t)\|_{\widetilde{E}}.$$

Using Proposition 1 on the left and the triangle inequality on the right, we find

$$\|x\cos t - y\sin t\|_E \le \|x+iy\|_{\widetilde{E}} \le \|x\cos t - y\sin t\|_E + \|x\sin t + y\cos t\|_E.$$

The result now follows immediately. ■

PROPOSITION 3. *Let $\widetilde{E}$ be a complexification of the real Banach space E. Among all the reasonable complexification norms on $\widetilde{E}$, the smallest is given by*

$$\|x+iy\|_{\mathrm{T}} := \sup_{0\le t\le 2\pi}\|x\cos t - y\sin t\|.$$

All other complexification norms $\|\cdot\|$ on $\widetilde{E}$ are equivalent to $\|\cdot\|_{\mathrm{T}}$. Indeed, for any $x, y\in E$,

$$\|x+iy\|_{\mathrm{T}} \le \|x+iy\| \le 2\|x+iy\|_{\mathrm{T}}.$$

We omit the simple proof. Verifying that $\|\cdot\|_{\mathrm{T}}$ is a reasonable complexification norm is straightforward, and the inequalities follow at once from Proposition 2. After Proposition 10, we will give an example to show it is possible to have $\|x+iy\| = 2\|x+iy\|_{\mathrm{T}}$.

The norm $\|\cdot\|_{\mathrm{T}}$ was first considered by A. E. Taylor [21] (see also [32]) and has since reappeared in several guises. We shall refer to $(\widetilde{E}, \|\cdot\|_{\mathrm{T}})$ as the *Taylor complexification* of E.

There is a useful alternative description of $\|x+iy\|_{\mathrm{T}}$:

$$\text{(1)}\qquad \begin{aligned} \|x+iy\|_{\mathrm{T}} &= \sup_{0\le t\le 2\pi} \|x\cos t - y\sin t\| \\ &= \sup_{0\le t\le 2\pi}\ \sup_{\|\varphi\|_{E^*}\le 1} |\varphi(x)\cos t - \varphi(y)\sin t| \\ &= \sup_{\|\varphi\|_{E^*}\le 1} \sqrt{\varphi(x)^2+\varphi(y)^2}. \end{aligned}$$

Taylor's norm also appears in a very natural way when we think in terms of Banach lattices. Recall that a real Banach space E can be viewed as a subspace of $C_{\mathbb{R}}(B_{E^*})$, the space of continuous real-valued functions on the weak*-compact set B_{E^*}, the closed unit ball of E^*. Each $x \in E$ is identified with a function $f_x \in C_{\mathbb{R}}(B_{E^*})$ given by

$$f_x(\varphi) = \varphi(x) \qquad \forall \varphi \in B_{E^*}.$$

Now $C_{\mathbb{R}}(B_{E^*})$ is a Banach lattice and so we can complexify it using the lattice complexification norm discussed in the introduction. This induces a norm on $\widetilde{E}$ which is nothing other than Taylor's norm:

$$\begin{aligned} \|x+iy\|_{C_{\mathbb{C}}(B_{E^*})} &= \|f_x + if_y\|_{C_{\mathbb{C}}(B_{E}^*)} = \|(|f_x|^2+|f_y|^2)^{1/2}\|_{C_{\mathbb{R}}(B_{E^*})} \\ &= \sup_{\|\varphi\|_{E^*}\le 1} (|f_x(\varphi)|^2 + |f_y(\varphi)|^2)^{1/2} \\ &= \sup_{\|\varphi\|_{E^*}\le 1} \sqrt{\varphi(x)^2+\varphi(y)^2}. \end{aligned}$$

In this context, it is worth remarking that Taylor's complexification of a $C(K)$-space coincides with the lattice complexification discussed in the introduction. Other natural interpretations of $\|\cdot\|_{\mathrm{T}}$ will soon be described. For now, we focus on another feature of the Taylor complexification, namely that it is a general complexification procedure whose definition is not tied to any specific characteristic of the real Banach space E which is being complexified. Moreover, this procedure allows us to extend continuous linear maps between real Banach spaces to complex linear maps between their complexifications without increasing the norm. On the algebraic level, there is no choice about how to extend. If $L : E \to F$ is a linear map between the real vector spaces E and F, there is a unique complex-linear extension $\widetilde{L} : \widetilde{E} \to \widetilde{F}$ given by

$$\widetilde{L}(x+iy) = L(x) + iL(y).$$

PROPOSITION 4. *Let E and F be real Banach spaces. If $L \in \mathcal{L}(E;F)$, then $\widetilde{L} \in \mathcal{L}((\widetilde{E}, \|\cdot\|_{\mathrm{T}}); (\widetilde{F}, \|\cdot\|_{\mathrm{T}}))$ and $\|\widetilde{L}\| = \|L\|$.*

Proof. Since $\widetilde{L}$ extends L, we have $\|\widetilde{L}\| \ge \|L\|$. On the other hand, if $x, y \in E$, then

$$\begin{aligned} \|\widetilde{L}(x+iy)\|_{\mathrm{T}} &= \|L(x)+iL(y)\|_{\mathrm{T}} = \sup_{0\le t\le 2\pi} \|L(x)\cos t - L(y)\sin t\|_F \\ &= \sup_{0\le t\le 2\pi} \|L(x\cos t - y\sin t)\|_F \\ &\le \|L\| \sup_{0\le t\le 2\pi} \|x\cos t - y\sin t\|_E \\ &= \|L\|\cdot\|x+iy\|_{\mathrm{T}}, \end{aligned}$$

and so $\|\widetilde{L}\| \le \|L\|$. ∎

Taylor's procedure is just one of infinitely many procedures with similar properties.

DEFINITION. A *natural complexification procedure* ν is a way of defining a reasonable complexification norm $\|\cdot\|_\nu$ on the complexification $\widetilde{E}$ of any real Banach space which has the property that

(v) if E, F are real Banach spaces and $L \in \mathcal{L}(E;F)$, then the complex-linear extension $\widetilde{L} : (\widetilde{E}, \|\cdot\|_\nu) \to (\widetilde{F}, \|\cdot\|_\nu)$ has the same norm as L.

We say that $\|\cdot\|_\nu$ is a *natural complexification norm* on $\widetilde{E}$ and that $(\widetilde{E}, \|\cdot\|_\nu)$ is a *natural complexification* of E. Further, we write $\|\widetilde{L}\|_\nu$ for the norm of $\widetilde{L}$ as an element of $\mathcal{L}((\widetilde{E}, \|\cdot\|_\nu); (\widetilde{F}, \|\cdot\|_\nu))$.

There are many interesting examples of such procedures which have been used in the literature.

Examples of natural complexification procedures

(a) *The ordered pair approach.* Let E be a real Banach space. It is tempting to try to define a natural complexification norm on $E \oplus iE$ by setting

$$n_p(x+iy) := (\|x\|^p + \|y\|^p)^{1/p}, \qquad 1 \le p < \infty,$$

with the usual modification when $p = \infty$. This attempt is doomed because the homogeneity condition fails: only in exceptional circumstances is it true that $n_p(\lambda(x+iy)) = |\lambda| n_p(x+iy)$ for each $\lambda \in \mathbb{C}$. It is also tempting to try to get round this problem by working with

$$\widetilde{n}_p(x+iy) := \sup_{0\le t\le 2\pi} n_p(e^{it}(x+iy)), \qquad 1 \le p \le \infty.$$

Indeed, $\widetilde{n}_p$ is a norm on $\widetilde{E}$ for each $1 \le p \le \infty$, but unless $2 \le p \le \infty$, it does not satisfy the reasonable norm condition (iii). One more adjustment is necessary.

DEFINITION. Let E be a real Banach space and let $1 \le p \le \infty$. For each $x, y \in E$ define
$$\|x+iy\|_{(p)} := 2^{\min(1/2-1/p,0)} \sup_{0\le t\le 2\pi} (\|x\cos t - y\sin t\|^p + \|x\sin t + y\cos t\|^p)^{1/p}.$$
In the case $p = \infty$, we simply set
$$\|x+iy\|_{(\infty)} := \sup_{0\le t\le 2\pi} \max\{\|x\cos t - y\sin t\|, \|x\sin t + y\cos t\|\}.$$

Notice that for each $1 \le p \le \infty$, we have
$$\|x+iy\|_{(p)} = 2^{\min(1/2-1/p,0)} \widetilde{n}_p(x+iy).$$
A simple calculus exercise reveals that
$$\sup_{0\le t\le 2\pi} (|\cos t|^p + |\sin t|^p)^{1/p} = \begin{cases} 1 & (2 \le p < \infty), \\ 2^{1/p-1/2} & (1 \le p \le 2). \end{cases}$$
With this in hand it is simple to check that the next proposition is true.

PROPOSITION 5. *Let $1 \le p \le \infty$. If E is a real Banach space, then $(E \oplus iE, \|\cdot\|_{(p)})$ is a natural complexification of E.*

Note that $\|\cdot\|_{(\infty)}$ is just another manifestation of Taylor's norm. It is also interesting to observe that $\|\cdot\|_{(2)}$ is the norm used by Lindenstrauss and Tzafriri [18, p. 81]. We shall often denote the Lindenstrauss–Tzafriri norm by $\|\cdot\|_{\mathrm{LT}}$. Moreover, modulo the correction factor $1/\sqrt{2}$, $\|\cdot\|_{(1)}$ is the norm chosen by Alexiewicz and Orlicz [1].

We have already commented that Taylor's complexification of a $C(K)$-space coincides with the Banach lattice complexification. For $L_2(\mu)$-spaces, the Lindenstrauss–Tzafriri procedure gives the Banach lattice complexification. However, the Alexiewicz–Orlicz procedure is not natural for $L_1(\mu)$-spaces. Instead, a procedure due to Bochnak [8] must be used. We describe this soon.

Later, it will be useful to have some elementary relationships involving the norms $\|\cdot\|_{(p)}$.

PROPOSITION 6. *Let $1 \le p \le \infty$ and let E be a real Banach space. For every $z = x + iy \in \widetilde{E}$,*
$$\|z\|_{(p)} \le \|z\|_{(2)} \le 2^{|1/2-1/p|}\|z\|_{(p)}.$$

Proof. First, for $p \ge 2$, an application of Hölder's inequality gives
$$\begin{aligned}\|z\|_{(2)} &= \sup_t(\|x\cos t - y\sin t\|^2 + \|x\sin t + y\cos t\|^2)^{1/2} \\ &\le 2^{1/2-1/p}\sup_t(\|x\cos t - y\sin t\|^p + \|x\sin t + y\cos t\|^p)^{1/p} \\ &= 2^{1/2-1/p}\|z\|_{(p)}.\end{aligned}$$

Then, for $p \le 2$, monotonicity of the l_p norms gives
$$\begin{aligned}\|z\|_{(2)} &= \sup_t(\|x\cos t - y\sin t\|^2 + \|x\sin t + y\cos t\|^2)^{1/2} \\ &\le \sup_t(\|x\cos t - y\sin t\|^p + \|x\sin t + y\cos t\|^p)^{1/p} = 2^{1/p-1/2}\|z\|_{(p)}.\end{aligned}$$
The other inequality is proved similarly. ■

(b) *Tensor product approach.* We denote $\mathbb{R}^2$ with the Euclidean norm by l_2^2. When E is a real Banach space we shall work only with complexification norms on $\widetilde{E} = E \otimes l_2^2$. Our first objective is to show that all reasonable complexification norms are reasonable tensor product norms on $E \otimes l_2^2$ in the sense of Schatten [29]. This means that we have to show that
$$\begin{aligned}\|x \otimes a\|_{\widetilde{E}} &= \|x\| \cdot \|a\| \quad \forall x \in E,\ \forall a \in l_2^2, \\ \|\varphi \otimes b\|_{(\widetilde{E})^*} &= \|\varphi\| \cdot \|b\| \quad \forall \varphi \in E^*,\ \forall b \in (l_2^2)^* = l_2^2.\end{aligned}$$

This requires some knowledge of the nature of $(\widetilde{E})^*$ when $\widetilde{E}$ is a natural complexification of E. It is straightforward to verify that, on the vector space level, $(\widetilde{E})^*$ is a complexification of E^*. Accordingly we write $(\widetilde{E})^* = \{\varphi + i\psi : \varphi, \psi \in E^*\}$. The duality is given by
$$(\varphi + i\psi)(x+iy) := (\varphi(x) - \psi(y)) + i(\varphi(y) + \psi(x)) \quad \forall x, y \in E.$$

PROPOSITION 7. *If $\widetilde{E}$ is a reasonable complexification of the real Banach space E, then $(\widetilde{E})^*$ is a reasonable complexification of E^*.*

Proof. We define $j : E^* \to (\widetilde{E})^*$ by $j(\varphi) = \varphi + i0$. The algebraic properties (i) and (ii) required of j are evidently true, so we turn to the reasonable norm conditions (iii) and (iv). First, if $\varphi \in E^*$ then
$$\begin{aligned}\|j(\varphi)\|_{(\widetilde{E})^*} &= \sup\{|(\varphi + i0)(x+iy)| : \|x+iy\|_{\widetilde{E}} \le 1\} \\ &= \sup\{\sqrt{\varphi(x)^2 + \varphi(y)^2} : \|x+iy\|_{\widetilde{E}} \le 1\}.\end{aligned}$$
Since we know from Proposition 3 that Taylor's norm is the smallest of the natural complexification norms, it follows that
$$\|j(\varphi)\|_{(\widetilde{E})^*} \le \|\varphi\|_{E^*}.$$
On the other hand,
$$\begin{aligned}\|j(\varphi)\|_{(\widetilde{E})^*} &\ge \sup\{|(\varphi + i0)(x+i0)| : \|x\|_E \le 1\} \\ &= \sup\{|\varphi(x)| : \|x\|_E \le 1\} = \|\varphi\|_{E^*}.\end{aligned}$$
This proves that condition (iii) holds for the map j.

To check condition (iv) note that if $\varphi, \psi \in E^*$ and $x, y \in E$, then
$$\begin{aligned}|(\varphi - i\psi)(x+iy)| &= |(\varphi(x) + \psi(y)) + i(\varphi(y) - \psi(x))| \\ &= |(\varphi(x) + \psi(y)) - i(\varphi(y) - \psi(x))| = |(\varphi + i\psi)(x-iy)|.\end{aligned}$$

Since $\|x+iy\|_{\widetilde{E}} = \|x-iy\|_{\widetilde{E}}$, it follows at once that

$$\|\varphi + i\psi\|_{(\widetilde{E})^*} = \|\varphi - i\psi\|_{(\widetilde{E})^*}. \ \blacksquare$$

PROPOSITION 8. *Let E be a real Banach space. All reasonable complexification norms on $\widetilde{E} = E \otimes l_2^2$ are reasonable tensor product norms.*

Proof. Let $x \in E$ and $a = a_1e_1 + a_2e_2 \in l_2^2$. Then

$$x \otimes a = a_1\, x \otimes e_1 + a_2\, x \otimes e_2 = (a_1 + ia_2)(x \otimes e_1).$$

Thus, by property (iii) of reasonable complexifications,

$$\|x \otimes a\|_{\widetilde{E}} = \|(a_1 + ia_2)(x \otimes e_1)\|_{\widetilde{E}} = |a_1 + ia_2|\|x \otimes e_1\|_{\widetilde{E}} = \|a\|_{l_2^2}\|x\|_E.$$

Similarly, since $(\widetilde{E})^*$ is a reasonable complexification of E^*, we find that

$$\|\varphi \otimes b\|_{(\widetilde{E})^*} = \|\varphi\|_{E^*}\|b\|_{l_2^2} \quad \forall \varphi \in E^*,\ \forall b \in l_2^2. \ \blacksquare$$

Grothendieck [14] introduced a class of "tensor norms" which are defined on the tensor product of *any* pair of Banach spaces, not just the single pair required for the discussion of reasonable norms. If E and F are Banach spaces and α is a tensor norm, we write $E \otimes_\alpha F$ for the vector space $E \otimes F$ equipped with the norm α. The defining properties of a tensor norm α are the reasonableness conditions

(a) $\|x \otimes y\|_{E\otimes_\alpha F} = \|x\|_E\|y\|_F$, $\forall x \in E$, $\forall y \in F$,
(b) $\|\varphi \otimes \psi\|_{(E\otimes_\alpha F)^*} = \|\varphi\|_{E^*}\|\psi\|_{F^*}$, $\forall \varphi \in E^*$, $\forall \psi \in F^*$;

together with the requirement that

(c) if $u : E_1 \to E_2$ and $v : F_1 \to F_2$ are continuous linear maps between Banach spaces, then

$$\|u \otimes v\|_{\mathcal{L}(E_1\otimes_\alpha F_1;E_2\otimes_\alpha F_2)} = \|u\|_{\mathcal{L}(E_1;E_2)}\|v\|_{\mathcal{L}(F_1;F_2)}.$$

PROPOSITION 9. *Let E be a real Banach space. If $\widetilde{E} = E \otimes_\alpha l_2^2$ where α is a tensor norm, then it is a natural complexification of E.*

Proof. We need to stop to check that $E \otimes_\alpha l_2^2$ is a complex Banach space. The only point which requires care is the verification that

$$\|\lambda(x+iy)\|_{E\otimes_\alpha l_2^2} = |\lambda| \cdot \|x+iy\|_{E\otimes_\alpha l_2^2}$$

whenever $x, y \in E$ and $\lambda \in \mathbb{C}$. It is certainly enough to do this when $\lambda = e^{it}$ with $t \in \mathbb{R}$. But if we write $u : l_2^2 \to l_2^2$ for the linear map with matrix

$$\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix},$$

then $\|u\| = 1$ and so

$$\begin{aligned}
\|e^{it}(x+iy)\|_{E\otimes_\alpha l_2^2} &= \|(x\cos t - y\sin t) + i(x\sin t + y\cos t)\|_{E\otimes_\alpha l_2^2} \\
&= \|x \otimes (e_1\cos t + e_2\sin t) + y \otimes (-e_1\sin t + e_2\cos t)\|_{E\otimes_\alpha l_2^2} \\
&= \|(\mathrm{id}_E \otimes u)(x \otimes e_1 + y \otimes e_2)\|_{E\otimes_\alpha l_2^2} \\
&\le \|\mathrm{id}_E \otimes u\|_{\mathcal{L}(E\otimes_\alpha l_2^2;E\otimes_\alpha l_2^2)}\|x \otimes e_1 + y \otimes e_2\|_{E\otimes_\alpha l_2^2} \\
&= \|x+iy\|_{E\otimes_\alpha l_2^2}.
\end{aligned}$$

Since this holds for any real t, we can infer that

$$\|e^{it}(x+iy)\|_{E\otimes_\alpha l_2^2} = \|x+iy\|_{E\otimes_\alpha l_2^2}.$$

Let $j : E \to E \otimes_\alpha l_2^2$, $x \mapsto x \otimes e_1$, be the usual map. Condition (iii) is simple to verify: if $x \in E$, then

$$\|j(x)\|_{E\otimes_\alpha l_2^2} = \|x \otimes e_1\|_{E\otimes_\alpha l_2^2} = \|x\|_E\|e_1\|_{l_2^2} = \|x\|_E.$$

For condition (iv), we need to recognize that the linear map $v : l_2^2 \to l_2^2$ defined by $v(e_1) = e_1$, $v(e_2) = -e_2$ has norm 1. Then, if $x, y \in E$, we have

$$\begin{aligned}
\|x-iy\|_{E\otimes_\alpha l_2^2} &= \|x \otimes e_1 - y \otimes e_2\|_{E\otimes_\alpha l_2^2} \\
&= \|(\mathrm{id}_E \otimes v)(x \otimes e_1 + y \otimes e_2)\|_{E\otimes_\alpha l_2^2} \\
&\le \|\mathrm{id}_E \otimes v\|_{\mathcal{L}(E\otimes_\alpha l_2^2;E\otimes_\alpha l_2^2)}\|x \otimes e_1 + y \otimes e_2\|_{E\otimes_\alpha l_2^2} \\
&= \|x+iy\|_{E\otimes_\alpha l_2^2}.
\end{aligned}$$

It follows at once that $\|x+iy\|_{E\otimes_\alpha l_2^2} = \|x-iy\|_{E\otimes_\alpha l_2^2}$.

Finally, if $L \in \mathcal{L}(E;F)$, then $\widetilde{L} : E \otimes_\alpha l_2^2 \to F \otimes_\alpha l_2^2$ is given by $\widetilde{L} = L \otimes \mathrm{id}_{l_2^2}$. It follows at once from property (c) of tensor norms that $\|\widetilde{L}\| = \|L\|$. $\blacksquare$

Within the framework of tensor norms, Taylor's norm appears in a very prominent position. It is the smallest of all tensor norms, the injective tensor norm ε, which we now define. Let E, F be real Banach spaces and let $t = \sum x_k \otimes y_k$ be an element of $E \otimes F$. Then

$$\|t\|_{E\otimes_\varepsilon F} := \sup\Big\{\Big|\sum \varphi(x_k)\psi(y_k)\Big| : \|\varphi\|_{E^*} \le 1,\ \|\psi\|_{F^*} \le 1\Big\}.$$

PROPOSITION 10. *Let E be a real Banach space. The natural complexification $\widetilde{E} = E \otimes_\varepsilon l_2^2$ is Taylor's complexification.*

Proof. Let $x, y \in E$. Then

$$\begin{aligned}\|x\otimes e_1 + y\otimes e_2\|_{E\otimes_\varepsilon l_2^2} &= \sup\{|\varphi(x)a_1 + \varphi(y)a_2| : \|\varphi\|_{E^*}\le 1,\ \|(a_1,a_2)\|_{l_2^2}\le 1\}\\ &= \sup\{\sqrt{\varphi(x)^2+\varphi(y)^2} : \|\varphi\|_{E^*}\le 1\}. \ \blacksquare\end{aligned}$$

At the other end of the scale is the largest tensor norm, the projective norm π. If E, F are real Banach spaces and $t \in E\otimes F$, then

$$\|t\|_{E\otimes_\pi F} := \inf\Big\{\sum \|x_k\|\cdot\|y_k\| : t = \sum x_k\otimes y_k\Big\}.$$

The complexification $\widetilde{E} = E\otimes_\pi l_2^2$ was used by Bochnak [8]. As $(E\otimes_\varepsilon l_2^2)^* = E^*\otimes_\pi l_2^2$ and $(E\otimes_\pi l_2^2)^* = E^*\otimes_\varepsilon l_2^2$, a duality argument shows that this is the largest possible natural complexification norm. We will often denote the Bochnak norm by $\|\cdot\|_{\mathrm{B}}$. It is Bochnak's procedure which gives the Banach lattice complexification of $L_1(\mu)$-spaces.

Bochnak's norm allows us to give an example where $\|x+iy\| = 2\|x+iy\|_{\mathrm{T}}$. For this we work with $E = l_2^2$. Consider $e_1 + ie_2$. Clearly, $\|e_1+ie_2\|_{\mathrm{T}} = 1$, while $\|e_1+ie_2\|_{\mathrm{B}}$ is just the norm of

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

in $l_2^2\otimes_\pi l_2^2$ —in other words, the trace class norm [14]. Thus $\|e_1+ie_2\|_{\mathrm{B}} = 2$.

(c) *Linear operator approach.* We again use the Euclidean norm on $\mathbb{R}^2$, and work with natural complexification norms on $\mathcal{L}(l_2^2; E)$. Ideal norms in the sense of Pietsch [23] are natural in this context. Recall that if $a \in l_2^2$ and $x \in E$, the operator $a\otimes x \in \mathcal{L}(l_2^2;E)$ is given by

$$(a\otimes x)(b) = (a,b)x \qquad \forall b\in l_2^2,$$

where (a,b) denotes the inner product of a, b. If α is an ideal norm then

(a) for every $a\in l_2^2$, $x\in E$ we have $\alpha(a\otimes x) = \|a\|\cdot\|x\|$, and

(b) for every real Banach space F and $u\in\mathcal{L}(l_2^2;l_2^2)$, $T\in\mathcal{L}(l_2^2;E)$, $v\in\mathcal{L}(E;F)$ we have

$$\alpha(vTu)\le \|v\|\alpha(T)\|u\|.$$

PROPOSITION 11. *Let E be a real Banach space. If $\widetilde{E} = \mathcal{L}(l_2^2;E)$ is equipped with an ideal norm α, then it is a natural complexification of E.*

Proof. The only point which requires thought in checking that $(\widetilde{E},\alpha)$ is a complex Banach space is the verification that

$$\alpha(\lambda(x+iy)) = |\lambda|\alpha(x+iy)$$

whenever $x,y\in E$ and $\lambda\in\mathbb{C}$. It is enough to check this when $\lambda = e^{it}$ with $t\in\mathbb{R}$. But the linear map $u : l_2^2\to l_2^2$ with matrix

$$\begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix}$$

has norm 1, and

$$\begin{aligned}\alpha(e^{it}(x+iy)) &= \alpha(T_{x\cos t - y\sin t,\ x\sin t + y\cos t})\\ &= \alpha(T_{x,y}u)\le\alpha(T_{x,y})\|u\| = \alpha(x+iy).\end{aligned}$$

It follows that $\alpha(e^{it}(x+iy)) = \alpha(x+iy)$, as required.

Now if $j : E\to\mathcal{L}(l_2^2;E)$, $x\mapsto T_{x,0}$, is the usual map, it is easy to verify the natural complexification norm condition (iii):

$$\alpha(j(x)) = \alpha((1,0)\otimes x) = \|(1,0)\|_{l_2^2}\|x\| = \|x\|.$$

For condition (iv), just observe that since $v : l_2^2\to l_2^2$, defined by $v(e_1) = e_1$, $v(e_2) = -e_2$, has norm 1, we have

$$\alpha(x-iy) = \alpha(T_{x,-y}) = \alpha(T_{x,y}v)\le\alpha(T_{x,y})\|v\| = \alpha(x+iy),$$

for every $x,y\in E$. Hence $\alpha(x+iy) = \alpha(x-iy)$.

Finally, if $L\in\mathcal{L}(E;F)$ then $\widetilde{L} : (\mathcal{L}(l_2^2;E),\alpha)\to(\mathcal{L}(l_2^2;F),\alpha)$ is given by $\widetilde{L}(T) = LT$ for each $T\in\mathcal{L}(l_2^2;E)$. Since

$$\alpha(\widetilde{L}(T)) = \alpha(LT)\le\|L\|\alpha(T)$$

we find that

$$\|\widetilde{L}\|\le\|L\|.$$

Since $\widetilde{L}$ extends L, there is equality of norms, and condition (v) is true. $\blacksquare$

There are many examples of complexifications arising from ideal norms. Taylor's complexification corresponds to the usual operator norm which was used by G. Pisier [24], while Bochnak's complexification is given by the integral norm. In addition, the p-summing norms π_p may be used [11].

The equality of the 2-summing norm π_2 and the Hilbert–Schmidt norm on $\mathcal{L}(l_2^2;H)$ when H is a Hilbert space tells us that the canonical lattice complexification of a Hilbert space is obtained using the 2-summing norm. This is also obtained with the Lindenstrauss–Tzafriri norm, so for Hilbert spaces

$$\sqrt{\|x\|^2+\|y\|^2} = \|x+iy\|_{\mathrm{LT}} = \pi_2(x+iy).$$

On general Banach spaces it is simple to check that

$$\sqrt{\|x\|^2+\|y\|^2}\le\|x+iy\|_{\mathrm{LT}}\le\pi_2(x+iy),$$

and hence that for $1 \le p \le 2$,

$$\sqrt{\|x\|^2+\|y\|^2} \le \pi_p(x+iy).$$

In fact, there is a more general relationship.

PROPOSITION 12. *Let E be a real Banach space and let $1 \le p < \infty$. Then, for each $x, y \in E$, we have*

$$\|x+iy\|_{(p)} \le \begin{cases} \pi_p(x+iy) & (2 \le p < \infty), \\ \pi_2(x+iy) & (1 \le p \le 2). \end{cases}$$

Proof. Let $t \in \mathbb{R}$. Then

$$\begin{aligned} &\|x\cos t - y\sin t\|^p + \|x\sin t + y\cos t\|^p \\ &\quad = \|T_{x,y}(\cos t, -\sin t)\|^p + \|T_{x,y}(\sin t, \cos t)\|^p \\ &\quad \le \pi_p^p(x+iy) \sup_{a^2+b^2=1} (|a\cos t - b\sin t|^p + |a\sin t + b\cos t|^p) \\ &\quad = \pi_p^p(x+iy) \sup_s (|\cos(s+t)|^p + |\sin(s+t)|^p). \end{aligned}$$

Since this is true for any t, and

$$\sup_u (|\cos u|^p + |\sin u|^p)^{1/p} = 1 \quad (2 \le p < \infty),$$

the result follows at once for $2 \le p < \infty$. To establish the remaining case, note that by Proposition 6 we have $\|x+iy\|_{(p)} \le \|x+iy\|_{(2)}$ for $1 \le p \le 2$, and apply what was just proved. ■

Inherent problems with complexification procedures. It is unfortunate that no one complexification is completely suited to all situations. For instance, if F is a subspace of the real Banach space E, then there are some natural complexification procedures for which $\widetilde{F}$ is isometrically a subspace of $\widetilde{E}$, and some for which this is not the case. Specifically, it is immediate from the definition that $(\widetilde{F}, \|\cdot\|_{\mathrm{T}})$ is isometrically a subspace of $(\widetilde{E}, \|\cdot\|_{\mathrm{T}})$, and, more generally, for any $1 \le p < \infty$, $(\widetilde{F}, \|\cdot\|_{(p)})$ is isometrically a subspace of $(\widetilde{E}, \|\cdot\|_{(p)})$. On the other hand, this does not work for the Bochnak complexification.

Problems also arise with quotient spaces. If F is a quotient space of the real Banach space E, then because of the general properties of the projective tensor product, $(\widetilde{F}, \|\cdot\|_{\mathrm{B}})$ will be isometrically a quotient of $(\widetilde{E}, \|\cdot\|_{\mathrm{B}})$. However, this isometry will in general be lost with other complexification procedures, such as Taylor's.

Finally, even though we have already observed that if ν is a natural complexification procedure and E is a real Banach space, then $(\widetilde{E}, \|\cdot\|_\nu)^*$ is a reasonable complexification of E^*, it will not generally be isometric to $((E^*)^\sim, \|\cdot\|_\nu)$ under the natural isomorphism. To prove this we first state without proof a simple identity.

LEMMA 13. *Let E be a real Banach space. For any f and g in E^*, we have*

$$\sup_{\|x\|^2+\|y\|^2=1} |(f+ig)(x+iy)| = \sup_{a^2+b^2=1} (\|af+bg\|^2 + \|bf-ag\|^2)^{1/2}. \tag{2}$$

PROPOSITION 14. *Let ν be a natural complexification procedure. Assume that either*

(a) *for any real Banach space E, $\|x+iy\|_\nu \le \sqrt{\|x\|^2+\|y\|^2}$ $\forall x, y \in E$,*

or

(b) *for any real Banach space E, $\|x+iy\|_\nu \ge \sqrt{\|x\|^2+\|y\|^2}$ $\forall x, y \in E$.*

Then the natural isomorphism between $((E^*)^\sim, \|\cdot\|_\nu)$ and $((\widetilde{E})^*, \|\cdot\|_\nu)$ cannot be an isometry unless E is a Hilbert space. Further, if E is a Hilbert space and $((E^*)^\sim, \|\cdot\|_\nu)$ is naturally isometrically isomorphic to $((\widetilde{E})^*, \|\cdot\|_\nu)$, then $\|\cdot\|_\nu$ must be the Lindenstrauss–Tzafriri norm.

REMARK. The comments before Proposition 12 show that condition (b) holds for the Lindenstrauss–Tzafriri, p-summing ($1 \le p \le 2$) and Bochnak norms. Evidently, (a) holds for the Taylor norm.

Proof (of Proposition 14). Suppose that $(E^*)^\sim$ and $(\widetilde{E})^*$ are naturally isometrically isomorphic and that (a) holds. Let $\varphi, \psi \in E^*$. Applying (a) and Lemma 13 we find

$$\begin{aligned} \|\varphi\|^2 + \|\psi\|^2 \ge \|\varphi + i\psi\|^2_{(E^*)^\sim} &= \|\varphi + i\psi\|^2_{(\widetilde{E})^*} \\ &= \sup_{\|x+iy\|_{\widetilde{E}}=1} |(\varphi+i\psi)(x+iy)|^2 \\ &\ge \sup_{\|x\|^2+\|y\|^2=1} |(\varphi+i\psi)(x+iy)|^2 \\ &= \sup_{a^2+b^2=1} (\|a\varphi+b\psi\|^2 + \|b\varphi - a\psi\|^2). \end{aligned}$$

By taking $a = b = 1/\sqrt{2}$, these inequalities imply

$$\|\varphi\|^2 + \|\psi\|^2 \ge \tfrac{1}{2}(\|\varphi+\psi\|^2 + \|\varphi-\psi\|^2).$$

Observe that if (b) is satisfied, then a similar argument yields the reverse inequality. In both cases we have a characteristic property of Hilbert spaces (see [10, p. 117]), so E is a Hilbert space. Our aim is now to show that $\|x+iy\|_\nu = \|x+iy\|_{\mathrm{LT}} = \sqrt{\|x\|^2+\|y\|^2}$ for every x, y in the Hilbert space E.

Assume that this is not so. Let $x,y\in E$. Then, still assuming (a) holds,

$$\|x+iy\|_\nu=\sup_{\|\varphi+i\psi\|_{(\widetilde E)^*}\le 1}|(\varphi+i\psi)(x+iy)|\ge\sup_{\|\varphi+i\psi\|_{(\widetilde E)^*}\le 1}|\varphi(x)-\psi(y)|$$
$$\ge\sup_{\|\varphi\|^2+\|\psi\|^2\le 1}|\varphi(x)-\psi(y)|=(\|x\|^2+\|y\|^2)^{1/2}\ge\|x+iy\|_\nu.$$

Thus $\|x+iy\|_\nu=\sqrt{\|x\|^2+\|y\|^2}$. A similar argument applies when (b) holds. ■

3. Complex extension of real-valued polynomials and multilinear forms on real Banach spaces. J. Bochnak and J. Siciak (see Theorem 3 in [9]) observed that when E and F are real Banach spaces, then each $L\in\mathcal{L}(^nE;F)$ has a unique complex extension $\widetilde L\in\mathcal{L}(^n\widetilde E;\widetilde F)$, defined by

$$\widetilde L(x_1^0+ix_1^1,\ldots,x_n^0+ix_n^1)=\sum i^{\sum_{j=1}^n\varepsilon_j}L(x_1^{\varepsilon_1},\ldots,x_n^{\varepsilon_n}),$$

where x_k^0, x_k^1 are vectors in E, and the summation is extended over the 2^n independent choices of $\varepsilon_k=0,1$ $(1\le k\le n)$. The norm of $\widetilde L$ depends on the norms used on $\widetilde E$ and $\widetilde F$, but continuity is always assured.

In the context of polynomials (see also [31, p. 313]), any $P\in\mathcal{P}(^nE;F)$ has a unique complex extension $\widetilde P\in\mathcal{P}(^n\widetilde E;\widetilde F)$, given by

$$(*)\qquad \widetilde P(x+iy)=\sum_{k=0}^{[n/2]}(-1)^k\binom{n}{2k}L(x^{n-2k}y^{2k})+i\sum_{k=0}^{[(n-1)/2]}(-1)^k\binom{n}{2k+1}L(x^{n-(2k+1)}y^{2k+1})$$

for x, y in E, where $P=\widehat L$ for some $L\in\mathcal{L}^s(^nE;F)$. Here, for $l+m=n$,

$$L(x^ly^m)=L(\underbrace{x,\ldots,x}_{l\text{ times}},\underbrace{y,\ldots,y}_{m\text{ times}}).$$

In general, any continuous polynomial $P:E\to F$ of degree n has a unique complex extension $\widetilde P:\widetilde E\to\widetilde F$. If $P(x)=\sum_{k=0}^nP_k(x)$, then $\widetilde P(x+iy)=\sum_{k=0}^n\widetilde P_k(x+iy)$. In the special case of a finite-dimensional space $(\mathbb{R}^N,\|\cdot\|)$, the complexification of a polynomial P on $\mathbb{R}^N$ is the polynomial $\widetilde P$ in N complex variables defined by

$$\widetilde P(x+iy)=P(x_1+iy_1,\ldots,x_N+iy_N),$$

for $x=(x_1\ldots,x_N)$ and $y=(y_1\ldots,y_N)$ in $\mathbb{R}^N$.

It would be good to be able to say that $\|L\|=\|\widetilde L\|$ and $\|P\|=\|\widetilde P\|$. Unfortunately, this is rarely the case. In this part of the paper we investigate the relationship between $\|L\|$, $\|\widetilde L\|$ and $\|P\|$, $\|\widetilde P\|$ for various natural complexification norms. Where it is important to distinguish different norms on $\widetilde L$ we will use the notation $\|\widetilde L\|_{\nu_1\to\nu_2}$ to denote the norm of $\widetilde L$ as a mapping from $(\widetilde E,\|\cdot\|_{\nu_1})$ to $(\widetilde F,\|\cdot\|_{\nu_2})$. When $\nu_1=\nu_2=\nu$, we shall simply write $\|\widetilde L\|_\nu$. Analogous notation will be used for polynomials.

The scalar-valued case is the easiest to handle. Indeed, to compare the norms of L and $\widetilde L$ where $L\in\mathcal{L}(^nE)$, it is enough to work with the real, continuous n-linear forms $\operatorname{Re}\widetilde L$, $\operatorname{Im}\widetilde L$ on $\widetilde E$.

Proposition 15. *Let E be a real Banach space. For any $L\in\mathcal{L}(^nE)$, $P\in\mathcal{P}(^nE)$ and any natural complexification procedure ν, we have*

$$\|\widetilde L\|_\nu=\|\operatorname{Re}\widetilde L\|_\nu=\|\operatorname{Im}\widetilde L\|_\nu\quad\textit{and}\quad\|\widetilde P\|_\nu=\|\operatorname{Re}\widetilde P\|_\nu=\|\operatorname{Im}\widetilde P\|_\nu.$$

Proof. For $x_k+iy_k\in\widetilde E$ $(1\le k\le n)$, we can find a real number t such that

$$e^{it}\widetilde L(x_1+iy_1,\ldots,x_n+iy_n)=|\widetilde L(x_1+iy_1,\ldots,x_n+iy_n)|.$$

Then

$$\begin{aligned}|\widetilde L(x_1+iy_1,\ldots,x_n+iy_n)|&=\widetilde L(e^{it/n}(x_1+iy_1),\ldots,e^{it/n}(x_n+iy_n))\\&=\operatorname{Re}\widetilde L(e^{it/n}(x_1+iy_1),\ldots,e^{it/n}(x_n+iy_n))\\&\le\|\operatorname{Re}\widetilde L\|_\nu\|e^{it/n}(x_1+iy_1)\|_\nu\ldots\|e^{it/n}(x_n+iy_n)\|_\nu\\&=\|\operatorname{Re}\widetilde L\|_\nu\|x_1+iy_1\|_\nu\ldots\|x_n+iy_n\|_\nu.\end{aligned}$$

In other words, $\|\widetilde L\|_\nu\le\|\operatorname{Re}\widetilde L\|_\nu$. Since the converse inequality is obvious, we conclude that $\|\widetilde L\|_\nu=\|\operatorname{Re}\widetilde L\|_\nu$. Similarly $\|\widetilde L\|_\nu=\|\operatorname{Im}\widetilde L\|_\nu$. The same argument works for homogeneous polynomials. ■

It is obvious that $\|L\|\le\|\widetilde L\|_\nu$, $\|P\|\le\|\widetilde P\|_\nu$ for any $L\in\mathcal{L}(^nE)$, $P\in\mathcal{P}(^nE)$ respectively and any natural complexification procedure ν. We now discuss the problem of finding constants M_1, M_2, depending only on n, such that $\|\widetilde L\|_\nu\le M_1\|L\|$, $\|\widetilde P\|_\nu\le M_2\|P\|$. In fact, for any continuous polynomial $P(x)=\sum_{k=0}^nP_k(x)$ of degree n on E, we find the best possible constant M in the more general inequality

$$(3)\qquad \|\widetilde P_n\|_\nu\le M\|P\|.$$

For polynomials in one variable, inequality (3) is a well-known theorem due to Chebyshev. If $P(x)=\sum_{k=0}^na_kx^k$ is a polynomial with real coefficients and $\|P\|=\max_{-1\le x\le 1}|P(x)|$, Chebyshev's inequality states that

$$(4)\qquad |a_n|\le 2^{n-1}\|P\|.$$

V. Markov obtained estimates for the other coefficients of P (see [22, p. 56]). In particular, for $n \geq 2$,

$$(5)\qquad |a_{n-1}| \leq 2^{n-2}\|P\|.$$

There is equality in (4) and (5) if $P = T_n$ or T_{n-1}, respectively, where $T_n(x) = \cos(n \arccos x)$ is the nth Chebyshev polynomial of the first kind.

The Chebyshev–Markov inequalities have been generalized for polynomials in many variables. If $P(x_1, \ldots, x_m) = \sum_{k=0}^{n} P_k(x_1, \ldots, x_m)$ is a polynomial of degree $\leq n$ in m variables,

$$(4')\qquad \|\widetilde{P}_n\|_{\infty,\mathbb{C}} \leq 2^{n-1}\|P\|_{\infty,\mathbb{R}},$$

$$(5')\qquad \|\widetilde{P}_{n-1}\|_{\infty,\mathbb{C}} \leq 2^{n-2}\|P\|_{\infty,\mathbb{R}},$$

where, for $j = n, n-1$,

$$\|\widetilde{P}_j\|_{\infty,\mathbb{C}} = \max_{\substack{0\leq\theta_k\leq 2\pi\\ 1\leq k\leq m}} |P_j(e^{i\theta_1}, \ldots, e^{i\theta_m})|,$$

$$\|P\|_{\infty,\mathbb{R}} = \max_{\substack{-1\leq x_k\leq 1\\ 1\leq k\leq m}} |P(x_1, \ldots, x_m)|.$$

Inequality (4′) is due to C. Visser [34]. H.-J. Rack [25], using a modification of the argument in [34], proved (5′) (see also [26] and [27]). There is equality in (4′) or (5′) if $P(x_1 \ldots, x_m) = \sum_{k=1}^{m} T_n(x_k)$ or $\sum_{k=1}^{m} T_{n-1}(x_k)$, respectively.

Our main result is a generalization of the previous inequalities for polynomials on any real Banach space. For the proof we adapt the technique given in [34].

PROPOSITION 16. *Let E be a real Banach space and let $P : E \to \mathbb{R}$, $P(x) = \sum_{k=0}^{n} P_k(x)$, be a scalar-valued polynomial of degree $\leq n$ $(n \geq 1)$. If ν is any natural complexification procedure, then*

$$(6)\qquad \|\widetilde{P}_n\|_\nu \leq 2^{n-1}\|P\|,$$

$$(7)\qquad \|\widetilde{P}_{n-1}\|_\nu \leq 2^{n-2}\|P\| \quad (n \geq 2).$$

The constants cannot generally be improved.

Proof. Let $z = x + iy \in (\widetilde{E}, \|\cdot\|_\nu)$ have norm 1. If we define

$$(8)\qquad f(t) := \widetilde{P}\left(\frac{ze^{it} + \overline{z}e^{-it}}{2}\right) = P(x\cos t - y\sin t) = \sum_{k=-n}^{n} a_k e^{ikt},$$

then $f(t)$ is a trigonometric polynomial of degree $\leq n$. Notice that

$$(9)\qquad a_n = \frac{1}{2^n}\widetilde{P}_n(z), \quad a_{-n} = \overline{a}_n = \frac{1}{2^n}\widetilde{P}_n(\overline{z}),$$

$$(10)\qquad a_{n-1} = \frac{1}{2^{n-1}}\widetilde{P}_{n-1}(z), \quad a_{-(n-1)} = \overline{a}_{n-1} = \frac{1}{2^{n-1}}\widetilde{P}_{n-1}(\overline{z}).$$

Since $\sup_t \|x\cos t - y\sin t\| = \|z\|_{\mathrm{T}} \leq \|z\|_\nu = 1$, we have

$$(11)\qquad |f(t)| \leq \|P\| \quad \forall t \in \mathbb{R}.$$

Now, using (8), (9) and the easily verified formula

$$(12)\qquad \frac{1}{2n}\sum_{p=0}^{2n-1}(-1)^p e^{ikp\pi/n} = \begin{cases} 1 & \text{if } k = n \pmod{2n},\\ 0 & \text{otherwise,}\end{cases}$$

we deduce that

$$\begin{aligned}\frac{1}{2n}\sum_{p=0}^{2n-1}(-1)^p f\left(t + p\frac{\pi}{n}\right) &= \frac{1}{2n}\sum_{p=0}^{2n-1}(-1)^p \sum_{k=-n}^{n} a_k e^{ik(t+p\pi/n)}\\ &= \sum_{k=-n}^{n} a_k e^{ikt}\left[\frac{1}{2n}\sum_{p=0}^{2n-1}(-1)^p e^{ikp\pi/n}\right]\\ &= a_n e^{int} + a_{-n}e^{-int} = \frac{1}{2^n}\widetilde{P}_n(z)e^{int} + \frac{1}{2^n}\widetilde{P}_n(\overline{z})e^{-int}.\end{aligned}$$

Bringing (11) into play, we find that

$$\sup_t |\widetilde{P}_n(z)e^{int} + \widetilde{P}_n(\overline{z})e^{-int}| \leq 2^n\|P\|,$$

and since

$$\begin{aligned}\sup_t |\widetilde{P}_n(z)e^{int} + \widetilde{P}_n(\overline{z})e^{-int}| &= \sup_t |\widetilde{P}_n(z)e^{int} + \overline{\widetilde{P}_n(z)}e^{-int}|\\ &= 2\sup_t |\mathrm{Re}\,\widetilde{P}_n(z)\cos nt - \mathrm{Im}\,\widetilde{P}_n(z)\sin nt| = 2|\widetilde{P}_n(z)|,\end{aligned}$$

inequality (6) is true.

For the proof of (7) we argue as in the case of (6), but use (8), (10) and formula (12) with $n-1$ in place of n.

The assertion that the constants are optimal is true because of the classical Chebyshev–Markov results. ■

It follows from this proposition that

$$(13)\qquad \|P_n\| \leq 2^{n-1}\|P\| \quad \text{and} \quad \|P_{n-1}\| \leq 2^{n-2}\|P\|.$$

But these results are consequences of V. Markov's estimates for the coefficients of a polynomial in one real variable (see [22, p. 56]). In fact, we can find sharp estimates for the norm of every P_k $(1 \leq k \leq n)$. Indeed, if

$P : E \to \mathbb{R}$, $P(x) = \sum_{k=0}^n P_k(x)$, then for every x in the unit ball of E,

$$p(t) = P(tx) := \sum_{k=0}^n P_k(x)t^k$$

is a polynomial in one real variable for which $\|p\| \le \|P\|$. Then, by applying the classical result to the coefficients of p, we immediately get estimates for the norms of the P_k's.

Using the first inequality of (13), we can prove a generalization of Markov's inequality for the nth derivative of a polynomial of degree n. This result gives a positive answer to part of a question posed by L. A. Harris in his commentary on problem 74 in The Scottish Book [20].

PROPOSITION 17. *Let E and F be real Banach spaces and let $P : E \to F$, $P(x) = \sum_{k=0}^n P_k(x)$, be a polynomial of degree $\le n$. Then*

$$\|\widehat{D}^n P\| \le T_n^{(n)}(1)\|P\|.$$

Proof. It is elementary to see that $\widehat{D}^n P(x)y = n!P_n(y)$, where $D^n P(x)$ is the nth Fréchet derivative of P at x, and $\widehat{D}^n P(x)$ is the nth homogeneous polynomial associated with $D^n P(x)$. Hence, it follows immediately from (13) that $\|\widehat{D}^n P\| \le 2^{n-1}n!\|P\| = T_n^{(n)}(1)\|P\|$. ∎

Since Markov's inequality for the first derivative holds on any real Banach space (see [28]), it would be interesting to know if $\|\widehat{D}^k P\| \le T_n^{(k)}(1)\|P\|$ $(1 < k < n)$ for polynomials on any real Banach space. For more details we refer to The Scottish Book [20, problem 74].

So far we have focused on extensions of general polynomials. In fact, we have analogous results for any $P_n \in \mathcal{P}(^nE)$ and $L \in \mathcal{L}(^nE)$.

PROPOSITION 18. *Let E be a real Banach space and let $P \in \mathcal{P}(^nE)$, and $L \in \mathcal{L}(^nE)$. Then, for any natural complexification procedure ν,*

$$\|\widetilde{P}\|_\nu \le 2^{n-1}\|P\| \quad \text{and} \quad \|\widetilde{L}\|_\nu \le 2^{n-1}\|L\|. \tag{14}$$

Proof. The statement about L can be proved by repeating the proof of Proposition 16, replacing (8) by

$$\begin{aligned} f(t) &:= \widetilde{L}\left(\frac{z_1e^{it} + \overline{z}_1e^{-it}}{2}, \ldots, \frac{z_ne^{it} + \overline{z}_ne^{-it}}{2}\right) \\ &= L(x_1\cos t - y_1\sin t, \ldots, x_n\cos t - y_n\sin t) = \sum_{k=-n}^n a_ke^{ikt} \end{aligned}$$

for $z_k = x_k + iy_k \in \widetilde{E}$, $\|z_k\|_\nu = 1$ $(1 \le k \le n)$, noting that $a_n = \frac{1}{2^n}\widetilde{L}(z_1, \ldots, z_n)$ and $a_{-n} = \overline{a}_n = \frac{1}{2^n}\widetilde{L}(\overline{z}_1, \ldots, \overline{z}_n)$. ∎

Interestingly, the constant 2^{n-1} which occurs in inequality (14) is necessary. In addition, the constant 2^{n-2} in inequality (7) is needed even when we restrict attention to polynomials of degree *exactly* n.

EXAMPLE 1. *Let $1 \le m \le n$. Define $P_m \in \mathcal{P}(^ml_2^2)$ by*

$$P_m(x) = \mathrm{Re}(x_1 + ix_2)^m \quad \forall x = (x_1, x_2) \in \mathbb{R}^2,$$

and write L_m for the associated symmetric m-linear form. Then, if we use the Taylor complexification,

(a) $\|\widetilde{P}_n\|_{\mathrm{T}} = 2^{n-1}\|P_n\|$,
(b) $\|\widetilde{L}_n\|_{\mathrm{T}} = 2^{n-1}\|L_n\|$,
(c) $\lim_{t\to0^+}\|\widetilde{P}_{n-1}\|_{\mathrm{T}}/\|tP_n + P_{n-1}\| = 2^{n-2}$.

Proof. Obviously, $\|P_m\| = L_m(1,0) = 1$, and by an old result (see, for instance, Theorem 4 in [15]), $\|L_m\| = \|P_m\| = 1$. Since, for $x = (x_1, x_2) \in \mathbb{R}^2$, $y = (y_1, y_2) \in \mathbb{R}^2$, we have $L_m(x^{m-2k}y^{2k}) = \mathrm{Re}(x_1 + ix_2)^{m-2k}(y_1 + iy_2)^{2k}$, formula $(*)$ implies that

$$\mathrm{Re}\,\widetilde{P}_m(x + iy) = \mathrm{Re}\left\{\sum_{k=0}^{[m/2]}(-1)^k\binom{m}{2k}(x_1 + ix_2)^{m-2k}(y_1 + iy_2)^{2k}\right\}.$$

Choose $x = (1,0)$, $y = (0,1)$. Then $\|x + iy\|_{\mathrm{T}} = \sup_\theta\|(\cos\theta, \sin\theta)\|_{l_2^2} = 1$, and

$$\mathrm{Re}\,\widetilde{P}_m((1,0) + i(0,1)) = \sum_{k=0}^{[m/2]}\binom{m}{2k} = 2^{m-1}.$$

Therefore $\|\widetilde{P}_m\|_{\mathrm{T}} = \|\mathrm{Re}\,\widetilde{P}_m\|_{\mathrm{T}} \ge 2^{m-1}$. In other words,

$$\|\widetilde{L}_m\|_{\mathrm{T}} \ge \|\widetilde{P}_m\|_{\mathrm{T}} \ge 2^{m-1}\|P_m\| = 2^{m-1}\|L_m\|.$$

Since, by (14), all the above inequalities must in fact be equalities, we infer that

$$\|\widetilde{L}_m\|_{\mathrm{T}} = 2^{m-1}\|L_m\| \quad \text{and} \quad \|\widetilde{P}_m\|_{\mathrm{T}} = 2^{m-1}\|P_m\|.$$

Taking $m = n$ we get (a) and (b). To obtain (c), we just show that for $t > 0$,

$$\|tP_n + P_{n-1}\| = 1 + t.$$

Since $(tP_n + P_{n-1})(1,0) = 1 + t$, the norm is certainly at least $1 + t$. On the other hand, if $|x_1 + ix_2| \le 1$,

$$|\mathrm{Re}(t(x_1 + ix_2)^n + (x_1 + ix_2)^{n-1})| \le |t(x_1 + ix_2)^n + (x_1 + ix_2)^{n-1}| \le 1 + t,$$

so the norm cannot exceed $1 + t$. ∎

We give one more example, using polynomials on finite-dimensional l_∞'s, where the constant 2^{n-1} in inequality (6) is achieved.

EXAMPLE 2. *Define polynomials* $P_k \in \mathcal{P}(^{2^k} l_\infty^{2^k})$ *inductively by*

$$P_1(x_1,x_2) = x_1^2 - x_2^2,$$
$$P_{k+1}(x_1,\ldots,x_{2^{k+1}}) = P_k(x_1,\ldots,x_{2^k})^2 - P_k(x_{2^k+1},\ldots,x_{2^{k+1}})^2 \quad (k\ge 1).$$

Then $\|P_k\| = 1$, *but* $\|\widetilde{P}_k\|_{\mathrm{T}} = 2^{2^k-1}$.

Proof. By induction, it is easy to see that $\|P_k\| = 1$. Another induction argument shows how to produce $z = (z_1,\ldots,z_{2^k})$ in the unit ball of the Taylor complexification of $l_\infty^{2^k}$ so that $\widetilde{P}_k(z) = 2^{2^k-1}$.

If $k = 1$, set $z_1 = 1$, $z_2 = i$. Then $\widetilde{P}_1(z_1,z_2) = 2$. If $z_1,\ldots,z_{2^k}$ of modulus one have been defined so that $\widetilde{P}_k(z_1,\ldots,z_{2^k}) = 2^{2^k-1}$, we put $z_{2^k+j} = z_j e^{i\pi/2^{k+1}}$ $(1\le j\le 2^k)$, and then

$$\begin{aligned}\widetilde{P}_{k+1}(z_1,\ldots,z_{2^{k+1}}) &= \widetilde{P}_k(z_1,\ldots,z_{2^k})^2 - \widetilde{P}_k(z_{2^k+1},\ldots,z_{2^{k+1}})^2\\ &= 2^{2(2^k-1)} - 2^{2(2^k-1)}e^{i\pi} = 2^{2^{k+1}-1}. \ \blacksquare\end{aligned}$$

REMARK. The proof of Proposition 16 does not provide good estimates for the norms of $\widetilde{P}_{n-k}$, $k\ge 2$. For instance, if $k=2$, then a_{n-2} in equation (8) cannot be expressed just in terms of $\widetilde{P}_{n-2}(z)$. We have $a_{n-2} = \frac{n}{2^n}\widetilde{L}_n(z^{n-1}\overline{z}) + \frac{1}{2^{n-2}}\widetilde{P}_{n-2}(z)$, where $P_n = \widehat{L}_n$ for some $L_n \in \mathcal{L}^s(^nE)$. Because of the extra term

$$\frac{n}{2^n}\widetilde{L}_n(z^{n-1}\overline{z}),$$

we cannot obtain a good inequality analogous to (6) and (7) for the norm of $\widetilde{P}_{n-2}$.

The upper bounds in Propositions 16 and 18 are valid for any natural complexification procedure, and we just saw that they cannot be improved for Taylor's procedure. However, Taylor's norm is the smallest natural complexification norm, so it is reasonable to ask whether improved upper bounds can be obtained by using different procedures. In the case of multilinear maps, dramatic improvements can be achieved. The biggest improvement naturally comes from using the Bochnak norm, and Bochnak proved [8, p. 276] that for any $L\in\mathcal{L}(^nE)$ we have $\|\widetilde{L}\|_{\mathrm{B}} = \|L\|$. Substantial improvements on Proposition 18 are also available for other natural complexification procedures.

PROPOSITION 19. *Let E be a real Banach space and let $1\le p\le\infty$. For any $n\ge 2$ and any $L\in\mathcal{L}(^nE)$,*

$$\|\widetilde{L}\|_{(p)} \le \begin{cases} 2^{n/2-1/2}\|L\| & \text{if } 1\le p\le 4/3,\\ 2^{n/2-2/p'}\|L\| & \text{if } 4/3\le p\le 2,\\ 2^{n/p'-1}\|L\| & \text{if } 2\le p\le\infty.\end{cases}$$

Proof. We first investigate the case $n=2$. For $x_1,x_2,y_1,y_2\in E$, we have

$$\begin{aligned}|\mathrm{Re}\,\widetilde{L}(x_1+iy_1, x_2+iy_2)| &= |L(x_1,x_2) - L(y_1,y_2)|\\ &\le \|L\|(\|x_1\|\cdot\|x_2\| + \|y_1\|\cdot\|y_2\|).\end{aligned}$$

Now, for $1\le p\le 4/3$, Proposition 1 shows that

$$\|x_1\|\cdot\|x_2\| + \|y_1\|\cdot\|y_2\| \le (\|x_1\|+\|y_1\|)\|x_2+iy_2\|_{(p)}$$

and so, using Hölder's inequality,

$$\begin{aligned}\|x_1\|\cdot\|x_2\| + \|y_1\|\cdot\|y_2\| &\le 2^{1/p'}(\|x_1\|^p + \|y_1\|^p)^{1/p}\|x_2+iy_2\|_{(p)}\\ &\le 2^{1/2}\|x_1+iy_1\|_{(p)}\|x_2+iy_2\|_{(p)}.\end{aligned}$$

Thus, for $1\le p\le 4/3$, Proposition 15 gives $\|\widetilde{L}\|_{(p)} \le 2^{1/2}\|L\|$.

Next, for $4/3\le p\le 2$, Hölder's inequality and the monotonicity of the l_p norms give

$$\begin{aligned}\|x_1\|\cdot\|x_2\| + \|y_1\|\cdot\|y_2\| &\le (\|x_1\|^p + \|y_1\|^p)^{1/p}(\|x_2\|^{p'} + \|y_2\|^{p'})^{1/p'}\\ &\le (\|x_1\|^p + \|y_1\|^p)^{1/p}(\|x_2\|^p + \|y_2\|^p)^{1/p}\\ &\le 2^{2(1/p-1/2)}\|x_1+iy_2\|_{(p)}\|x_2+iy_2\|_{(p)}.\end{aligned}$$

Thus, for such p, we have, by Proposition 15, $\|\widetilde{L}\|_{(p)} \le 2^{1-2/p'}\|L\|$.

Finally, we consider $2\le p\le\infty$. Here, since $1/p+1/p\le 1$, the generalized form of Hölder's inequality gives

$$\begin{aligned}\|x_1\|\cdot\|x_2\| + \|y_1\|\cdot\|y_2\| &\le 2^{1-2/p}(\|x_1\|^p + \|y_1\|^p)^{1/p}(\|x_2\|^p + \|y_2\|^p)^{1/p}\\ &\le 2^{2/p'-1}\|x_1+iy_1\|_{(p)}\|x_2+iy_2\|_{(p)},\end{aligned}$$

and another use of Proposition 15 yields $\|\widetilde{L}\|_{(p)} \le 2^{2/p'-1}\|L\|$.

For higher values of n, we proceed by induction. Write

$$K_n^{(p)} = \begin{cases} 2^{n/2-1/2} & \text{if } 1\le p\le 4/3,\\ 2^{n/2-2/p'} & \text{if } 4/3\le p\le 2,\\ 2^{n/p'-1} & \text{if } 2\le p\le\infty.\end{cases}$$

If $L\in\mathcal{L}(^nE)$, then for fixed $x\in E$, $F_x(x_1,\ldots,x_{n-1}) := L(x_1,\ldots,x_{n-1},x)$ defines a continuous $(n-1)$-linear form F_x on E. Now, for $x_k+iy_k\in E$ $(1\le k\le n)$, the induction hypothesis and Proposition 15 give

$$\begin{aligned}&|\mathrm{Re}\,\widetilde{L}(x_1+iy_1,\ldots,x_n+iy_n)|\\ &\quad = |\mathrm{Re}\,\widetilde{F}_{x_n}(x_1+iy_1,\ldots,x_{n-1}+iy_{n-1})\\ &\qquad - \mathrm{Im}\,\widetilde{F}_{y_n}(x_1+iy_1,\ldots,x_{n-1}+iy_{n-1})|\\ &\quad\le (\|\mathrm{Re}\,\widetilde{F}_{x_n}\|_{(p)} + \|\mathrm{Im}\,\widetilde{F}_{y_n}\|_{(p)})\|x_1+iy_1\|_{(p)}\cdots\|x_{n-1}+iy_{n-1}\|_{(p)}\\ &\quad\le K_{n-1}^{(p)}\|L\|(\|x_n\|+\|y_n\|)\|x_1+iy_1\|_{(p)}\cdots\|x_{n-1}+iy_{n-1}\|_{(p)}.\end{aligned}$$

Hence, by Hölder's inequality,

$$\|x_n\|+\|y_n\| \le 2^{1/p'}(\|x_n\|^p+\|y_n\|^p)^{1/p} \le \begin{cases} 2^{1/2}\|x_n+iy_n\|_{(p)} & \text{if } 1\le p\le 2,\\ 2^{1/p'}\|x_n+iy_n\|_{(p)} & \text{if } 2\le p\le\infty. \end{cases}$$

The induction step is now immediate. ∎

The constant given in Proposition 19 is sharp when $p\ge 2$. Indeed, let $P_n: l_2^2\to\mathbb{R}$ be the polynomial defined in Example 1, and set $x=(2^{-1/p},0)$, $y=(0,2^{-1/p})$ for a given $p\ge 2$. Then $\|x+iy\|_{(p)}=1$ and

$$\operatorname{Re}\widetilde{P}_n(x+iy)=\sum_{k=0}^{[n/2]}\binom{n}{2k}2^{-n/p}=2^{n-1}2^{-n/p}=2^{n/p'-1}.$$

Hence $\|\widetilde{P}_n\|_{(p)}=\|\operatorname{Re}\widetilde{P}_n\|_{(p)}\ge 2^{n/p'-1}$, and so

$$\|\widetilde{L}_n\|_{(p)}\ge\|\widetilde{P}_n\|_{(p)}\ge 2^{n/p'-1}=2^{n/p'-1}\|P_n\|=2^{n/p'-1}\|L_n\|.$$

Since the reverse inequality is true by Proposition 19, we get equality.

For $p\le 2$, essentially this example shows that the constant must be at least $2^{n/2-1}$.

The techniques we have just used for multilinear forms do not adapt to homogeneous polynomials of degree greater than 2. In fact, loss of norm is in general inevitable when extending homogeneous polynomials of degree at least 4, even if Bochnak's complexification procedure is used.

Example 3. *Let $P_{4n}: l_\infty^4\to\mathbb{R}$ be given by*

$$P_{4n}(x)=[(x_1^2-x_2^2)^2-(x_3^2-x_4^2)^2]^n\quad \forall x=(x_1,x_2,x_3,x_4)\in l_\infty^4.$$

Then $\|P_{4n}\|=1$ for all n, while $\lim_{n\to\infty}\|\widetilde{P}_{4n}\|_{\mathrm{B}}=\infty$.

Proof. Since $|a^2-b^2|\le 1$ whenever $a,b\in[-1,1]$, it is clear that $\|P_{4n}\|=1$ for all $n\in\mathbb{N}$. Now consider $z_0=\big(1,i,\frac{1}{\sqrt2}e^{i\pi/4},\frac{1}{\sqrt2}e^{3i\pi/4}\big)\in\widetilde{l}_\infty^4$. Since $z_0=\big(1,0,\frac12,-\frac12\big)+i\big(0,1,\frac12,\frac12\big)=x_0+iy_0$, Proposition 2 gives

$$\begin{aligned}\|z_0\|_{\mathrm{B}}&\le\inf_t\{\|x_0\cos t-y_0\sin t\|_\infty+\|x_0\sin t+y_0\cos t\|_\infty\}\\ &\le\frac{1}{\sqrt2}(\|x_0-y_0\|_\infty+\|x_0+y_0\|_\infty)\\ &\le\frac{1}{\sqrt2}(\|(1,-1,0,-1)\|_\infty+\|(1,1,1,0)\|_\infty)=\sqrt2.\end{aligned}$$

Then,

$$\|\widetilde{P}_{4n}\|_{\mathrm{B}}\ge\frac{|\widetilde{P}_{4n}(z_0)|}{\|z_0\|_{\mathrm{B}}^{4n}}\ge\frac{\big[(1-i^2)^2-\frac14(i+i)^2\big]^n}{4^n}=\Big(\frac54\Big)^n,$$

and so $\lim_{n\to\infty}\|\widetilde{P}_{4n}\|_{\mathrm{B}}=\infty$. ∎

This example can be modified to obtain faster divergence to infinity, but the modification we are able to make seems far from optimal. To obtain a similar result for polynomials of degree greater than, but not a multiple of 4, it is enough to consider

$$\begin{aligned}&P_{4n+1}: l_\infty^5\to\mathbb{R}\quad\text{given by}\quad P_{4n+1}(x)=[(x_1^2-x_2^2)^2-(x_3^2-x_4^2)^2]^n x_5,\\ &P_{4n+2}: l_\infty^6\to\mathbb{R}\quad\text{given by}\quad P_{4n+2}(x)=[(x_1^2-x_2^2)^2-(x_3^2-x_4^2)^2]^n x_5x_6,\\ &P_{4n+3}: l_\infty^7\to\mathbb{R}\quad\text{given by}\quad P_{4n+3}(x)=[(x_1^2-x_2^2)^2-(x_3^2-x_4^2)^2]^n x_5x_6x_7.\end{aligned}$$

Although the general situation is unsatisfactory for polynomials, some improvements on Proposition 18 are possible when natural complexification procedures other than Taylor's are used.

Proposition 20. *Let E be a real Banach space and let n be an even number. If $P\in\mathcal{P}(^nE)$, then*

$$\|\widetilde{P}\|_{(2)}\le 2^{n-2}\|P\|.$$

Proof. This is a simple modification of the proof of Proposition 16. Notice that $\cos(t+\frac{\pi}{2})=-\sin t$, $\sin(t+\frac{\pi}{2})=\cos t$. In the notation of the proof of Proposition 16,

$$\begin{aligned}&\Big|f\Big(t+\frac{p\pi}{n}\Big)\Big|+\Big|f\Big(t+\frac{(p+n/2)\pi}{n}\Big)\Big|\\ &\qquad=\Big|P\Big(x\cos\Big(t+\frac{p\pi}{n}\Big)-y\sin\Big(t+\frac{p\pi}{n}\Big)\Big)\Big|\\ &\qquad\quad+\Big|P\Big(x\sin\Big(t+\frac{p\pi}{n}\Big)+y\cos\Big(t+\frac{p\pi}{n}\Big)\Big)\Big|\\ &\qquad\le\|P\|\sup_s(\|x\cos s-y\sin s\|^n+\|x\sin s+y\cos s\|^n)\\ &\qquad\le\|P\|\cdot\|x+iy\|_{(2)}^n.\end{aligned}$$

Now follow the proof of Proposition 16 to obtain, for $\|z\|_{(2)}=1$,

$$\sup_t|\widetilde{P}(z)e^{int}+\widetilde{P}(\bar z)e^{-int}|\le 2^{n-1}\|P\|,$$

and then

$$|\widetilde{P}(z)|\le 2^{n-2}\|P\|.\ ∎$$

The same estimate holds for any natural complexification norm which dominates $\|\cdot\|_{(2)}$. In particular, it holds for Bochnak's norm. Notice that for such norms, homogeneous polynomials of degree 2 can be complexified without increasing their norms. However, we now show that if Taylor's complexification is used there is no real Banach space for which all 2-homogeneous polynomials can be complexified without increase of norm. We need a preliminary result.

LEMMA 21. *Let E be a real Banach space, and let $n \in \mathbb{N}$. For any $P \in \mathcal{P}(^nE)$,*
$$\|DP\| \le n\|\widetilde{P}\|_{\mathrm{T}}.$$
In particular, if $P \in \mathcal{P}(^2E)$, then $\|L\| \le \|\widetilde{P}\|_{\mathrm{T}}$, where $L \in \mathcal{L}^s(^2E)$ with $\widehat{L} = P$.

Proof. Let $t_n(\theta) := \widetilde{P}(x\cos\theta + iy\sin\theta)$, where x, y are unit vectors in E. Then $t_n(\theta)$ is a complex trigonometric polynomial of degree $\le n$. Since
$$\|x\cos\theta + iy\sin\theta\|_{\mathrm{T}} = \sup_{\phi}\|x\cos\theta\cos\phi + y\sin\theta\sin\phi\| \le 1,$$
we have $|t_n(\theta)| \le \|\widetilde{P}\|_{\mathrm{T}}$ for all real θ. Now, using Bernstein's inequality for trigonometric polynomials, we deduce that
$$|inL(x^{n-1}y)| = |t_n'(0)| \le n\sup_{\theta}|t_n(\theta)| \le n\|\widetilde{P}\|_{\mathrm{T}},$$
where $L \in \mathcal{L}^s(^nE)$ with $\widehat{L} = P$. Hence $|L(x^{n-1}y)| \le \|\widetilde{P}\|_{\mathrm{T}}$ for $\|x\| = \|y\| = 1$, and the result follows. ∎

PROPOSITION 22. *There is no real Banach space E for which $\|P\| = \|\widetilde{P}\|_{\mathrm{T}}$ for every $P \in \mathcal{P}(^2E)$.*

Proof. Suppose that $\|P\| = \|\widetilde{P}\|_{\mathrm{T}}$ for every $P \in \mathcal{P}(^2E)$. Then, by Lemma 21,
$$\|P\| = \|L\| = \|\widetilde{P}\|_{\mathrm{T}} \quad \forall P \in \mathcal{P}(^2E),$$
where $L \in \mathcal{L}^s(^2E)$ with $\widehat{L} = P$, and this implies that E is a real Hilbert space (see [5]). But when E is a real Hilbert space, Example 1 shows that there is a $P \in \mathcal{P}(^2E)$ with $\|\widetilde{P}\|_{\mathrm{T}} = 2\|P\|$. This contradiction completes the proof. ∎

At least for 2-dimensional spaces, there is an analogous result for bilinear forms. To reach this, it is useful to have some notation. Let E be a real Banach space and write
$$\mathcal{L}(n;E) = \inf\{K > 0 : \|\widetilde{L}\|_{\mathrm{T}} \le K\|L\| \ \forall L \in \mathcal{L}(^nE)\},$$
$$\mathcal{P}(n;E) = \inf\{K > 0 : \|\widetilde{P}\|_{\mathrm{T}} \le K\|P\| \ \forall P \in \mathcal{P}(^nE)\}.$$
The following result can easily be verified. We spare the reader the details.

LEMMA 23. *If E and F are isomorphic real Banach spaces, then*
$$\mathcal{K}(n;F) \le (\mathrm{d}(E,F))^n\mathcal{K}(n;E),$$
where $\mathrm{d}(E,F)$ denotes the Banach–Mazur distance, and $\mathcal{K} = \mathcal{L}$ or $\mathcal{P}$.

PROPOSITION 24. *There is no two-dimensional real Banach space E for which $\|\widetilde{L}\|_{\mathrm{T}} = \|L\|$, for every $L \in \mathcal{L}(^2E)$.*

The proof hinges on another example.

EXAMPLE 4. *Let $L \in \mathcal{L}^s(^2l_1^2)$ be defined by*
$$L(x,y) = x_1y_1 + x_1y_2 + x_2y_1 - x_2y_2$$
for $x = (x_1,x_2)$, $y = (y_1,y_2)$ in $\mathbb{R}^2$. Then $\|\widetilde{L}\|_{\mathrm{T}} > \|L\|$.

Proof. As usual, we write $P = \widehat{L}$. Obviously, $\|L\| = \|P\| = 1$. On the other hand,
$$\widetilde{P}(x+iy) = (x_1+iy_1)^2 + 2(x_1+iy_1)(x_2+iy_2) - (x_2+iy_2)^2.$$
For $x = (1/\sqrt{2},0)$, $y = (0,1/\sqrt{2})$ we have $\|x+iy\|_{\mathrm{T}} = 1$, and $|\widetilde{P}(x+iy)| = |1+i| = \sqrt{2}$. Therefore $\|\widetilde{L}\|_{\mathrm{T}} > \|L\|$. ∎

Proof of Proposition 24. Let E be a 2-dimensional real Banach space. Suppose $\|L\| = \|\widetilde{L}\|_{\mathrm{T}}$ for every $L \in \mathcal{L}(^2E)$. Let $F = l_2^2$ in Lemma 23 and refer to Example 1 to see that
$$2 = \mathcal{L}(2;l_2^2) \le (\mathrm{d}(E,l_2^2))^2.$$
Therefore $\mathrm{d}(E,l_2^2) = \sqrt{2}$, and this implies that $E = l_1^2$ (see Proposition 37.4 in [33]). Hence, we can only have $\|L\| = \|\widetilde{L}\|_{\mathrm{T}}$ for all $L \in \mathcal{L}(^2E)$ if $E = l_1^2$. However, Example 4 provides an $L \in \mathcal{L}^s(^2l_1^2)$ with $\|\widetilde{L}\|_{\mathrm{T}} > \|L\|$. ∎

4. Complex extension of vector-valued homogeneous polynomials and multilinear maps.

Let E be a real Banach space and let $L : E \times \ldots \times E \to E^*$ be a continuous n-linear map. This can be viewed as a continuous $(n+1)$-linear map
$$M : E \times \ldots \times E \to \mathbb{R}, \quad M(x_1,\ldots,x_{n+1}) = L(x_1,\ldots,x_n)(x_{n+1}) \quad \forall x_1,\ldots,x_{n+1} \in E.$$
In view of this, it is not surprising that the change in norm for complexifications of vector-valued multilinear maps should potentially be worse than what we find in the scalar-valued case.

PROPOSITION 25. *Let E and F be real Banach spaces and let ν be a natural complexification procedure.*

(a) *Let $P : E \to F$, $P(x) = \sum_{k=0}^n P_k(x)$, be a vector-valued polynomial of degree $\le n$. Then*
$$\|\widetilde{P}_n\|_\nu \le 2^n\|P\|, \tag{15}$$
$$\|\widetilde{P}_{n-1}\|_\nu \le 2^{n-1}\|P\|. \tag{16}$$

(b) *Let $L \in \mathcal{L}(^nE;F)$. Then $\|\widetilde{L}\|_\nu \le 2^n\|L\|$.*

In the case of the Taylor complexification, the constants can be reduced by a factor of 2.

Proof. The proof of these inequalities is very similar to that of Proposition 16. The only difference is that in the last step we just have to notice that

$$\sup_t \|\widetilde{P}_n(z)e^{int} + \widetilde{P}_n(\bar{z})e^{-int}\| = 2\|\widetilde{P}_n(z)\|_{\mathrm{T}} \ge \|\widetilde{P}_n(z)\|_\nu. \ \blacksquare$$

We do not know if the constants in Proposition 25 are best possible. However, it is possible to arrange for equality if different complexification procedures are used for E and F.

EXAMPLE 5. *If $E = F = l_2^2$, let $P \in \mathcal{P}(^nE;F)$ be defined by*

$$P(x) = (\mathrm{Re}(x_1+ix_2)^n, \mathrm{Im}(x_1+ix_2)^n) \quad \forall x = (x_1,x_2) \in l_2^2.$$

Then, writing L for the symmetric n-linear map associated with P,

$$\|\widetilde{L}\|_{\mathrm{T}\to\mathrm{B}} = \sup\{\|\widetilde{L}(z_1,\ldots,z_n)\|_{\mathrm{B}} : \|z_1\|_{\mathrm{T}} \le 1,\ldots,\|z_n\|_{\mathrm{T}} \le 1\} = 2^n\|L\|,$$
$$\|\widetilde{P}\|_{\mathrm{T}\to\mathrm{B}} = \sup\{\|\widetilde{P}(z)\|_{\mathrm{B}} : \|z\|_{\mathrm{T}} \le 1\} = 2^n\|P\|.$$

Proof. It is easily seen that $\|L\| = \|P\| = 1$. Since
$L(x^{n-2k}y^{2k}) = (\mathrm{Re}(x_1+ix_2)^{n-2k}(y_1+iy_2)^{2k}, \mathrm{Im}(x_1+ix_2)^{n-2k}(y_1+iy_2)^{2k})$,
formula $(*)$ implies that

$$\mathrm{Re}\,\widetilde{P}(x+iy) = \sum_{k=0}^{[n/2]}(-1)^k\binom{n}{2k}(\mathrm{Re}(x_1+ix_2)^{n-2k}(y_1+iy_2)^{2k}, \mathrm{Im}(x_1+ix_2)^{n-2k}(y_1+iy_2)^{2k}),$$

for $x = (x_1,x_2) \in l_2^2$, $y = (y_1,y_2) \in l_2^2$. Choose $x = e_1$, $y = e_2$. Then

$$\|x+iy\|_{\mathrm{T}} = \sup_\theta \|(\cos\theta, \sin\theta)\|_{l_2^2} = 1,$$

and

$$\mathrm{Re}\,\widetilde{P}(e_1+ie_2) = \sum_{k=0}^{[n/2]}\binom{n}{2k}e_1 = 2^{n-1}e_1.$$

Similarly, we can prove that

$$\mathrm{Im}\,\widetilde{P}(e_1+ie_2) = \sum_{k=0}^{[(n-1)/2]}\binom{n}{2k+1}e_2 = 2^{n-1}e_2.$$

Therefore $\|\widetilde{P}\|_{\mathrm{T}\to\mathrm{B}} \ge \|\widetilde{P}(e_1+ie_2)\|_{\mathrm{B}} = 2^{n-1}\|e_1+ie_2\|_{\mathrm{B}} = 2^n$. In other words,

$$\|\widetilde{L}\|_{\mathrm{T}\to\mathrm{B}} \ge \|\widetilde{P}\|_{\mathrm{T}\to\mathrm{B}} \ge 2^n\|P\| = 2^n\|L\|.$$

But, as in the proof of Proposition 25, we can show that

$$\|\widetilde{P}\|_{\mathrm{T}\to\mathrm{B}} \le 2^n\|P\|, \quad \|\widetilde{L}\|_{\mathrm{T}\to\mathrm{B}} \le 2^n\|L\|.$$

Hence

$$\|\widetilde{L}\|_{\mathrm{T}\to\mathrm{B}} = \|\widetilde{P}\|_{\mathrm{T}\to\mathrm{B}} = 2^n\|P\| = 2^n\|L\|. \ \blacksquare$$

When specific natural complexification norms, other than the Taylor norm, are used, we can obtain other results similar to the scalar-valued case, but with slightly worse constants. We single out one special case which has been investigated in the past.

PROPOSITION 26. *Let H_1, H_2 be real Hilbert spaces. If $L \in \mathcal{L}(^nH_1;H_2)$, then*

$$\|\widetilde{L}\|_{(2)} \le 2^{(n-1)/2}\|L\|. \tag{17}$$

In addition, if $P \in \mathcal{P}(^nH_1;H_2)$, then

$$\|\widetilde{P}\|_{(2)} \le 2^{(n-1)/2}\|P\|. \tag{18}$$

The constant $2^{(n-1)/2}$ is best possible.

Proof. It is enough to observe that $\mathcal{L}(^nH_1;H_2)$ and $\mathcal{L}(^{n+1}H_1^n \times H_2;\mathbb{R})$ are isometric . Since the $\|\cdot\|_{(2)}$ norm on the complexification of H_1, H_2 gives the natural lattice complexification, the result follows from the techniques used to prove Proposition 19. ■

REMARKS. (i) Inequality (17), and therefore (18), for $L \in \mathcal{L}^s(^nH_1;H_2)$ and $P \in \mathcal{P}(^nH_1;H_2)$ was found by A. E. Taylor [31, pp. 313–314] using a different technique. Observe, however, that our inequality (17) is true for *any* n-linear mapping $L : H_1^n \to H_2$; symmetry is not required.

(ii) D. H. Hyers, in his expository article on polynomial operators (see [16, p. 435]), states incorrectly that the extension $\widetilde{L} : \widetilde{H}_1^n \to \widetilde{H}_2$ of any n-linear operator $L : H_1^n \to H_2$ preserves its norm, i.e. $\|\widetilde{L}\|_{(2)} = \|L\|$. In fact, this is only possible in general for bounded bilinear forms.

5. An application. In Proposition 16 we found lower bounds for the norm of a polynomial of degree at most n on a real Banach space in terms of the norm of its n- or $(n-1)$-homogeneous part; see also inequalities (4′) and (5′). In [2] lower bounds for the sup-norm of a polynomial of degree at most n in many variables were obtained in terms of coefficients of its n-homogeneous part. Using our result we can improve one of the estimates in [2].

A polynomial $P(x_1,\ldots,x_m)=\sum_{k=0}^n P_k(x_1,\ldots,x_m)$ in m variables can be written in the form

$$P(x_1,\ldots,x_m)=\sum_{j=1}^m a_j x_j^n+\sum_{|k|\le n} a_k x_1^{k_1}\ldots x_m^{k_m}, \tag{19}$$

with $k=(k_1,\ldots,k_m)$, $|k|=k_1+\ldots+k_m$, where in the last terms all k's with $|k|=n$ have at least two non-zero components. It was shown in [3] (see also [4] and Theorem 1.1 in [2]) that

$$\sum_{j=1}^m |a_j|\le \|\widetilde{P}_n\|_{\infty,\mathbb{C}}.$$

In other words, the complex sup-norm of the nth homogeneous polynomial P_n is bounded below by the sum of the absolute values of its leading coefficients. In [2, Theorem 1.6], a similar, but worse, lower bound was established for the real sup-norm of P. Our next result is an improvement on this.

PROPOSITION 27. *Let P be a polynomial of degree n, in m variables, with real coefficients, written in the form* (19). *Then*

$$\sum_{j=1}^m |a_j|\le 2^{n-1}\|P\|_{\infty,\mathbb{R}},$$

and the constant is best possible.

Proof. By inequality (4′), we have

$$\|\widetilde{P}_n\|_{\infty,\mathbb{C}}\le 2^{n-1}\|P\|_{\infty,\mathbb{R}}.$$

If we combine this with the lower bound given above for $\|\widetilde{P}_n\|_{\infty,\mathbb{C}}$, we find

$$\sum_{j=1}^m |a_j|\le 2^{n-1}\|P\|_{\infty,\mathbb{R}}.$$

This inequality is sharp for the polynomial $P(x_1,\ldots,x_m)=\sum_{k=1}^m T_n(x_k)$, where T_n is the nth Chebyshev polynomial of the first kind. ■

6. Complex extensions of non-homogeneous polynomials. Finally, we discuss the problem of comparing the norms of a not necessarily homogeneous polynomial and its complex extension. For polynomials in one variable, a classical result due to S. Bernstein [7] (see also [19, p. 42]) states that

$$|P(z)|\le (a+b)^n \max_{-1\le x\le 1}|P(x)|, \tag{20}$$

where a and b are the semi-axes of an ellipse passing through the point z with foci at the points 1 and -1. If we consider the ellipse with $a=\sqrt{2}$,

$b=1$, and apply the maximum modulus principle, inequality (20) implies

$$\|\widetilde{P}\|_{\infty,\mathbb{C}}\le (1+\sqrt{2})^n\|P\|_{\infty,\mathbb{R}}. \tag{21}$$

J. Siciak [30] generalized inequality (21) for any complex polynomial of degree at most n on $\mathbb{C}^N$. His result improves Theorem 2.1 in [2]. In the more general case of polynomials on Banach lattices, an analogous inequality was found by M. Lacruz (see Theorem 5.7.7 in [17]).

If P is a polynomial of degree n in one variable with *real* coefficients, a result of P. Erdős [13] gives an improvement of (21):

$$\|\widetilde{P}\|_{\infty,\mathbb{C}}\le |T_n(i)|\cdot\|P\|_{\infty,\mathbb{R}}, \tag{22}$$

and the constant $|T_n(i)|$ is best possible. R. Duffin and A. C. Schaeffer [12] also gave an improvement of (20) in the case where P has real coefficients. If we argue as in the proof of (21), we obtain a result which implies (22) when n is even.

Our final result is a generalization of (22) for polynomials on any real Banach space. For the proof, which is similar to that of Theorem 5.7.7 in [17], we use inequality (22) and the following easy extension of a well-known polynomial inequality (see Lemma 5.7.3 in [17]).

LEMMA 28. *Let E be a complex Banach space and let $P:E\to\mathbb{C}$ be a polynomial of degree n. Then*

$$\|P\|\le \frac{1}{r^n}\max\{|P(x)|:\|x\|\le r\}.$$

PROPOSITION 29. *Let P be a polynomial of degree n on a real Banach space E. Then*

$$\|\widetilde{P}\|_{\mathrm{T}}\le 2^{n/2}|T_n(i)|\cdot\|P\|. \tag{23}$$

Proof. Fix $\varepsilon>0$. By Lemma 28, there exists $z=x+iy\in\widetilde{E}$ with $\|z\|_{\mathrm{T}}\le 1/\sqrt{2}$ and

$$|\widetilde{P}(z)|=|\widetilde{P}(x+iy)|\ge(1-\varepsilon)(1/\sqrt{2})^n\|\widetilde{P}\|_{\mathrm{T}}.$$

Since the polynomial $q(t):=P(x+ty)$ has real coefficients, inequality (22) gives

$$\max_{|z|=1}|q(z)|\le|T_n(i)|\max_{-1\le t\le 1}|q(t)|.$$

But, for $-1\le t\le 1$ and $\varphi\in E^*$ we have

$$\begin{aligned}\|x+ty\|&=\sup_{\|\varphi\|=1}|\varphi(x)+t\varphi(y)|\\&\le\sqrt{1+t^2}\sup_{\|\varphi\|=1}\sqrt{\varphi^2(x)+\varphi^2(y)}\le\sqrt{2}\|x+iy\|_{\mathrm{T}}\le 1,\end{aligned}$$

and this shows that $\max_{-1\le t\le 1}|q(t)| \le \|P\|$. Therefore

$$(1-\varepsilon)(1/\sqrt{2})^n\|\widetilde{P}\|_{\mathrm{T}} \le |\widetilde{P}(x+iy)| = |q(i)| \le \max_{|z|=1}|q(z)|$$
$$\le |T_n(i)| \max_{-1\le t\le 1}|q(t)| \le |T_n(i)|\cdot\|P\|.$$

So, for every $\varepsilon > 0$ we have

$$\|\widetilde{P}\|_{\mathrm{T}} \le (1-\varepsilon)^{-1}2^{n/2}|T_n(i)|\cdot\|P\|. \blacksquare$$

Observe that our constant $2^{n/2}|T_n(i)|$ is less than or equal to $\frac{1}{2}[(2+\sqrt{2})^n + (2-\sqrt{2})^n]$. It would be interesting to know if the constant in (23) can be replaced by $|T_n(i)|$. Notice that in the case of n-homogeneous polynomials the best constant 2^{n-1} is smaller than $|T_n(i)|$.

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Departamento de Análisis
Facultad de Matemáticas
Universidad Complutense de Madrid
Madrid 28040, Spain
E-mail: gustavo@sunam1.mat.ucm.es

Mathematics Department
National Technical University
Zografou Campus 157 80
Athens, Greece
E-mail: ysarant@math.ntua.gr

Department of Mathematics and Computer Science
Kent State University
Kent, Ohio 44242
U.S.A.
E-mail: tonge@mcs.kent.edu

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