again denotes the (periodic) shift by $h$ on $L_q(L_q) := L_q([0,1], L_q(\Omega,\nu))$ then $(\mathrm{R}_q)$ for $X = L_q(\Omega,\nu)$ is equivalent to

$$\|(I - S_h)K\| \to 0 \quad \text{as } h \to 0.$$

Hence, again by interpolation, $K$ satisfies $(\mathrm{R}_p)$ on $L_p(L_p)$. Indeed, by [2], Theorem 5.1.2, we have

$$\|(I - S_h)K\|_{L_p(L_p)} \le \|(I - S_h)K\|^{1-\theta}_{L_q(L_q)} \|(I - S_h)K\|^{\theta}_{L_2(L_2)}$$

where in the case $1 < p < 2$ we choose $q = 1$ and $\theta = 2/p'$ and in the case $2 < p < \infty$ we choose some $q$ with $p < q < \infty$ and $\theta = 2(q-p)/(p(q-2))$. Since we have $\|(I - S_h)K\|_{L_q(L_q)} \le 2C$, an appeal to Theorem 2.4 completes the proof. ■

## References


[1] H. Alt, *Lineare Funktionalanalysis*, Springer, Berlin, 1992.

[2] J. Bergh and J. Löfström, *Interpolation Spaces*, Springer, Berlin, 1976.

[3] E. B. Davies, *One-Parameter Semigroups*, Academic Press, London, 1980.

[4] A. Driouich and O. El-Mennaoui, *On the inverse formula of Laplace transforms*, preprint.

[5] O. El-Mennaoui and K. J. Engel, *On the characterization of eventually norm continuous semigroups in Hilbert space*. Arc. Math. (Basel) 63 (1994), 437–440.

[6] —, *Towards a characterization of eventually norm continuous semigroups on Banach spaces*, Quaestiones Math. 19 (1996), 183–190.

[7] J. Martinez and J. Mazon, *$C_0$-semigroups norm continuous at infinity*, Semigroup Forum 52 (1996), 213–224.

[8] J. van Neerven, *The Asymptotic Behaviour of Semigroups of Linear Operators*, Birkhäuser, Basel, 1996.

[9] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer, New York, 1983.

[10] L. Weis, *A short proof for the stability theorem for positive semigroups on $L_p(\nu)$*, Proc. Amer. Math. Soc. (1998), to appear.

[11] —, *Integral operators and changes of density*, Indiana Univ. Math. J. 31 (1982), 83–96.

[12] P. You, *Characteristic conditions for $c_0$-semigroups with continuity in the uniform operator topology for $t > 0$*, Proc. Amer. Math. Soc. 116 (1992), 991–997.



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# Function spaces and spectra of elliptic operators on a class of hyperbolic manifolds

by

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**Abstract.** The paper deals with quarkonial decompositions and entropy numbers in weighted function spaces on hyperbolic manifolds. We use these results to develop a spectral theory of related Schrödinger operators in these hyperbolic worlds.

## 1. INTRODUCTION

A bounded connected domain $\Omega$ in $\mathbb{R}^n$ is called a *d-domain* if, roughly speaking, its boundary $\partial\Omega$ is an inner Minkowski $d$-set. Here $n-1 \le d < n$, with $d = n-1$ in the case of a Lipschitzian boundary, whereas $n-1 < d < n$ indicates fractal distortions. We convert $\Omega$ in a non-compact hyperbolic manifold $M$ of bounded geometry and with positive injectivity radius by introducing the Riemannian metric

$$ds^2 = g^2(x)\, dx^2, \quad x \in \Omega, \tag{1.1}$$

where $g(x)$ is a positive $C^\infty$ function in $\Omega$ with

$$(\mathrm{dist}(x, \partial\Omega))^{-1} \sim g(x), \quad x \in \Omega, \tag{1.2}$$

where "$\sim$" means that the quotient of the two functions involved can be estimated from above and from below by positive constants which are independent of $x \in \Omega$. Based on [Tri86] and [Tri87] we developed in [Tri92], Ch. 7, a theory of two scales of function spaces $F^s_{pq}(M)$ and $B^s_{pq}(M)$ on (abstract) Riemannian manifolds with bounded geometry and positive injectivity radius. These scales include (fractional) Sobolev spaces, (classical) Besov spaces, Hölder–Zygmund spaces and (inhomogeneous) Hardy spaces. Under the above more special circumstances there is no problem to introduce weighted spaces $F^s_{pq}(M, g^\varkappa)$ with $\varkappa \in \mathbb{R}$. The paper deals with the following topics:

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A. Quarkonial (or subatomic) decompositions of some spaces $F^s_{pq}(M, g^{\varkappa})$. This extends what has been done in [Tri97], Sect. 14, and in particular in [Tri98], from the euclidean case to the above Riemannian case.

B. Based on A we estimate the entropy numbers $e_k$ of the compact embedding

$$\mathrm{id} : F^{s_1}_{p_1q_1}(M, g^{\varkappa_1}) \to F^{s_2}_{p_2q_2}(M, g^{\varkappa_2}) \tag{1.3}$$

where $0 < p_1 \le p_2 < \infty$, $0 < q_1 \le \infty$, $0 < q_2 \le \infty$,

$$\delta = \left(s_1 - \frac{n}{p_1}\right) - \left(s_2 - \frac{n}{p_2}\right) > 0, \quad \varkappa = \varkappa_1 - \varkappa_2 > 0. \tag{1.4}$$

We obtain

$$e_k \sim k^{-(s_1-s_2)/n}, \qquad k \in \mathbb{N}, \text{ if } \varkappa > \delta d/n, \tag{1.5}$$

$$e_k \sim k^{-\varkappa/d+1/p_2-1/p_1}, \quad k \in \mathbb{N}, \text{ if } \varkappa < \delta d/n. \tag{1.6}$$

Of course the interpretation of "$\sim$" is as in (1.2) now with respect to $k \in \mathbb{N}$. This behaviour is surprisingly similar to the euclidean case, where $M, g(x), d$ is replaced by $\mathbb{R}^n, (1+|x|^2)^{1/2}, n$. We refer to [ET96], 4.3.2, pp. 170–171, based on [HaT94], and recently substantially extended in [Har98] (limiting cases).

C. On the basis of A and B, and the technique used in [Tri97], one can now develop a spectral theory of weighted (fractal) pseudodifferential operators on the above hyperbolic manifold $M$. This will not be done here in detail. We restrict ourselves to an example. It is well known that the Laplace–Beltrami operator $-\Delta_g$ with its domain of definition

$$\mathrm{dom}(-\Delta_g) = H^2(M) = F^2_{2,2}(M) \tag{1.7}$$

is self-adjoint in $L_2(M)$ $(= F^0_{2,2}(M))$ and bounded from below. Let $\varrho \in \mathbb{R}$ be such that

$$\mathrm{spec}(-\Delta_g + \varrho\,\mathrm{id}) \subset [1, \infty). \tag{1.8}$$

Then we are interested in the negative spectrum of the relatively compact perturbation

$$H_\beta = -\Delta_g + \varrho\,\mathrm{id} - \beta g^{-\varkappa}, \quad \varkappa > 0,\ \beta > 0, \tag{1.9}$$

of $-\Delta_g + \varrho\,\mathrm{id}$. In other words we ask for the behaviour of

$$N_\beta = \sharp\{\mathrm{spec}(H_\beta) \cap (-\infty, 0]\} \quad \text{as } \beta \to \infty. \tag{1.10}$$

Problems of this type attracted a lot of attention in the euclidean setting (i.e. with $\mathbb{R}^n$ in place of $M$). They originate from (euclidean) quantum mechanics and the semi-classical limit $\hbar \to 0$ (Planck's constant tending to zero) and $\beta \sim \hbar^{-2}$, considered there. Let $|x|_g$ be the Riemannian distance of $x \in M$ to a fixed off-point, say, $0 \in M$. Then there are two positive continuous functions

$$m(x) \sim 1, \quad M(x) \sim 1, \quad x \in \Omega, \tag{1.11}$$

such that

$$g(x) = M(x)\, 2^{m(x)|x|_g}, \quad |x|_g \ge 1. \tag{1.12}$$

Hence the potential $g^{-\varkappa}(x)$ in (1.9) is of exponential decay measured against $|x|_g$. We obtain

$$N_\beta \sim \beta^{d/\varkappa} \quad \text{if } 0 < \varkappa < 2d/n \ (\beta \to \infty), \tag{1.13}$$

$$N_\beta \sim \beta^{n/2} \quad \text{if } \varkappa > 2d/n \ (\beta \to \infty), \tag{1.14}$$

which might be considered as the main result of this paper. Again there is a striking similarity to the euclidean case where $M, g, d$ is replaced by $\mathbb{R}^n, (1+|x|^2)^{-\varkappa/2}, n$. We refer to [ET96], 5.4.7–5.4.9, pp. 236–242, based on [HaT94*], and extended in [Har98]. To imitate the hydrogen atom in $\mathbb{R}^n$ (where $n = 3$ has physical relevance) we put $\varkappa = 1$ in (1.9) and multiply near the off-point with the "Coulomb" potential $|x|_g^{-1}$, hence

$$H^\beta = -\Delta_g + \varrho\,\mathrm{id} - \beta(g \min(1, |x|_g))^{-1}. \tag{1.15}$$

It turns out that for $n \ge 3$ the local perturbation $|x|_g^{-1}$ does not influence (1.13) and that

$$N_\beta \sim \beta^d, \quad \beta \to \infty,\ n \ge 3. \tag{1.16}$$

Hence the physicists in the hyperbolic world $(M, g)$ claim that one can find out by local measurement (1.16) how fractal $(n-1 \le d < n)$ the invisible boundary of their infinite world might be (under the assumption $\varkappa = 1$).

We collect definitions, results and further comments in Section 2. Proofs are given in Section 3. Here we are in a comfortable situation. The paper is an application of [Tri98] combined with the techniques developed in [Tri97] in connection with fractal pseudodifferential operators. With these results the proofs are not so complicated.

## 2. DEFINITIONS, RESULTS, COMMENTS

### 2.1. Manifolds

**2.1.1.** *Preliminaries.* Let $\Omega$ be a bounded connected domain in $\mathbb{R}^n$. Then $\mathrm{dist}(x, \partial\Omega)$ denotes the distance of $x \in \Omega$ to the boundary $\partial\Omega$ of $\Omega$. It is well known (and can be checked easily) that there is a positive $C^\infty$ function $g(x)$ in $\Omega$ with

$$g(x) \sim (\mathrm{dist}(x, \partial\Omega))^{-1}, \quad |D^\gamma g(x)| \le c_\gamma g^{1+|\gamma|}(x), \quad x \in \Omega,\ \gamma \in \mathbb{N}^n_0, \tag{2.1}$$

for some positive constants $c_\gamma$. We use "$\sim$" for two positive functions $a(x)$ and $b(x)$ or two sequences of positive numbers $a_k$ and $b_k$ (say, $k \in \mathbb{N}$) if there are two positive numbers $c$ and $C$ such that

$$ca(x) \le b(x) \le Ca(x) \quad \text{or} \quad ca_k \le b_k \le Ca_k$$

for all admitted variables $x$ or $k$. We equip $\Omega$ with the Riemannian metric

$$ds^2 = g^2(x)\, dx^2. \tag{2.2}$$

Then we obtain a non-compact hyperbolic manifold, denoted by $M$, of bounded geometry and with positive injectivity radius. Details about the notation used may be found in [Tri92], 7.2.1, pp. 281–285, and the references given there. We used this type of interplay between Riemannian and euclidean metrics in [Tri88] to study pseudodifferential operators.

**2.1.2.** *The covering.* Let

$$\Omega_j = \{x \in \Omega : 2^{-j-1} < \operatorname{dist}(x, \partial\Omega) \le 2^{-j}\}, \qquad j \in \mathbb{N}_0. \tag{2.3}$$

We assume, without restriction of generality, that $\Omega_j \neq \emptyset$ for all $j \in \mathbb{N}_0$ and $\operatorname{dist}(x, \partial\Omega) < 1$ for any $x \in \Omega$. For a fixed small positive number $c$ we cover $\overline{\Omega}_j$ by balls $B_{jm}$ of radius $c2^{-j}$ centred in $\overline{\Omega}_j$ such that

$$B_{jm} \subset \Omega_{j-1} \cup \Omega_j \cup \Omega_{j+1}, \qquad j \in \mathbb{N}_0, \tag{2.4}$$

(with $\Omega_{-1} = \emptyset$) and $m = 1, \ldots, M_j$. The covering

$$\Omega = \bigcup_{j=0}^{\infty} \bigcup_{m=1}^{M_j} B_{jm} \tag{2.5}$$

is assumed to be locally finite: there is $N \in \mathbb{N}$ such that at most $N$ balls involved in (2.5) have a non-empty intersection. Furthermore we assume that there is a number $\lambda$ with $0 < \lambda < 1$ such that all the balls $\lambda B_{jm}$ are disjoint. Here $\lambda B_{jm}$ stands for the ball with the same centre as $B_{jm}$ and of radius $\lambda$ times the radius of $B_{jm}$.

**2.1.3.** DEFINITION. Let $n-1 \le d < n$. The above domain $\Omega$, or likewise the related Riemannian manifold $(M, g)$, is called a *d-domain* if there is a covering of the above type such that

$$M_j \sim 2^{jd}, \qquad j \in \mathbb{N}_0. \tag{2.6}$$

**2.1.4.** REMARK. If $\Omega$ has a Lipschitz boundary then $d = n-1$. If $n-1 < d < n$ then $\partial\Omega$ is a fractal. Notation of this type is useful in connection with the spectral theory of the (euclidean) Laplacian $-\Delta$ in bounded domains with fractal boundary [Lap91, HeL97, EvH93, Ber98]. In the notation introduced there $d$ is the inner Minkowski dimension of $\Omega$. We refer to [Fal85], [Fal90], [Mat95] for the background of fractal geometry. A short description of some relevant aspects may also be found in [Tri97], including what is called a $d$-set.



**2.1.5.** *Examples: Thorny star-like d-domains.* The prototype of an $(n-1)$-domain in the above context is the Poincaré $n$-ball, that is, the unit ball $B = \Omega$ in $\mathbb{R}^n$ equipped with the Riemannian metric

$$ds^2 = (1 - |x|^2)^{-2}\, dx^2, \qquad |x| < 1. \tag{2.7}$$

Let $Q$ be the unit cube in $\mathbb{R}^{n-1}$ and let $0 < s < 1$. In [Tri97], Sect. 16, pp. 119–123, we constructed a positive function $x_n = f(x')$ with $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$, compact support in $Q$, and $f \in C^s(\mathbb{R}^{n-1})$ (Hölder spaces) such that its graph $\{(x', f(x')) : x' \in \mathbb{R}^{n-1}\}$ is an $(n-s)$-set in $\mathbb{R}^n$. As indicated in [Tri97], p. 123, with obvious modifications one can replace $Q$ by the unit sphere $\{x \in \mathbb{R}^n : |x| = 1\}$ and $x_n$ by the radial direction. Then one obtains a rather thorny star-like (with respect to the origin) simply connected $d$-domain where $d = n - s$.

## 2.2. Function spaces

**2.2.1.** *Basic notation.* Let $S(\mathbb{R}^n)$ be the Schwartz space of all complex-valued, rapidly decreasing, infinitely differentiable functions on $\mathbb{R}^n$. By $S'(\mathbb{R}^n)$ we denote the dual space of all tempered distributions on $\mathbb{R}^n$. Furthermore, $L_p(\mathbb{R}^n)$ with $0 < p \le \infty$ is the usual quasi-Banach space with respect to the Lebesgue measure, quasi-normed by

$$\|f \mid L_p(\mathbb{R}^n)\| = \Big( \int_{\mathbb{R}^n} |f(x)|^p\, dx \Big)^{1/p}, \tag{2.8}$$

with the usual modification if $p = \infty$. If $\varphi \in S(\mathbb{R}^n)$ then

$$\widehat{\varphi}(\xi) = (F\varphi)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\xi x} \varphi(x)\, dx, \qquad \xi \in \mathbb{R}^n, \tag{2.9}$$

denotes the Fourier transform of $\varphi$. As usual, $\varphi^\vee$ or $F^{-1}\varphi$ stands for the inverse Fourier transform, given by $\varphi^\vee(\xi) = \widehat{\varphi}(-\xi)$. Both $F$ and $F^{-1}$ are extended to $S'(\mathbb{R}^n)$ in the standard way. Let $\varphi \in S(\mathbb{R}^n)$ with

$$\varphi(x) = 1 \quad \text{if } |x| \le 1, \qquad \varphi(y) = 0 \quad \text{if } |y| \ge 3/2. \tag{2.10}$$

We put $\varphi_0 = \varphi$, $\varphi_1(x) = \varphi(x/2) - \varphi(x)$, and

$$\varphi_k(x) = \varphi_1(2^{-k+1}x), \qquad x \in \mathbb{R}^n,\ k \in \mathbb{N}. \tag{2.11}$$

Then, since

$$1 = \sum_{k=0}^{\infty} \varphi_k(x) \quad \text{for all } x \in \mathbb{R}^n, \tag{2.12}$$

the $\varphi_k$ form a dyadic resolution of unity. Recall that $(\varphi_k \widehat{f})^\vee(x)$ is an entire analytic function on $\mathbb{R}^n$ for any $f \in S'(\mathbb{R}^n)$.

**2.2.2.** *The F-scale.* Let $s \in \mathbb{R}$ and $0 < p \le \infty$, $0 < q \le \infty$ (with $q = \infty$ if $p = \infty$). Then $F^s_{pq}(\mathbb{R}^n)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$\|f \mid F^s_{pq}(\mathbb{R}^n)\| = \Big\| \Big( \sum_{j=0}^{\infty} 2^{jsq} |(\varphi_j \widehat{f})^\vee(\cdot)|^q \Big)^{1/q} \Big| L_p(\mathbb{R}^n) \Big\| \tag{2.13}$$

(with the usual modification if $q = \infty$) is finite. The theory of these spaces has been developed in [Tri83], [Tri92], and more recently in [ET96], [RuS96], [Tri97]. In particular they are quasi-Banach spaces which are independent of $\varphi$ (have equivalent quasi-norms with the somewhat sloppy omission of $\varphi$ on the left-hand side of (2.13)). We mention that

$$H^s_p(\mathbb{R}^n) = F^s_{p,2}(\mathbb{R}^n), \quad s \in \mathbb{R},\ 1 < p < \infty, \tag{2.14}$$

are the usual (fractional) Sobolev spaces with the (classical) Sobolev spaces

$$W^k_p(\mathbb{R}^n) = H^k_p(\mathbb{R}^n), \quad k \in \mathbb{N}_0,\ 1 < p < \infty, \tag{2.15}$$

as a subclass. Furthermore,

$$\mathcal{C}^s(\mathbb{R}^n) = F^s_{\infty\infty}(\mathbb{R}^n), \quad s > 0, \tag{2.16}$$

are the Hölder–Zygmund spaces. In what follows one might always think in terms of the special cases (2.14)–(2.16), accepting that there is a generalisation $F^s_{pq}(\mathbb{R}^n)$. In connection with manifolds the $F$-scale has always preference compared with the $B$-scale consisting of the spaces $B^s_{pq}(\mathbb{R}^n)$, where $s \in \mathbb{R}$, $0 < p \le \infty$, $0 < q \le \infty$. We refer to [Tri92], Ch. 7. We only mention that $B^s_{pp} = F^s_{pp}$.

**2.2.3.** Definition. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$. Let $s \in \mathbb{R}$, $0 < p \le \infty$, $0 < q \le \infty$ (with $q = \infty$ if $p = \infty$).

(i) We set

$$F^s_{pq}(\Omega) = \{f \in D'(\Omega) : \text{there is a } g \in F^s_{pq}(\mathbb{R}^n) \text{ with } g|\Omega = f\}, \tag{2.17}$$

$$\|f \mid F^s_{pq}(\Omega)\| = \inf \|g \mid F^s_{pq}(\mathbb{R}^n)\|, \tag{2.18}$$

where the infimum is taken over all $g \in F^s_{pq}(\mathbb{R}^n)$ such that the restriction $g|\Omega$ coincides in $D'(\Omega)$ with $f$. Furthermore, $\mathring{F}^s_{pq}(\Omega)$ is the completion of $D(\Omega)$ in $F^s_{pq}(\Omega)$.

(ii) We also set

$$\widetilde{F}^s_{pq}(\Omega) = \{f \in D'(\Omega) : \text{there is a } g \in F^s_{pq}(\mathbb{R}^n) \text{ with } \operatorname{supp} g \subset \overline{\Omega} \text{ and } f = g|\Omega\}, \tag{2.19}$$

$$\|f \mid \widetilde{F}^s_{pq}(\Omega)\| = \inf \|g \mid F^s_{pq}(\mathbb{R}^n)\| \tag{2.20}$$

where the infimum is taken over all $g$ admitted in (2.19). Furthermore

$$\widetilde{F}^s_{pq}(\overline{\Omega}) = \{f \in F^s_{pq}(\mathbb{R}^n) : \operatorname{supp} f \subset \overline{\Omega}\}. \tag{2.21}$$



**2.2.4.** *Properties.* Of course, $D(\Omega) = C_0^\infty(\Omega)$ stands for the collection of all complex $C^\infty$ functions with compact support in $\Omega$, whereas $D'(\Omega)$ is the dual space of complex distributions. $F^s_{pq}(\Omega)$ and $\widetilde{F}^s_{pq}(\Omega)$ are quasi-Banach spaces in $D'(\Omega)$, while $\widetilde{F}^s_{pq}(\overline{\Omega})$ is a closed subspace of $F^s_{pq}(\mathbb{R}^n)$. Under the additional assumption that $\Omega$ is a bounded $C^\infty$ domain we studied these spaces in [Tri98]. We list some of the properties proved there which are useful later on (always assuming that $\Omega$ is bounded and $C^\infty$). Recall $a_+ = \max(a, 0)$ if $a \in \mathbb{R}$.

(i) If $0 < p < \infty$, $-\infty < s < 1/p$, $0 < q < \infty$, then

$$F^s_{pq}(\Omega) = \mathring{F}^s_{pq}(\Omega). \tag{2.22}$$

(ii) If $0 < p \le \infty$, $0 < q \le \infty$ (with $q = \infty$ if $p = \infty$ ), and

$$\max(1/p - 1, n(1/p - 1)) < s < \infty, \tag{2.23}$$

then

$$\widetilde{F}^s_{pq}(\Omega) = \widetilde{F}^s_{pq}(\overline{\Omega}). \tag{2.24}$$

(iii) If $0 < p < \infty$, $0 < q < \infty$, and

$$s > n(1/p - 1)_+ \quad \text{with} \quad s - 1/p \notin \mathbb{N}_0, \tag{2.25}$$

then

$$\mathring{F}^s_{pq}(\Omega) = \widetilde{F}^s_{pq}(\Omega). \tag{2.26}$$

Proof, comments, and references may be found in [Tri98].

**2.2.5.** *Spaces on M.* Let $\Omega$ be a bounded connected domain in $\mathbb{R}^n$. Furnishing $\Omega$ with the Riemannian metric (2.1), (2.2), we obtain a non-compact hyperbolic manifold with bounded geometry and positive injectivity radius, denoted by $M$ or $(M, g)$. On manifolds of this type we introduced in [Tri92], Ch. 7, in an abstract way, spaces $F^s_{pq}(M)$ where $s \in \mathbb{R}$, $0 < p \le \infty$, $0 < q \le \infty$ (with $q = \infty$ if $p = \infty$). We do not repeat this definition here, but we assume in the sequel that this is done by using a resolution of unity related to the covering (2.5), where the balls $B_{jm}$ are approximately congruent with respect to the Riemannian metric. This resolution of unity will be described below. Details may also be found in Step 1 in 3.1. Under these circumstances it makes sense to speak about weights on $M$ and related weighted spaces.

**2.2.6.** Definition. Let $(M, g)$ be the above manifold and let $\varkappa \in \mathbb{R}$. Then

$$F^s_{pq}(M, g^\varkappa) = \{f \in D'(\Omega) : g^\varkappa f \in F^s_{pq}(M)\}, \tag{2.27}$$

$$\|f \mid F^s_{pq}(M, g^\varkappa)\| = \|g^\varkappa f \mid F^s_{pq}(M)\|. \tag{2.28}$$

**2.2.7.** *Resolution of unity.* Let $\Omega$ be covered according to (2.5). We may assume that there is a related resolution of unity,

$$\sum_{j=0}^{\infty}\sum_{m=1}^{M_j}\varphi_{jm}(x)=1 \quad \text{if } x\in\Omega, \tag{2.29}$$

of $C^\infty$ functions $\varphi_{jm}(x)$ with

$$\operatorname{supp}\varphi_{jm}\subset B_{jm} \tag{2.30}$$

and, for suitable $c_\gamma>0$,

$$|D^\gamma\varphi_{jm}(x)|\le c_\gamma 2^{j|\gamma|}, \quad \text{where } \gamma\in\mathbb{N}_0^n. \tag{2.31}$$

Recall the abbreviations

$$\sigma_p=\left(\frac{1}{p}-1\right)_+ \quad\text{and}\quad \sigma_{pq}=\left(\frac{1}{\min(p,q)}-1\right)_+ \tag{2.32}$$

where $0<p\le\infty$, $0<q\le\infty$.

**2.2.8.** THEOREM. *Let either*

$$1<p\le\infty,\quad 1<q\le\infty \quad (q=\infty \textit{ if } p=\infty),\quad s\in\mathbb{R}, \tag{2.33}$$

*or*

$$0<p\le\infty,\quad 0<q\le\infty \quad (q=\infty \textit{ if } p=\infty),\quad s>\sigma_{pq}. \tag{2.34}$$

*Let $\varkappa\in\mathbb{R}$ and let $F^s_{pq}(M,g^\varkappa)$ be the space introduced in 2.2.6. Then $f\in D'(\Omega)$ belongs to $F^s_{pq}(M,g^\varkappa)$ if, and only if,*

$$\Big(\sum_{j=0}^{\infty}\sum_{m=1}^{M_j}2^{j(\varkappa-s+n/p)p}\|\varphi_{jm}f\mid F^s_{pq}(\mathbb{R}^n)\|^p\Big)^{1/p}<\infty \tag{2.35}$$

*(with the usual modification if $p=q=\infty$). Furthermore, (2.35) is an equivalent quasi-norm.*

**2.2.9.** REMARK. The theorem gives a perfect link between the weighted spaces $F^s_{pq}(M,g^\varkappa)$ on the above hyperbolic manifold $M$ and related spaces on $\mathbb{R}^n$. For our later purposes (2.34) would be sufficient. But we incorporated (2.33) to make it clear that (2.35) holds in particular for the interesting weighted (fractional) Sobolev spaces

$$H^s_p(M,g^\varkappa)=F^s_{p,2}(M,g^\varkappa), \quad 1<p<\infty,\ s\in\mathbb{R}. \tag{2.36}$$

On the other hand, the restrictions (2.33), (2.34) allow a direct application of the pointwise multiplier assertions in [Tri98], 3.9. But there is hardly any doubt that the theorem holds for all $0<p\le\infty$, $0<q\le\infty$ ($q=\infty$ if $p=\infty$), and $s\in\mathbb{R}$. Furthermore, (2.35) indicates how the spaces $F^s_{pq}(M,g^\varkappa)$ and the corresponding spaces on $\Omega$ introduced in 2.2.3 and described in 2.2.4 are related.



**2.2.10.** THEOREM. *Let $\Omega$ be a bounded $C^\infty$ domain in $\mathbb{R}^n$ and let $(M,g)$, or, for short, $M$, be the related Riemannian manifold introduced in* 2.1.1.

(i) *Let*

$$0<p\le\infty,\quad 0<q\le\infty \quad (q=\infty \textit{ if } p=\infty),\quad s>\sigma_{pq}. \tag{2.37}$$

*Then*

$$F^s_{pq}(M,g^{s-n/p})=\widetilde{F}^s_{pq}(\Omega) \tag{2.38}$$

*(with equivalent quasi-norms).*

(ii) *Let*

$$0<p<\infty,\quad 0<q<\infty,\quad s>\sigma_{pq},\quad s-1/p\notin\mathbb{N}_0. \tag{2.39}$$

*Then*

$$F^s_{pq}(M,g^{s-n/p})=\mathring{F}^s_{pq}(\Omega). \tag{2.40}$$

(iii) *Let*

$$1<p\le\infty,\quad 1<q\le\infty \quad (q=\infty \textit{ if } p=\infty),\quad -\infty<s<1/p. \tag{2.41}$$

*Then*

$$F^s_{pq}(M,g^{s-n/p})=F^s_{pq}(\Omega). \tag{2.42}$$

**2.2.11.** REMARK. The above theorem holds under the additional assumption that the bounded domain in $\Omega$ is $C^\infty$. In particular $\Omega$ is an $(n-1)$-domain according to 2.1.3. This restriction comes from [Tri98] where we dealt exclusively with bounded $C^\infty$ domains. Combined with the two lifts described below, in this case any space $F^s_{pq}(M,g^\varkappa)$ can be reduced to spaces on $\Omega$ in the euclidean setting.

**2.2.12.** *Lifts.* Let $(M,g)$ be the above hyperbolic non-compact manifold with bounded geometry and positive injectivity radius. Let $-\Delta_g$ be the corresponding Laplace–Beltrami operator. let $0<p\le\infty$, $0<q\le\infty$ ($q=\infty$ if $p=\infty$), and $s\in\mathbb{R}$. Then for any sufficiently large $\varrho\in\mathbb{R}$,

$$-\Delta_g+\varrho\,\mathrm{id}:F^{s+2}_{pq}(M)\to F^s_{pq}(M) \tag{2.43}$$

is an isomorphic mapping (lift). This assertion is proved in [Tri92], Theorem 7.4.3, p. 316, based on [Tri86], [Tri87]. The Laplace–Beltrami operator $-\Delta_g$ and, more generally, pseudodifferential operators on Riemannian manifolds attracted a lot of attention: (essential) self-adjointness in $L_2(M)$, dependence on $p$ of the spectrum in $L_p(M)$ with $1<p<\infty$, mapping properties etc. We refer to [Tri88], [Dav89], Ch. 5, [Tay89], [Shu92], [Stu93], and most recently [Skr98*]. We need here an extension of (2.43) to $F^s_{pq}(M,g^\varkappa)$ and mapping properties of the pointwise multiplication operator $G_\mu$,

$$G_\mu: f\mapsto g^\mu f, \quad \mu\in\mathbb{R},\ f\in D'(\Omega). \tag{2.44}$$

**2.2.13.** THEOREM. *Let $0 < p \le \infty$, $0 < q \le \infty$ ($q = \infty$ if $p = \infty$), $s \in \mathbb{R}$ and $\varkappa \in \mathbb{R}$. Let $F^s_{pq}(M, g^\varkappa)$ be the spaces defined in* 2.2.6.

(i) *If $\varrho \in \mathbb{R}$ is sufficiently large, then*

$$-\Delta_g + \varrho\,\mathrm{id} : F^{s+2}_{pq}(M, g^\varkappa) \to F^s_{pq}(M, g^\varkappa) \tag{2.45}$$

*is an isomorphic mapping.*

(ii) *Let $\mu \in \mathbb{R}$. Then*

$$G_\mu : F^s_{pq}(M, g^\varkappa) \to F^s_{pq}(M, g^{\varkappa-\mu}) \tag{2.46}$$

*is an isomorphic mapping.*

**2.2.14.** REMARK. If $\varkappa = 0$, then (2.45) is covered by (2.43). However, the arguments in [Tri92] can be extended to all $\varkappa \in \mathbb{R}$. In other words: We take part (i) of the above theorem for granted. Part (ii) follows from Definition 2.2.6.

**2.2.15.** *Embeddings.* We recall basic facts on embeddings between $F^s_{pq}$ spaces on $\mathbb{R}^n$ and on bounded domains $\Omega$ in $\mathbb{R}^n$. Let

$$-\infty < s_2 < s_1 < \infty,\quad 0 < p_1 \le p_2 \le \infty,\quad 0 < q_1 \le \infty,\quad 0 < q_2 \le \infty \tag{2.47}$$

($q_1 = \infty$ if $p_1 = \infty$, and $q_2 = \infty$ if $p_2 = \infty$). Then

$$F^{s_1}_{p_1q_1}(\mathbb{R}^n) \subset F^{s_2}_{p_2q_2}(\mathbb{R}^n) \tag{2.48}$$

if, and only if, $\delta \ge 0$, where

$$\delta = \left(s_1 - \frac{n}{p_1}\right) - \left(s_2 - \frac{n}{p_2}\right). \tag{2.49}$$

The same assertion holds with $\Omega$ in place of $\mathbb{R}^n$. Since $\Omega$ is bounded we know in addition that

$$F^{s_1}_{p_1q_1}(\Omega) \subset F^{s_2}_{p_2q_2}(\Omega) \quad \text{is compact} \tag{2.50}$$

if, and only if, $\delta > 0$. We refer to [Tri83], 2.7.1, and [ET96], 3.3.1. Using Theorems 2.2.8 and 2.2.13(i) one has the following counterpart.

**2.2.16.** PROPOSITION. *Let $M$ be a manifold as in* 2.1.1 *and let the spaces $F^{s_1}_{p_1q_1}(M, g^{\varkappa_1})$ and $F^{s_2}_{p_2q_2}(M, g^{\varkappa_2})$ be defined by* 2.2.6 *under the condition* (2.47) *and $\varkappa_1 \in \mathbb{R}$, $\varkappa_2 \in \mathbb{R}$. Let $\delta$ be given by* (2.49). *Then*

$$F^{s_1}_{p_1q_1}(M, g^{\varkappa_1}) \subset F^{s_2}_{p_2q_2}(M, g^{\varkappa_2}) \tag{2.51}$$

(*continuous embedding*) *if, and only if, $\delta \ge 0$ and $\varkappa_1 \ge \varkappa_2$. Furthermore,* (2.51) *is compact if, and only if, $\delta > 0$ and $\varkappa_1 > \varkappa_2$.*

**2.2.17.** REMARK. This proposition is the direct counterpart of the corresponding assertions for weighted spaces on $\mathbb{R}^n$ as developed in [HaT94] and [ET96], 4.2.3, p. 160. As indicated in (1.3)–(1.6) it is one of the main aims of this paper to calculate the entropy numbers of these compact embeddings.



For this purpose we need first quarkonial (subatomic) representations of the related spaces.

## 2.3. Quarkonial decompositions

**2.2.1.** *Preliminaries.* As already mentioned, this paper is a continuation of [Tri98]. This applies in particular to quarkonial (or subatomic) decompositions. In [Tri98], 2.5, we gave a description of quarkonial decompositions of the spaces $F^s_{pq}(\mathbb{R}^n)$ and added a few comments and references about atomic and subatomic decompositions. This will not be repeated here. Whereas atoms in (euclidean) function spaces are quite fashionable nowadays, there are only a few papers dealing with atomic decompositions for function spaces on Riemannian manifolds. We refer in particular to [Skr96], [Skr97], [Skr98].

Let $\Omega$ be a bounded $C^\infty$ domain in $\mathbb{R}^n$ and let $\widetilde{F}^s_{pq}(\Omega)$ be the (euclidean) spaces introduced in (2.19), (2.20). If $p, q, s$ are restricted by (2.37) then we have (2.38), and (2.35) reduces to

$$\Big(\sum_{j=0}^{\infty}\sum_{m=1}^{M_j} \|\varphi_{jm} f \mid F^s_{pq}(\mathbb{R}^n)\|^p\Big)^{1/p} < \infty. \tag{2.52}$$

This coincides with [Tri98], Theorem 2.2.2. This assertion is the basis to prove the quarkonial decomposition for the spaces $\widetilde{F}^s_{pq}(\Omega)$ obtained in [Tri98], Theorem 2.5.7. In other words, if one accepts (2.35) under the restriction (2.34) then one can take over the arguments from [Tri98]. The only point is that one has to adapt the quarks to account for the additional factor $2^{j(\varkappa-s+n/p)}$ in (2.35). But this causes no trouble.

**2.3.2.** *Some notation.* We follow essentially [Tri98], 2.5.6. Let $\Omega$ be a bounded connected domain in $\mathbb{R}^n$. There are positive numbers $c_l$ ($l = 1, \ldots, 8$) and $c_\gamma$ ($\gamma \in \mathbb{N}^n_0$), (irregular) lattices

$$\{x^{j,m} : m = 1, \ldots, N_j\} \subset \Omega \quad \text{for } j \in \mathbb{N}_0, \tag{2.53}$$

and subordinate resolutions of unity $\{\psi_{jm} : m = 1, \ldots, N_j\}$ with the following properties:

(i) we have

$$c_1 \le N_j\, 2^{-jn} \le c_2 \quad \text{for } j \in \mathbb{N}_0; \tag{2.54}$$

$$|x^{j,m_1} - x^{j,m_2}| \ge c_3 2^{-j} \quad \text{for } j \in \mathbb{N}_0,\ m_1 \ne m_2; \tag{2.55}$$

$$\mathrm{dist}(K_{jm}, \partial\Omega) \ge c_4 2^{-j} \quad \text{for } j \in \mathbb{N}_0,\ m = 1, \ldots, N_j, \tag{2.56}$$

with

$$K_{jm} = \{y : |y - x^{j,m}| \le c_5 2^{-j}\} \subset \Omega. \tag{2.57}$$

(ii) $\psi_{jm}(x)$ are $C^\infty$ functions with

$$\operatorname{supp}\psi_{jm} \subset K_{jm} \quad \text{for } j \in \mathbb{N}_0,\ m = 1,\dots,N_j, \tag{2.58}$$

$$|D^\gamma \psi_{jm}(x)| \le c_\gamma 2^{j|\gamma|} \quad \text{for } \gamma \in \mathbb{N}_0^n,\ m = 1,\dots,N_j, \tag{2.59}$$

and

$$c_6 \le \sum_{m=1}^{N_j} \psi_{jm}(x) \le c_7 \quad \text{if } x \in \Omega \text{ with } \operatorname{dist}(x,\partial\Omega) \ge c_8 2^{-j} \tag{2.60}$$

for all $j \in \mathbb{N}_0$. For short, we call $\{\psi_{jm}\}$ a *family of approximative resolutions of unity in $\Omega$* if one finds lattices $\{x^{j,m}\}$ with all the properties (2.54)–(2.60). Let $\chi_{jm}(x)$ be the characteristic function of $K_{jm}$ in (2.57). Then $\chi^{(p)}_{jm}(x) = 2^{jn/p}\chi_{jm}(x)$ with $0 < p \le \infty$ is the $p$-normalised version. As usual in connection with atoms and quarks for $F^s_{pq}$ spaces we introduce the sequence space

$$f^\Omega_{pq} = \Big\{\lambda : \|\lambda \mid f^\Omega_{pq}\| = \Big\|\Big(\sum_{j=0}^{\infty}\sum_{m=1}^{N_j} |\lambda_{jm}\chi^{(p)}_{jm}(\cdot)|^q\Big)^{1/q} \Big| L_p(\Omega)\Big\| < \infty\Big\} \tag{2.61}$$

where

$$\lambda = \{\lambda_{jm} \in \mathbb{C} : j \in \mathbb{N}_0,\ m = 1,\dots,N_j\} \tag{2.62}$$

are the related sequences. We put

$$\lambda^\gamma = \{\lambda^\gamma_{jm} \in \mathbb{C} : j \in \mathbb{N}_0,\ m = 1,\dots,N_j\} \quad \text{for } \gamma \in \mathbb{N}_0^n. \tag{2.63}$$

**2.3.3.** *Normalised $\gamma$-quarks.* Let $s \in \mathbb{R}$, $0 < p \le \infty$, $\gamma \in \mathbb{N}_0^n$, and $\varkappa \in \mathbb{R}$. Let $\{\psi_{jm}\}$ be a family of approximative resolutions of unity in $\Omega$ as described above. Then

$$\begin{aligned}(\gamma qu)_{jm}(x) &= 2^{-j(s-n/p)+j|\gamma|}\, 2^{-k(\varkappa-s+n/p)}(x - x^{j,m})^\gamma \psi_{jm}(x)\\ &= 2^{-k\varkappa} 2^{-(j-k)(s-n/p)} 2^{j|\gamma|}(x - x^{j,m})^\gamma \psi_{jm}(x)\end{aligned} \tag{2.64}$$

are the normalised elementary building blocks we are looking for. Here

$$k = k(j,m) \quad \text{such that} \quad x^{j,m} \in \Omega_k \tag{2.65}$$

according to (2.3). In [Tri97], [Tri98], we called constructions of type (2.64) $\gamma$-quarks, which might explain the notation. Compared with [Tri98], (2.92), we now have the additional factor $2^{-k(\varkappa-s+n/p)}$ with (2.65) which comes from the related factor in (2.35). By [Tri97], Sect. 14, and [Tri98] we have

$$\|(\gamma qu)_{jm} \mid F^s_{pq}(M, g^\varkappa)\| \sim c(\gamma), \tag{2.66}$$

which means that the elementary building blocks in (2.64) are approximately normalised, where the related constants are independent of $j$ and $m$.

**2.3.4.** THEOREM. *Let $\Omega$ be a bounded connected domain in $\mathbb{R}^n$. Let $p, q, s$ satisfy* (2.34). *Let $\varkappa \in \mathbb{R}$. There are a number $\mu_0 > 0$ and a family $\{\psi_{jm}\}$ of approximative resolutions of unity in $\Omega$ as described above with the following property. Let $\mu > \mu_0$ and let $(\gamma qu)_{jm}(x)$ be given by* (2.64), (2.65). *Then $f \in D'(\Omega)$ belongs to $F^s_{pq}(M, g^\varkappa)$ if, and only if, it can be represented as*

$$f = \sum_{\gamma\in\mathbb{N}_0^n}\sum_{j=0}^{\infty}\sum_{m=1}^{N_j} \lambda^\gamma_{jm}(\gamma qu)_{jm}(x) \tag{2.67}$$

*with*

$$\sup_\gamma 2^{\mu|\gamma|}\|\lambda^\gamma \mid f^\Omega_{pq}\| < \infty. \tag{2.68}$$

*Furthermore, the infimum of the expression in* (2.68) *over all admissible representations* (2.67) *is an equivalent quasi-norm in $F^s_{pq}(M, g^\varkappa)$.*

**2.3.5.** REMARK. This is the counterpart of Theorem 2.5.7 in [Tri98]. In 2.5.8 of that paper we gave a discussion and related references. So we restrict ourselves here to a few comments to make it clear what is going on in the above theorem. If $\varkappa - s + n/p \ge 0$ then by (2.35) and $s > \sigma_{pq}$, and the related embeddings, it follows that

$$F^s_{pq}(M, g^\varkappa) \subset L_1(\Omega). \tag{2.69}$$

We may assume that $c_5$ in (2.57) is small. Hence,

$$2^{j|\gamma|}|(x - x^{j,m})^\gamma|\psi_{jm}(x) \le \psi_{jm}(x). \tag{2.70}$$

Then by (2.68) the expansion (2.67) converges absolutely in $L_1(\Omega)$. If $\varkappa$ is small then by (2.46) one has a corresponding assertion in the weighted space $L_1(\Omega, g^\varrho)$ for some $\varrho \in \mathbb{R}$. The number $\mu$ may be chosen large, but the equivalence constants in the above assertions depend in turn on $\mu$. A discussion of this effect has been given in [Tri97], 14.6, p. 97.

**2.3.6.** REMARK. Although it is clear by construction, we remark that there is a number $c > 0$ such that $k \le cj$ in (2.65). We may even assume $k \le j$ in (2.65) and (2.64).

**2.3.7.** REMARK. If $\varkappa = s - n/p$ then (2.35) reduces to (2.52). In this case we proved Theorem 2.3.4 in [Tri98] in a slightly different context. (If one starts from (2.52) then the assumption in [Tri98] that $\Omega$ is $C^\infty$ is not needed.) The different scaling in (2.35) is just the $k$-term in (2.64). In other words: Theorem 2.3.4 follows from Theorem 2.2.8 and the technique in [Tri98].

**2.3.8.** *Generalisation.* By the lifts in Theorem 2.2.13 one can start from Theorem 2.3.4 and derive quarkonial decompositions for all spaces $F^s_{pq}(M, g^\varkappa)$ defined in 2.2.5 and 2.2.6. We will not need this.

## 2.4. Entropy numbers

**2.4.1.** *Preliminaries.* Let $A, B$ be quasi-Banach spaces and let $T \in L(A,B)$. Put $U_A = \{a \in A : \|a\,|A\| \le 1\}$. Then for all $k \in \mathbb{N}$, the $k$th *entropy number* $e_k(T)$ of $T$ is defined by

$$(2.71)\quad e_k(T) = \inf\Big\{\varepsilon > 0 : T(U_A) \subset \bigcup_{j=1}^{2^{k-1}} (b_j + \varepsilon U_B) \text{ for some } b_1,\ldots,b_{2^{k-1}} \in B\Big\}.$$

Let $T \in L(A)$ be compact and let

$$(2.72)\qquad |\mu_1(T)| \ge |\mu_2(T)| \ge \ldots > 0$$

be the ordered sequence of its non-zero eigenvalues listed with algebraic multiplicity. Then

$$(2.73)\qquad |\mu_k(T)| \le \sqrt{2}\, e_k(T), \quad k \in \mathbb{N}$$

(Carl's inequality). We do not go into further details which may be found in [ET96], Ch. 1. There one can find proofs, references and (historical) comments. A short summary of what is needed may also be found in [Tri97], Sect. 6. We use (2.73) later on. First we are interested in entropy numbers of the compact embedding operator

$$(2.74)\qquad \mathrm{id} : F^{s_1}_{p_1q_1}(M, g^{\varkappa_1}) \to F^{s_2}_{p_2q_2}(M, g^{\varkappa_2})$$

of Proposition 2.2.16.

**2.4.2.** Theorem. *Let $n-1 \le d < n$. Let $\Omega$ be a connected bounded $d$-domain as in Definition* 2.1.3. *Let $(M,g)$ be the related non-compact hyperbolic manifold with the function spaces as given in Definition* 2.2.6. *Let $\varkappa_1 \in \mathbb{R}$, $\varkappa_2 \in \mathbb{R}$,*

$$(2.75)\qquad \begin{gathered} -\infty < s_2 < s_1 < \infty, \quad 0 < p_1 \le p_2 \le \infty, \\ 0 < q_1 \le \infty, \quad 0 < q_2 \le \infty \end{gathered}$$

*($q_1 = \infty$ if $p_1 = \infty$, and $q_2 = \infty$ if $p_2 = \infty$) with*

$$(2.76)\qquad \delta = \Big(s_1 - \frac{n}{p_1}\Big) - \Big(s_2 - \frac{n}{p_2}\Big) > 0, \quad \varkappa = \varkappa_1 - \varkappa_2 > 0.$$

*Then* id *in* (2.74) *is compact and for the related entropy numbers $e_k = e_k(\mathrm{id})$ we have*

$$(2.77)\qquad e_k \sim k^{-(s_1-s_2)/n}, \quad k \in \mathbb{N}, \ \text{if } \varkappa > \delta d/n,$$

$$(2.78)\qquad e_k \sim k^{-\varkappa/d + 1/p_2 - 1/p_1}, \quad k \in \mathbb{N}, \ \text{if } \varkappa < \delta d/n.$$

**2.4.3.** *Comment.* The proof given below is based on the quarkonial decomposition of Theorem 2.3.4 and the technique developed in [Tri97]. Based on these results, (2.73) paves the way to a related spectral theory described



in the next subsection. It is of interest to compare the above theorem with a corresponding theory with the replacement of

$$(2.79)\qquad F^s_{pq}(M, g^{\varkappa}) \quad \text{by} \quad F^s_{pq}(\mathbb{R}^n, w_{\varkappa}), \quad w_{\varkappa}(x) = (1+|x|^2)^{\varkappa/2}.$$

The results are very similar. As in the above behaviour of $e_k$ in (2.77), (2.78) there is also a splitting point which is then $\varkappa = \delta$. First results were obtained in [HaT94], which may also be found in [ET96], 4.3.2, pp. 170–171. The state of art in this euclidean setting may be found in [Har98] and [Har99] where special attention is paid to just these limiting cases. In the above hyperbolic setting $\varkappa = \delta d/n$ corresponds to this limiting case which we avoided so far. By comparison with the euclidean case it should also be possible to extend (2.75) to some cases with $p_2 < p_1$.

## 2.5. Spectral theory

**2.5.1.** *Preliminaries.* Let $\lambda > 0$, $0 \le \gamma \le 1$, and

$$(2.80)\qquad b(\cdot, D) \in \Psi^{-\lambda}_{1,\gamma}(\mathbb{R}^n)$$

be a pseudodifferential operator in $\mathbb{R}^n$ in the Hörmander class indicated. Let $1 \le r_1 \le \infty$, $1 \le r_2 \le \infty$, and

$$(2.81)\qquad b_1 \in L_{r_1}(\mathbb{R}^n, w_{\varkappa_1}), \quad b_2 \in L_{r_2}(\mathbb{R}^n, w_{\varkappa_2}), \quad \varkappa_1 \in \mathbb{R}, \ \varkappa_2 \in \mathbb{R},$$

$\varkappa = \varkappa_1 + \varkappa_2 > 0$. Based on [HaT94*] we developed in [ET96], 5.4, a spectral theory of the degenerate pseudodifferential operator

$$(2.82)\qquad B = b_2 b(\cdot, D) b_1$$

with the euclidean counterpart of Theorem 2.4.2 as starting point. More recent results in this direction may be found in [Har98]. In [Tri97], Ch. 5, we discussed a corresponding spectral theory for fractal pseudodifferential operators. Armed with Theorem 2.4.2 one can try to extend this theory from the euclidean case to the above hyperbolic manifolds. Pseudodifferential operators on manifolds with bounded geometry and positive injectivity radius, especially the Laplace–Beltrami operator, have been considered in [Dav89], [Tay89], [Shu92], [Stu93], [Skr98*], [Tri88], [Tri92], Ch. 7. Especially the mapping properties proved there can be taken as the starting point for a spectral theory of the hyperbolic counterpart of (2.82). This will not be done here. We restrict ourselves to a special case which is physically relevant in the euclidean setting.

**2.5.2.** *The set-up.* Let $(M,g)$ be the manifold as in 2.1.1. The related Laplace–Beltrami operator $-\Delta_g$ is self-adjoint in $L_2(M)$ with the Sobolev space $H^2(M) = F^2_{2,2}(M)$ as domain of definition. Furthermore,

$$(2.83)\qquad \mathrm{spec}(-\Delta_g + \varrho\,\mathrm{id}) \subset [1, \infty) \quad \text{for some } \varrho \in \mathbb{R}.$$

In addition we have the isomorphic mappings described in (2.43) and Theorem 2.2.13(i). The necessary references are given in 2.2.12 and 2.5.1. Let

$$H_\beta = -\Delta_g + \varrho\,\mathrm{id} - \beta g^{-\varkappa}, \quad \varkappa > 0,\ \beta > 0. \tag{2.84}$$

By Theorem 2.2.13,

$$B = (-\Delta_g + \varrho\,\mathrm{id})^{-1} \circ g^{-\varkappa} \tag{2.85}$$

is an isomorphic map from $L_2(M)$ onto

$$H^2(M, g^\varkappa) = F^2_{2,2}(M, g^\varkappa). \tag{2.86}$$

Furthermore, $B$, considered as an operator in $L_2(M)$, is compact. This follows from Theorem 2.4.2 in the case of $d$-domains. But this qualitative assertion also holds without this additional assumption. Hence, $H_\beta$ is a self-adjoint relatively compact perturbation of $-\Delta_g + \varrho\,\mathrm{id}$. Then the essential spectra of $H_\beta$ and $-\Delta_g + \varrho\,\mathrm{id}$ coincide ([EdE87], Theorem 2.1 on p. 418). We ask about the behaviour of the number of negative eigenvalues

$$N_\beta = \sharp\{\mathrm{spec}(H_\beta) \cap (-\infty, 0]\} \quad \text{as } \beta \to \infty. \tag{2.87}$$

At least in the euclidean setting problems of this type attracted a lot of attention. In quantum mechanics one considers the semi-classical limit $\hbar \to 0$ (the admittedly tiny Planck's *constant* $\hbar$ has the miraculous property of tending to zero). Physical reasoning (euclidean case) creates problems of type (2.84), (2.87) with $\beta \sim \hbar^{-2}$.

**2.5.3.** THEOREM. *Let $n-1 \le d < n$. Let $\Omega$, or likewise $(M,g)$, be a connected bounded $d$-domain as in Definition 2.1.3. Let $H_\beta$ and $N_\beta$ be given by (2.84) and (2.87). Then*

$$N_\beta \sim \beta^{d/\varkappa} \quad \text{if } 0 < \varkappa < 2d/n,\ \beta \to \infty, \tag{2.88}$$

$$N_\beta \sim \beta^{n/2} \quad \text{if } \varkappa > 2d/n,\ \beta \to \infty. \tag{2.89}$$

**2.5.4.** REMARK. Here "$\sim$" must be understood as explained in 2.1.1 with respect to $\beta$, $\beta \ge \beta_0$, where $\beta_0$ is a sufficiently large positive number. As already mentioned, the problem of the "negative spectrum" has been widely discussed in the euclidean setting. One finds background information and references in [ET96], Ch. 5. Related problems for degenerate elliptic problems have been treated in [HaT94*], [ET96], 5.4.7–5.4.9, pp. 236–242, and [Har98] (limiting cases). For fractal potentials we refer to [Tri97], Sect. 31. In any case, (2.89) is the expected "Weyl behaviour". Our interest in the above theorem is twofold. First, we wished to find out what is the appropriate problem for hyperbolic manifolds. Secondly, what is the influence of the invisible fractal boundary of this hyperbolic world?

**2.5.5.** *Hydrogen-like atoms in $(M,g)$.* It is well known that the euclidean version of (2.84) in $\mathbb{R}^n$ has physical relevance, at least if $n=3$. With $c|x|^{-1}$ in place of $g^{-\varkappa}(x)$ it describes the hydrogen atom. From this point of view it is desirable to replace the smooth "potential" $g^{-\varkappa}(x)$ by potentials admitting local singularities. This can be done by extending the "euclidean" technique developed in [ET96], Ch. 5, to the above hyperbolic case. Then one needs the full power of Theorem 2.4.2. This will not be done here. But we look at the special case of hydrogen-like atoms in the hyperbolic world of $d$-domains. We fix an off-point in $\Omega$, say $0 \in \Omega$, where the nucleus is located. Assume that locally the Coulomb potential is proportional to $|x|_g^{-1}$, where $|x|_g$ is the Riemannian distance of $x \in \Omega$ to 0. As will be proved in 3.7 there are two continuous functions $m(x)$ and $M(x)$ in $\Omega$, bounded from above and from below by positive constants,

$$m(x) \sim 1 \quad \text{and} \quad M(x) \sim 1, \quad x \in \Omega, \tag{2.90}$$

such that

$$g(x) = M(x)\, 2^{m(x)|x|_g}, \quad |x|_g \ge 1, \tag{2.91}$$

or

$$|x|_g \sim \log g(x), \quad |x|_g \ge 1. \tag{2.92}$$

Hence, measured against $|x|_g$ the potential $g^{-\varkappa}(x)$ in (2.84) is of exponential decay (as it should be in hyperbolic worlds). At least from the mathematical point of view and in analogy to the euclidean case the Coulomb potential for hydrogen-like atoms in the hyperbolic world should behave off the origin like $g^{-\varkappa}(x)$ for some $\varkappa > 0$ or like $g^{-\varkappa}(x)(\log g(x))^\sigma$. Whether there is any physical reasoning to specify $\varkappa$ and $\sigma$ is not clear. We take the simplest case $\varkappa = 1$ and $\sigma = 0$. Then we modify $H_\beta$ in (2.84) by

$$H^\beta = -\Delta_g + \varrho\,\mathrm{id} - \beta(g \min(1, |x|_g))^{-1}. \tag{2.93}$$

At least if $n \ge 3$ then the counterpart of $B$ in (2.85) is compact in $L_2(M)$. Hence

$$N^\beta = \sharp\{\mathrm{spec}(H^\beta) \cap (-\infty, 0]\}, \quad \beta > 0, \tag{2.94}$$

is finite.

**2.5.6.** COROLLARY. *Let $2 \le n-1 \le d < n$. Let $\Omega$, or $(M,g)$, be a connected bounded $d$-domain as in Definition 2.1.3. Let $H^\beta$ and $N^\beta$ be given by (2.93) and (2.94). Then*

$$N^\beta \sim \beta^d, \quad \beta \to \infty. \tag{2.95}$$

**2.5.7.** REMARK. As already mentioned, $H^\beta$ in (2.93) should be considered as an example. The technique indicated allows one to incorporate local perturbations belonging to $L_p^{\mathrm{loc}}(M)$ for some $p \ge 1$. More interesting are global perturbations of type $L_p(M, g^\sigma)$ or a replacement of $g^{-\varkappa}$ in (2.84) by $g^{-\varkappa}(\log g)^\sigma$. For this purpose one needs hyperbolic counterparts of the euclidean techniques in [HaT94*], [ET96], Ch. 5, and [Har98].

## 3. PROOFS

### 3.1. Proof of Theorem 2.2.8

STEP 1. Let $(M,g)$ be the manifold introduced in 2.1.1. According to [Tri88], 2.3.1, it is a complete connected Riemannian manifold with bounded geometry and positive injectivity radius. For the Riemannian background we refer to [Tri92], 7.2.1, pp. 281–285, but this is not really needed here since we reduce everything to the euclidean case. Let

$$s \in \mathbb{R}, \quad 0 < p \le \infty, \quad 0 < q \le \infty \quad (\text{with } q = \infty \text{ if } p = \infty), \tag{3.1}$$

and let $\{\varphi_{jm}\}$ be the resolution of unity given by (2.29)–(2.31). Let temporarily $x^{j,m}$ be the centres of the balls $B_{jm}$ in (2.30). Then $F^s_{pq}(M)$ is the collection of all $f \in D'(\Omega)$ such that

$$\Big(\sum_{j=0}^{\infty}\sum_{m=1}^{M_j} \|(\varphi_{jm}f)(2^{-j}\cdot + x^{j,m}) \mid F^s_{pq}(\mathbb{R}^n)\|^p\Big)^{1/p} \tag{3.2}$$

is finite. By [Tri88], 2.4.4, this definition is independent of the chosen resolution of unity (up to equivalent quasi-norms) and can be taken as an equivalent quasi-norm in $F^s_{pq}(M)$, originally introduced in [Tri92], 7.2.2, pp. 285–286, via geodesic coordinates.

STEP 2. Let $p,q$, and $s$ be restricted by (2.34). Let $B_\lambda$ be the ball of radius $\lambda > 0$ centred at the origin in $\mathbb{R}^n$. Let

$$h \in F^s_{pq}(\mathbb{R}^n) \quad \text{with } \operatorname{supp} h \subset B_\lambda \text{ and } 0 < \lambda \le 1. \tag{3.3}$$

Then

$$\|h(\lambda\cdot) \mid F^s_{pq}(\mathbb{R}^n)\| \sim \lambda^{s-n/p}\|h \mid F^s_{pq}(\mathbb{R}^n)\|, \tag{3.4}$$

where "$\sim$" means independence of $h$ with (3.3) *and of* $\lambda$. We refer to [Tri98], 3.9, formula (3.80). We apply this equivalence to (3.2) with $\lambda = c\,2^{-j}$ and obtain

$$\|f \mid F^s_{pq}(M)\| \sim \Big(\sum_{j=0}^{\infty}\sum_{m=1}^{M_j} 2^{-j(s-n/p)p}\|\varphi_{jm}f \mid F^s_{pq}(\mathbb{R}^n)\|^p\Big)^{1/p}. \tag{3.5}$$

This proves (2.35) with $\varkappa = 0$. Let $p,q$, and $s$ be again restricted by (2.34) and let

$$\eta \in C^\infty(\overline{B}_\lambda), \quad |D^\gamma\eta(x)| \le c_\gamma\lambda^{-|\gamma|}, \quad \gamma \in \mathbb{N}^n_0. \tag{3.6}$$

Then for $h$ given by (3.3),

$$\|\eta h \mid F^s_{pq}(\mathbb{R}^n)\| \le c\|h \mid F^s_{pq}(\mathbb{R}^n)\|, \tag{3.7}$$

where $c$ is independent of $h$ *and* $\lambda$ (but depends on finitely many $c_\gamma$). We refer to [Tri98], 3.9, formulas (3.81), (3.82). If $f \in F^s_{pq}(M,g^\varkappa)$ then we have (3.5) with $g^\varkappa f$ in place of $f$. We apply the translated inequality (3.7) to $h = \varphi_{jm}f$ and $\eta = 2^{-j\varkappa}g^\varkappa$ or $\eta = 2^{j\varkappa}g^{-\varkappa}$ and obtain

$$\|g^\varkappa\varphi_{jm}f \mid F^s_{pq}(\mathbb{R}^n)\| \sim 2^{j\varkappa}\|\varphi_{jm}f \mid F^s_{pq}(\mathbb{R}^n)\|, \tag{3.8}$$

where again $\sim$ is independent of $j$ and $m$. Here the needed estimates in (3.6) are covered by (2.1). This completes the proof of (2.35) under the assumption (2.34). If $p,q$ and $s$ are restricted by (2.33) then one can use the same duality arguments as in [Tri98], 3.9(iii), or directly the homogeneity assertions and pointwise multiplier properties proved there.



**3.2. Proof of Theorem 2.2.10.** By (2.35) we have

$$\|f \mid F^s_{pq}(M,g^{s-n/p})\| \sim \Big(\sum_{j=0}^{\infty}\sum_{m=1}^{M_j}\|\varphi_{jm}f \mid F^s_{pq}(\mathbb{R}^n)\|^p\Big)^{1/p}. \tag{3.9}$$

Since $\Omega$ is bounded and $C^\infty$ we obtain part (i) from [Tri98], Theorem 2.2.2. Now part (ii) is a special case of [Tri98], Theorem 2.4.2. Finally, part (iii) is a consequence of [Tri98], Theorem 2.3.3 and (3.9).

### 3.3. Proof of Proposition 2.2.16

STEP 1. Let $\delta \ge 0$ and $\varkappa_1 \ge \varkappa_2$. We wish to prove (2.51). We may assume in addition that both $p_1,q_1,s_1$ and $p_2,q_2,s_2$ satisfy (2.34): the general case can be reduced to this special case by iterative use of the lifts in (2.45). Then it follows by (2.48) and (3.4) that

$$2^{-j(s_2-n/p_2)}\|\varphi_{jm}f \mid F^{s_2}_{p_2q_2}(\mathbb{R}^n)\| \le c2^{-j(s_1-n/p_1)}\|\varphi_{jm}f \mid F^{s_1}_{p_1q_1}(\mathbb{R}^n)\| \tag{3.10}$$

where $c$ is independent of $j,m$, and $f$. Then (2.51) is a consequence of $\varkappa_1 \ge \varkappa_2$ and (2.35). If, in addition, $\delta > 0$, then we have (2.50). If $\varkappa_1 > \varkappa_2$ we see by the same reduction that the embedding (2.51) is even compact.

STEP 2. Assume that the embedding (2.51) is continuous, resp. compact. Then, by 2.2.15, $\delta \ge 0$, resp. $\delta > 0$. Again by (3.4) and the above assumption that $p_1,q_1,s_1$ and $p_2,q_2,s_2$ satisfy (2.34) we have

$$2^{-j(s_2-n/p_2)}\|\varphi_{jm} \mid F^{s_2}_{p_2q_2}(\mathbb{R}^n)\| \sim 2^{-j(s_1-n/p_1)}\|\varphi_{jm} \mid F^{s_1}_{p_1q_1}(\mathbb{R}^n)\| \tag{3.11}$$

where we may assume that $\sim$ is independent of $j$ and $m$. But then it follows that for continuity, resp. compactness, $\varkappa_1 \ge \varkappa_2$, resp. $\varkappa_1 > \varkappa_2$, is necessary.

**3.4. Proof of Theorem 2.3.4.** As indicated in 2.3.7, for $\varkappa = s - n/p$ (2.35) reduces to (2.52). Then we can argue as in [Tri98], 3.13, which proves the theorem in this special case. However, in general we have simply to take the factors $2^{k(\varkappa-s+n/p)}$ in (2.35) into account. This has been done in (2.64). The rest is unchanged compared with [Tri98], Theorem 2.5.7, and its proof in [Tri98], 3.13.

### 3.5. Proof of Theorem 2.4.2

STEP 1. We outline the proof of

$$e_k \le ck^{-(s_1-s_2)/n}, \qquad k \in \mathbb{N}, \text{ if } \varkappa > \delta d/n, \tag{3.12}$$

$$e_k \le ck^{-\varkappa/d+1/p_2-1/p_1}, \qquad k \in \mathbb{N}, \text{ if } \varkappa < \delta d/n, \tag{3.13}$$

for some $c > 0$. By Theorem 2.2.13 we may assume that Theorem 2.3.4 can be applied to the spaces involved. Then we have the quarkonial decomposition (2.67), (2.68) with the $\gamma$-quarks $(\gamma qu)_{jm}(x)$ given by (2.64). This brings us in the same position as in [Tri97], Proposition 20.5, pp. 162–165. First we remark that by the arguments given there the values of $q_1$ and $q_2$ in (2.75) are unimportant (we avoid limiting cases). Hence we may assume $q_1 = p_1$ and $q_2 = p_2$. Then we are in the $B$-scale and we can apply directly the arguments in [Tri97]. The crucial point of the proof in [Tri97], pp. 163–165, is the reduction

$$e_k \le ce_k(\mathrm{id}), \qquad k \in \mathbb{N}, \tag{3.14}$$

where in our case

$$\mathrm{id} : \ell_\infty[2^{\varrho_1|\gamma|}\ell_{p_1}(2^{r\varkappa}\ell_{p_1}^{L_r})] \to \ell_\infty[2^{\varrho_2|\gamma|}\ell_{p_2}(\ell_{p_2}^{L_r})] \tag{3.15}$$

is a compact embedding between the sequence spaces indicated. These spaces have been considered in Sect. 9 of [Tri97], including equivalence relations for the related entropy numbers $e_k(\mathrm{id})$. For details we refer to [Tri97], Sects. 8 and 9. But we have to explain what is meant in our context by the ingredients in (3.15). The numbers $\varrho_1$ and $\varrho_2$ originate from $\mu$ in (2.68) which can be chosen arbitrarily large. As in [Tri97] we may assume

$$\varrho_1 > \varrho_2 > 0 \quad \text{large}, \tag{3.16}$$

otherwise these numbers are immaterial. Furthermore, $\ell_{p_1}^{L_r}$ stands for an $L_r$-dimensional $\ell_{p_1}$-space, interpreted as a block in the matrix space $\ell_{p_1}(2^{r\varkappa}\ell_{p_1}^{L_r})$ with $r \in \mathbb{N}_0$. Here $\varkappa > 0$ has the same meaning as in (2.76). Crucial for the estimate of $e_k(\mathrm{id})$ in (3.14) is the knowledge of $L_r$. By the technique developed in [Tri97] we have to compare the normalising factors of the $\gamma$-quarks (2.64) with respect to the two spaces (2.74) involved. The quotient of these two factors is of interest. Then both $\delta$ and $\varkappa$ from (2.76) are coming in. To fix ideas we may assume by Theorem 2.2.13 that $\varkappa_2 = 0$ and hence $\varkappa = \varkappa_1 > 0$. If $r \in \mathbb{N}_0$ then we have to estimate the number $L_r$ of balls $K_{jm}$ given by (2.57) with

$$\mathrm{dist}(K_{jm}, \partial\Omega) \sim 2^{-l} \quad \text{for } l = 0, \dots, j, \tag{3.17}$$

and

$$r\varkappa \sim l\varkappa + (j-l)\delta. \tag{3.18}$$



Here we used $l \le j$ according to Remark 2.3.6 without restriction of generality. Hence $L_r$ is the number of those balls $K_{jm}$ where the quotient of the normalising factors of the related $\gamma$-quarks (2.64) is $\sim 2^{r\varkappa}$. For fixed $l$ with $l = 0, \dots, r$ we have to estimate the number $L_r^l$ of balls $K_{jm}$ of radius $\sim 2^{-j}$ which, say, intersect $\Omega_l$ given by (2.3) and satisfying (3.18). Hence

$$L_r^l \sim (\mathrm{vol}\,\Omega_l)2^{jn} \sim 2^{-l(n-d)}2^{jn} \sim 2^{ld}2^{(r-l)n\varkappa/\delta}. \tag{3.19}$$

We used (2.6). Summation over $r$ yields

$$L_r \sim 2^{r\varkappa n/\delta}\sum_{l=0}^{r} 2^{l(d-n\varkappa/\delta)}. \tag{3.20}$$

We obtain

$$L_r \sim 2^{rd} \qquad \text{if } d > n\varkappa/\delta, \tag{3.21}$$

$$L_r \sim 2^{rn\varkappa/\delta} \qquad \text{if } d < n\varkappa/\delta. \tag{3.22}$$

Now the entropy numbers of the compact embedding id in (3.15) follow from [Tri97], Theorem 9.2, p. 47,

$$e_k(\mathrm{id}) \sim k^{-\varkappa/d+1/p_2-1/p_1}, \qquad k \in \mathbb{N}, \text{ if } \varkappa < \delta d/n, \tag{3.23}$$

and

$$e_k(\mathrm{id}) \sim k^{-\delta/n+1/p_2-1/p_1} = k^{-(s_1-s_2)/n}, \qquad k \in \mathbb{N}, \text{ if } \varkappa > \delta d/n. \tag{3.24}$$

Together with (3.14) we obtain (3.12) and (3.13).

STEP 2. The main aim of Step 1 was to find out how the arguments given in [Tri97] must be changed if the $\gamma$-quarks in (2.64) have now the additional factor $2^{-k(\varkappa-s+n/p)}$. This is the point where both the Riemannian metric and the weights $g^\varkappa$ are coming in. We obtain (3.21) and (3.22). On this basis and (3.23), (3.24) one can now prove inequalities inverse to (3.12) and (3.13) in the same way as in [Tri97], pp. 167–168.

### 3.6. Proof of Theorem 2.5.3

STEP 1. As mentioned in 2.5.2, with $H^2(M)$ as domain of definition, $H_\beta$ in (2.84) is self-adjoint in $L_2(M)$. As remarked there, $B$ in (2.85) is compact and hence the essential spectra of $H_\beta$ and $-\Delta_g + \varrho\,\mathrm{id}$ coincide. We assume (2.83). To estimate $N_\beta$ in (2.87) (under the conditions of Theorem 2.5.3) from above we use the entropy version of the Birman–Schwinger principle as described in [Tri97], 31.1, p. 243. We obtain

$$N_\beta \le \sharp\{k \in \mathbb{N} : \sqrt{2}\,\beta e_k(B) \ge 1\}. \tag{3.25}$$

Here $e_k(B)$ are the entropy numbers of the compact operator $B$. By the above explanations we have

$$e_k(B) \sim e_k(\mathrm{id}), \qquad k \in \mathbb{N}, \tag{3.26}$$

with

$$\mathrm{id} : H^2(M, g^{\varkappa}) \to L_2(M). \tag{3.27}$$

By Theorem 2.4.2 with $s_1 = p_1 = p_2 = 2$, $s_2 = \varkappa_2 = 0$ and $\varkappa = \varkappa_1$ we obtain

$$e_k(\mathrm{id}) \sim k^{-\varkappa/d}, \quad k \in \mathbb{N}, \text{ if } 0 < \varkappa < 2d/n, \tag{3.28}$$

$$e_k(\mathrm{id}) \sim k^{-2/n}, \quad k \in \mathbb{N}, \text{ if } \varkappa > 2d/n. \tag{3.29}$$

We insert (3.28), resp. (3.29), via (3.26) in (3.25) to obtain

$$N_\beta \le c\beta^{d/\varkappa} \quad \text{if } 0 < \varkappa < 2d/n, \tag{3.30}$$

$$N_\beta \le c\beta^{n/2} \quad \text{if } \varkappa > 2d/n. \tag{3.31}$$

STEP 2. Recall that $H_\beta$ is self-adjoint in $L_2(M)$. This is needed now to prove the inverse inequalities to (3.30) and (3.31). We modify the scheme of [Tri97], p. 250, and shift the problem to quadratic forms and the Max-Min principle. By (2.84) the related quadratic form is given by

$$\begin{aligned}(H_\beta f, f)_{L_2(M)} &\sim \|f \mid H^1(M)\|^2 - \beta\|g^{-\varkappa/2} f \mid L_2(M)\|^2 \\ &= \|f \mid H^1(M)\|^2 - \beta\|f \mid L_2(M, g^{-\varkappa/2})\|^2.\end{aligned} \tag{3.32}$$

Let $r \in \mathbb{N}$. We apply the arguments in 3.5, Step 1, to

$$H^1(M) = F^1_{2,2}(M) \quad \text{and} \quad L_2(M, g^{-\varkappa/2}) = F^0_{2,2}(M, g^{-\varkappa/2}). \tag{3.33}$$

In particular we have $\delta = 1$ and $\varkappa/2$ in place of $\varkappa$ in (3.18) and (3.21), (3.22). Then we find an orthonormal system $\{\varphi_{jm}\}$ in $H^1(M)$, related to the balls $K_{jm}$ in (3.17), consisting of $L_r$ elements with pairwise disjoint supports such that

$$\|\varphi_{jm} \mid L_2(M, g^{-\varkappa/2})\| \sim 2^{-r\varkappa/2}. \tag{3.34}$$

We choose $\beta = c\,2^{r\varkappa}$ with $c > 0$ small (but independent of $r \in \mathbb{N}$) and insert finite linear combinations of these functions $\varphi_{jm}$ in (3.32). We find that for these functions the quadratic form is always negative. Hence $(H_\beta f, f)_{L_2(M)}$ with $\beta = c2^{r\varkappa}$ is negative on a subspace of $H^1(M)$ of dimension $L_r$. By the Max-Min principle (see [EdE87], pp. 489–492), it follows that

$$N_\beta \ge L_r, \quad \text{where} \quad \beta = c\,2^{r\varkappa}. \tag{3.35}$$

With $\delta = 1$ and $\varkappa/2$ in place of $\varkappa$ it follows by (3.21) and (3.22) that

$$N_\beta \ge c_1 2^{rd} = c_2\beta^{d/\varkappa} \quad \text{if } 0 < \varkappa < 2d/n, \tag{3.36}$$

$$N_\beta \ge c_1 2^{r\varkappa n/2} = c_2\beta^{n/2} \quad \text{if } \varkappa > 2d/n, \tag{3.37}$$

where $c_1$ and $c_2$ are positive numbers. Together with (3.30), (3.31) we obtain (2.88), (2.89).



**3.7. Proof of (2.91).** By (2.2) and (2.3) the Riemannian width of each slice $\Omega_j$ is approximately 1. If one starts from an outer point $x \in \Omega$ with, say, $g(x) \sim 2^j$, then one needs approximately $j$ steps of Riemannian length 1 to reach a given inner point, say, $0 \in \Omega$. Hence

$$g(x) \sim 2^j \sim 2^{m(x)|x|_g} \tag{3.38}$$

with a suitable function $m(x)$ bounded from above and from below by positive constants. This proves (2.91) with (2.90).

**3.9. Proof of Corollary 2.5.6.** Since $n \ge 3$ we always have

$$n < 2(n-1) \le 2d. \tag{3.39}$$

Hence with $\varkappa = 1$ we have

$$N_\beta \sim \beta^d, \quad \beta \to \infty, \tag{3.40}$$

in Theorem 2.5.3. It remains to check that the local perturbation in $H^\beta$ in (2.93) does not influence this assertion. But this is essentially a euclidean matter. It is covered by [ET96], Theorem 5.4.8, p. 239, and its proof.

**References**


[Ber98] G. Berger, *Eigenvalue distribution of elliptic operators of second order with Neumann boundary conditions in a snowflake domain*, preprint, Leipzig, 1998.

[Dav89] E. B. Davies, *Heat Kernels and Spectral Theory*, Cambridge Univ. Press, 1989.

[EdE87] D. E. Edmunds and W. D. Evans, *Spectral Theory and Differential Operators*, Oxford Univ. Press, 1987.

[ET96] D. E. Edmunds and H. Triebel, *Function Spaces, Entropy Numbers, Differential Operators*, Cambridge Univ. Press, 1996.

[EvH93] W. D. Evans and D. J. Harris, *Fractals, trees and the Neumann Laplacian*, Math. Ann. 296 (1993), 493–527.

[Fal85] K. J. Falconer, *The Geometry of Fractal Sets*, Cambridge Univ. Press, 1985.

[Fal90] —, *Fractal Geometry*, Wiley, Chichester, 1990.

[Har98] D. Haroske, *Some logarithmic function spaces, entropy numbers, applications to spectral theory*, Dissertationes Math. 373 (1998).

[Har99] —, *Embeddings of some weighted function spaces on $\mathbb{R}^n$; entropy and approximation numbers*, preprint, Jena, 1998.

[HaT94] D. Haroske and H. Triebel, *Entropy numbers in weighted function spaces and eigenvalue distributions of some degenerate pseudodifferential operators I*, Math. Nachr. 167 (1994), 131–156.

[HaT94*] —, —, *Entropy numbers in weighted function spaces and eigenvalue distributions of some degenerate pseudodifferential operators II*, ibid. 168 (1994), 109–137.

[HeL97] C. Q. He and M. L. Lapidus, *Generalized Minkowski content, spectrum of fractal drums, fractal strings and the Riemann zeta-function*, Mem. Amer. Math. Soc. 608 (1997).

[Lap91] M. L. Lapidus, *Fractal drums, inverse spectral problems for elliptic operators and a partial resolution of the Weyl-Berry conjecture*, Trans. Amer. Math. Soc. 325 (1991), 465–529.

[Mat95] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces*, Cambridge Univ. Press, 1995.

[RuS96] T. Runst and W. Sickel, *Sobolev Spaces of Fractional Order, Nemytzkij Operators, and Nonlinear Partial Differential Equations*, de Gruyter, Berlin, 1996.

[Shu92] M. A. Shubin, *Spectral theory of elliptic operators on non-compact manifolds*, Astérisque 207 (1992), 35–108.

[Skr96] L. Skrzypczak, *Heat semi-group and function spaces on symmetric spaces of non-compact type*, Z. Anal. Anwendungen 15 (1996), 881–899.

[Skr97] —, *Besov spaces on symmetric manifolds—the atomic decomposition*, Studia Math. 124 (1997), 215–238.

[Skr98] —, *Atomic decompositions on manifolds with bounded geometry*, Forum Math. 10 (1998), 19–38.

[Skr98*] —, *Mapping properties of pseudodifferential operators on manifolds with bounded geometry*, J. London Math. Soc., to appear.

[Stu93] K.-T. Sturm, *On the $L^p$-spectrum of uniformly elliptc operators on Riemannian manifolds*, J. Funct. Anal. 118 (1993), 442–453.

[Tay89] M. E. Taylor, *$L^p$-estimates on functions of the Laplace operator*, Duke Math. J. 58 (1989), 773–793.

[Tri83] H. Triebel, *Theory of Function Spaces*, Birkhäuser, Basel, 1983.

[Tri86] —, *Spaces of Besov-Hardy-Sobolev type on complete Riemannian manfolds*, Ark. Mat. 24 (1986), 299–337.

[Tri87] —, *Characterizations of function spaces on a complete Riemannian manifold with bounded geometry*, Math. Nachr. 130 (1987), 312–346.

[Tri88] —, *On a class of weighted function spaces and related pseudodifferential operators*, Studia Math. 90 (1988), 37–68.

[Tri92] —, *Theory of Function Spaces II*, Birkhäuser, Basel, 1992.

[Tri97] —, *Fractals and Spectra*, Birkhäuser, Basel, 1997.

[Tri98] —, *Decompositions of function spaces*, in: Progress in Nonlinear Differential Equations Appl. 35, Birkhäuser, 1999, 691–730.

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[7] R. Hill and A. James, *An index formula*, J. Differential Equations 15 (1982), 197–211.

[8] J. Kowalski, *Some remarks on $J(X)$*, in: Algebra and Analysis (Edmonton, 1973), E. Brook (ed.), Lecture Notes in Math. 867, Springer, Berlin, 1974, 115–124.

[Nov] A. S. Novikov, *An existence theorem for planar graphs*, preprint, Moscow University, 1980 (in Russian).

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