APPLICATIONES MATHEMATICAE
26,4 (1999), pp. 383–394

A. DRWALEWSKA and L. GAJEK (Łódź)

ON LOCALIZING GLOBAL PARETO SOLUTIONS IN A GIVEN CONVEX SET

Abstract. Sufficient conditions are given for the global Pareto solution of the multicriterial optimization problem to be in a given convex subset of the domain. In the case of maximizing real valued-functions, the conditions are sufficient and necessary without any convexity type assumptions imposed on the function. In the case of linearly scalarized vector-valued functions the conditions are sufficient and necessary provided that both the function is concave and the scalarization is increasing with respect to the cone generating the preference relation.

1. Introduction. The aim of this paper is to investigate a sufficient and necessary condition for a vector-valued function g to attain its weak Pareto maximum in a given convex subset V of a vector space X. Seemingly, the condition is somehow similar to that of Pshenichnyĭ (see [6]) who has given necessary and sufficient conditions for a continuous and concave real function to attain its maximum at a given point $x \in X$ in terms of the subgradient and of the cone of feasible directions. Pshenichnyĭ's result was extended by Swartz (see [9]) to the case of Pareto maxima for vector-valued functions.

Comparing with the Pshenichnyĭ and Swartz results, we do not assume that g is continuous. Moreover, we give a sufficient and necessary condition for the weak Pareto maximum of g to be attained on a given set $V \subseteq X$ not necessarily at a given point of X. Even when V consists of one point only, our condition differs from those of Pshenichnyĭ and Swartz. This can be easily seen in the case of real-valued functions where our condition is sufficient and necessary without imposing *any* convexity type conditions on g, contrary to the result of Pshenichnyĭ, which does not hold without such assumptions.

1991 *Mathematics Subject Classification*: 49K27, 49K30, 49J52, 49M49.
Key words and phrases: Pareto optimal solutions, feasible directions, dual cones, sufficient and necessary conditions for optimality.

We first fix the notation. Let X and Z be real, locally convex topological vector spaces, and let K be a closed convex cone in Z. Let K^0 denote the interior of K. We assume throughout the paper that $K^0 \neq \emptyset$. Let Ω be a subset of X, and $f : \Omega \to \mathbb{R}$ be a given function.

Fix a nonempty convex set $V \subset X$ and a point $x \notin V$. Let $\partial f(V, x) \subset X'$ (the dual space to X) be the set of all $x' \in X'$ such that

$$\exists v \in V,\ t \in (0,1] : \quad f(x_t) - f(x) \geq \langle x', x_t - x\rangle, \tag{1}$$

where $x_t = tv + (1-t)x$, i.e. $x_t - x = t(v - x)$. Clearly, $\partial f(V, x) \supseteq \partial f(W, x)$ for any $V \supseteq W$ and $x \in X$.

Recall that the *subdifferential* $\partial f(x)$ of f at x is defined as the set of all $x' \in X'$ satisfying

$$f(y) - f(x) \geq \langle x', y - x\rangle \quad \text{for all } y \in \Omega.$$

Observe that $\partial f(V, x) \supseteq \partial f(x)$ for any $\emptyset \neq V \subseteq \Omega$, $x \in \Omega\backslash V$ and $f : \Omega \to \mathbb{R}$.

For $x \notin V$, denote by $G(V, x)$ the set of all directions leading to V from x, i.e.

$$G(V, x) = \{u \in X : x + tu \in V \text{ for some } t > 0\}.$$

Clearly, $G(V, x) = \{t(v - x) : v \in V \text{ and } t > 0\}$ and because V is convex, $G(V, x)$ is a convex cone as well.

Let $G(V, x)^*$ denote the dual cone of $G(V, x)$, i.e.

$$G(V, x)^* = \{x' \in X' : \langle x', u\rangle \geq 0 \text{ for all } u \in G(V, x)\}.$$

A function $g : \Omega \to Z$ has a *weak Pareto maximum* (weak P-maximum) at $x \in \Omega$ if there exists no $y \in \Omega$ such that $g(y) - g(x) \in K^0$.

A function $\varphi : Z \to \mathbb{R}$ is *strictly monotone with respect to the cone* K^0 if for every $x, y \in X$ such that $y \in x + K^0$ we have $\varphi(y) > \varphi(x)$.

Let us recall Pshenichnyĭ's condition. It states that a continuous concave function f attains its maximum over a convex set Ω at $x \in \Omega$ iff

$$\partial[-f](x) \cap F(\Omega, x)^* \neq \emptyset,$$

where $F(\Omega, x)^*$ is the dual cone of $F(\Omega, x) = \{u \in X : \exists \tau > 0 \text{ such that } x + tu \in \Omega \text{ for all } 0 \leq t \leq \tau\}$, the cone of feasible directions to Ω at $x \in \Omega$.

Let V be a convex sequentially compact subset of Ω such that $\overline{\Omega \setminus V} \subseteq \Omega$. The main result of the paper is Theorem 3.1, which says that if there is a function $\varphi : Z \to \mathbb{R}$ which is strictly monotone with respect to K^0 and such that $\varphi \circ g$ is upper semicontinuous on Ω and for every $x \in \Omega \setminus V$,

$$\partial(\varphi \circ g)(V, x) \cap G(V, x)^* \neq \emptyset,$$

then g attains its weak P-maximum over Ω in V. The result need not hold when: 1) $\varphi \circ g$ is not upper semicontinuous, or 2) $\overline{\Omega \setminus V} \not\subseteq \Omega$, or 3) V is not a convex set, or 4) V is not sequentially compact, or 5) $\emptyset = \partial(\varphi \circ g)(V, x) \cap G(V, x)^*$ (see Examples 3.8–3.10 in Section 3).

The sufficient condition turns out to be necessary provided g is a real-valued function, no matter if g is concave or not (Proposition 3.6). The condition is no longer necessary if g is vector-valued and φ is linear (Example 3.4). When g is K-concave, i.e.

$$-g(tx+(1-t)y)+tg(x)+(1-t)g(y)\in -K \quad \text{for } x,y\in\Omega \text{ and } t\in[0,1],$$

and φ is linear the condition becomes again necessary (Proposition 3.7).

An open question is what other kind of correspondence between g and φ makes Theorem 3.1 a sufficient and necessary condition for a weak P-maximum of g to be in V.

2. Preliminary notions and results. To prove the main result we need some results on sequentially compact sets and countably orderable sets. Therefore we first recall some notions and results from [2], [4] and [8].

Let X be a nonempty set and $s\subset X^2$ be an arbitrary relation in X. We write xsy instead of $(x,y)\in s$; by s^* we mean the transitive closure of s, i.e. xs^*y iff $x=x_1s\ldots sx_n=y$ for some finite sequence $x_1,\ldots,x_n\in X$; by $\sim(xsy)$ we mean $(x,y)\notin s$. For every $x\in X$ and $U\subseteq X$ we denote by $U_s(x)$ the set $\{z\in U: xsz\}$. If X is a linear space, for every $x,y\in X$, $[x,y]$ means the set $\{z\in X:\exists t\in[0,1] \text{ such that } z=tx+(1-t)y\}$.

Definition 2.1 (see [4, p. 288]). X is called *countably orderable* with respect to the relation s if for every nonempty subset $W\subseteq X$ the existence of a relation η well ordering W and such that $\eta\subseteq s^*\cup \mathrm{id}$ implies that W is at most countable.

A number of examples of countably orderable sets together with applications to optimization are given in [4].

The following result has been proven in [4].

Theorem 2.2 (see [4, Theorem 4.1, p. 297]). *Let X be a nonempty set with relations $s,r\subseteq X\times X$. Let $U,V\subseteq X$ be nonempty sets such that $U\setminus V\neq\emptyset$, $U\setminus V$ is countably orderable with respect to r and r is transitive on $U\setminus V$. Assume that for every $u\in U\setminus V$ the set $U_s(u)$ is nonempty and the following conditions hold*:

(i) *for every sequence $(x_i)\subseteq U_s(u)\setminus V$ such that x_irx_{i+1} for every $i\in\mathbb{N}$ there is $x\in U_s(u)$ with x_irx for every $i\in\mathbb{N}$*;

(ii) *for every $x\in U_s(u)\setminus V$ there is $y\in U_s(u)$ for which xry, $x\neq y$ and $\sim(yrx)$.*

Then $U\cap V\neq\emptyset$ and for every $u\in U\setminus V$ there exists $v\in U\cap V$ such that usv.

Now we recall the definition of a sequentially compact set.

Definition 2.3 (see [2, p. 261]). A nonempty subset A of a topological Hausdorff space X is called *sequentially compact* if for every sequence $\{x_i\}$ in A there is a subsequence $\{x_{i_k}\} \subseteq \{x_i\}$ converging to some $x \in A$.

The following results have been given in [2].

Theorem 2.4 (see [2, Theorem 24, p. 262]). *The Cartesian product of a countable number of sequentially compact spaces is sequentially compact with Tikhonov's topology.*

Theorem 2.5 (see [2, Theorem 21, p. 262]). *Let X be a sequentially compact space. If there is a continuous function f from X onto Y and Y is a Hausdorff space*, *then Y is sequentially compact.*

Let X be a vector space and A be a subset of X.

Definition 2.6 (see [8, p. 47]). The intersection of all convex sets containing A is called the *convex hull* ($\mathrm{conv}(A)$) of A.

It is easy to check that

$$\mathrm{conv}(A) = \Big\{ y \in X : \\ y = \sum_{i=1}^{n} \alpha_i a_i,\ a_i \in A,\ \alpha_i \geq 0,\ \sum_{i=1}^{n} \alpha_i = 1,\ n \text{ arbitrary} \Big\}. \tag{2}$$

Lemma 2.7. *Let A be a sequentially compact and convex subset of a real locally convex topological space X and $x_0 \in X$. Then* $\mathrm{conv}(A \cup \{x_0\})$ *is sequentially compact.*

Proof. Since A is convex, it follows from (2) that

$$\mathrm{conv}(A \cup \{x_0\}) = \Big\{ y \in X : \\ y = \sum_{i=1}^{2} \alpha_i a_i,\ a_1 \in A,\ a_2 = x_0,\ \alpha_1 + \alpha_2 = 1,\ \alpha_1, \alpha_2 \geq 0 \Big\}.$$

Define

$$L := \{(\alpha_1, \alpha_2) : \alpha_1, \alpha_2 \geq 0 \text{ and } \alpha_1 + \alpha_2 = 1\} \subset \mathbb{R}^2.$$

It is easy to show that L is a sequentially compact subset of $\mathbb{R}^2$. Observe that $\mathrm{conv}(A \cup \{x_0\})$ is a continuous image of the sequentially compact set $L \times A \times \{x_0\} \subset \mathbb{R}^2 \times X \times X$ (see Theorem 2.4), so by Theorem 2.5, it is sequentially compact. ■

3. The main results. We start this section with presenting a general sufficient condition for a global Pareto maximum of a vector-valued function to be in a given convex subset of the domain.

THEOREM 3.1. *Let X and Z be locally convex topological vector spaces, Ω be a nonempty subset of X and $g : \Omega \to Z$. Let $V \neq \emptyset$ be a given sequentially compact, convex subset of Ω such that $\overline{\Omega \setminus V} \subseteq \Omega$. If a function $\varphi : g(\Omega) \to \mathbb{R}$ is such that $\varphi \circ g$ is upper semicontinuous on $\overline{\Omega \setminus V}$ and*

$$\emptyset \neq \partial(\varphi \circ g)(V, y) \cap G(V, y)^* \tag{3}$$

for every $y \in \Omega \setminus V$, then

$$\sup_{x \in \Omega} \varphi \circ g(x) = \sup_{v \in V} \varphi \circ g(v).$$

If, additionally, φ is strictly monotone with respect to K^0 and $\varphi \circ g$ attains its maximum over V at $v_0 \in V$, then v_0 is a global weak P-maximum of g over Ω.

Proof. Fix $\varepsilon > 0$ and a convex absorbing neighbourhood U of zero. Let μ_U denote the Minkowski functional relative to U. Define a relation $s \subset \Omega \times \Omega$ by

$$\forall x, y \in \Omega : \quad xsy \Leftrightarrow \varphi \circ g(x) - \varepsilon\mu_U(x, V) \le \varphi \circ g(y) - \varepsilon\mu_U(y, V),$$

where $\mu_U(x, V) = \inf_{v \in V} \mu_U(v - x)$. Define a relation $r \subset \Omega \times \Omega$ by

$$\forall x, y \in \Omega : \quad xry \Leftrightarrow \exists v \in V : y \in [v, x].$$

Since V is convex, r is transitive.

When $\Omega \setminus V = \emptyset$ the assertion trivially holds, so assume that $\Omega \setminus V \neq \emptyset$. First we show that $\Omega \setminus V$ is countably orderable with respect to r. Let $W \subseteq \Omega \setminus V$ be well ordered with respect to some relation $\eta \subseteq r \cup \mathrm{id}$. Let w_0 be the η-first element of W and denote $V - \mathrm{conv}(V \cup \{w_0\})$ by A. Clearly, $tA \subseteq A$ for every $t \in [0, 1]$. Since A is convex and absorbing (for $V - W$), it is easy to prove that the Minkowski functional of A has the following properties (cf. [8]):

1° $\mu_A(x) < \infty$ for all $x \in V - W$,
2° $\mu_A(x + y) \le \mu_A(x) + \mu_A(y)$ for all $x, y \in V - W$,
3° $\mu_A(\lambda x) = \lambda\mu_A(x)$ for all $x \in V - W$ and $\lambda \in \mathbb{R}_+$.

From Lemma 2.7 it also follows that A is sequentially compact.

Now we show, using the above properties of A and μ_A, that the conditions $x, y \in W$, $x \neq y$ and $x\eta y$ together force

$$\mu_A(x, V) > \mu_A(y, V),$$

where $\mu_A(x, V) = \inf_{v \in V} \mu_A(v - x)$. First we show that $\mu_A(x, V) > 0$ for all $x \in W$. Suppose that $\mu_A(x, V) = 0$ for some $x \in W$. Then for every $n > 0$ there is $v_n \in V$ such that $\mu_A(v_n - x) < 2^{-n}$, hence $v_n - x \in 2^{-n}A$. Now suppose that x is not the limit of $\{v_n\}$. Then there is a convex symmetric neighbourhood U_0 of zero and a subsequence $\{v_{n_k}\} \subseteq \{v_n\}$ such that $v_{n_k} - x \notin U_0$ for all n_k. The way the sequence $\{v_n\}$ was constructed implies that we can choose points $a_{n_k} \in A$ for which $v_{n_k} - x = 2^{-n_k} a_{n_k} \notin U_0$ for every $k \in \mathbb{N}$. On the other hand, A is sequentially compact, so $a_{n_{k_j}} \to a \in A$ for some subsequence $\{a_{n_{k_j}}\}$, and hence $\mu_{U_0}(a_{n_{k_j}} - a) \le 1$ for every j large enough. Thus for $j \to \infty$,

$$\infty \leftarrow 2^{n_{k_j}} \le \mu_{U_0}(a_{n_{k_j}}) \le \mu_{U_0}(a) + \mu_{U_0}(a_{n_{k_j}} - a) \le \mu_{U_0}(a) + 1 < \infty,$$

which is a contradiction. Therefore the sequence $\{v_n\}$ converges to x. Since V is sequentially compact, we have $x \in V$, which is a contradiction because $x \in W \subseteq \Omega \setminus V$. So $\mu_A(x, V) > 0$ for all $x \in W$.

Now let $x, y \in W$, $x \neq y$ and $x \eta y$. Then there are $t \in (0,1)$ and $u \in V$ such that $y = (1-t)x + tu$. We can take $v \in V$ such that

$$\mu_A(v - x) < (1+t)\mu_A(x, V).$$

Observe that since V is convex, we have $(1-t)v + tu \in V$ and by 3°,

$$\begin{aligned} \mu_A(y, V) &\le \mu_A((1-t)v + tu - y) \\ &= \mu_A((1-t)v + tu - (1-t)x - tu) \\ &= (1-t)\mu_A(v - x) < (1-t)(1+t)\mu_A(x, V) < \mu_A(x, V). \end{aligned}$$

Thus the set $\mu_A(W, V) \subseteq \mathbb{R}$ is well ordered by $\le$ and therefore it is at most countable. But the same concerns W, since $\mu_A(\cdot, V)$ is a one-to-one mapping on W. So $\Omega \setminus V$ is countably orderable with respect to r.

Now, fix $x_0 \in \Omega \setminus V$ and let $\Omega_s(x_0) = \{x \in \Omega : x_0 s x\}$. Consider a sequence $\{x_i\} \subset \Omega_s(x_0) \setminus V$ such that $x_i r x_{i+1}$ for all $i = 1, 2, \ldots$ Since r is transitive, $x_i \in \operatorname{conv}(V \cup \{x_1\})$ for all $i = 2, 3, \ldots$, and from Lemma 2.7, $\operatorname{conv}(V \cup \{x_1\})$ is sequentially compact. Choose a subsequence $\{x_{i_k}\} \subseteq \{x_i\}$ converging to some $x \in \operatorname{conv}(V \cup \{x_1\})$. Then $x_{i_{k+j}} \in \operatorname{conv}(V \cup \{x_{i_k}\})$ for every $j = 1, 2, \ldots$ Therefore $x \in \operatorname{conv}(V \cup \{x_{i_k}\})$ for all $k = 1, 2, \ldots$ and $x_{i_k} r x$ for all $k = 1, 2, \ldots$

Now, we show that $\mu_U(x, V) \ge \mu_U(y, V)$ for all $x, y \in \Omega \setminus V$ with $x \neq y$ and $x r y$.

1° If $\mu_U(x, V) = 0$ and $x \in \Omega \setminus V$, then for every $n \in \mathbb{N}$ there exists $v_n \in V$ such that $\mu_U(v_n - x) < 2^{-n}$. Since V is sequentially compact there is a subsequence $\{v_{n_k}\} \subseteq \{v_n\}$ converging to a point $v_0 \in V$. Consequently, for every $n_0 \in \mathbb{N}$, there is $k \ge n_0$ such that for $n_j > k$ we have $v_{n_j} \in 2^{-n_0} U + v_0$. Hence for $n_0 \to \infty$,

$$0 \le \mu_U(v_0 - x) \le \mu_U(v_0 - v_{n_j}) + \mu_U(v_{n_j} - x) \le \frac{1}{2^{n_0}} + \frac{1}{2^{n_j}} \to 0.$$

So there is $v_0 \in V$ such that $\mu_U(v_0 - x) = 0$, and finally,

$$\mu_U(v_0 - x) = \mu_U(x, V) = 0. \tag{4}$$

2° If $\mu_U(x,V) > 0$, then

(5) $$\forall t \in (0,1)\ \exists v_0 \in V: \quad \mu_U(v_0 - x) < (1+t)\mu_U(x,V).$$

From (4) and (5), we get in both cases 1° and 2°,

(6) $$\forall x \in \Omega \setminus V,\ t \in (0,1)\ \exists v_0 \in V: \quad \mu_U(v_0 - x) \le (1+t)\mu_U(x,V).$$

Let $x, y \in \Omega \setminus V$, $x \ne y$ and xry. By the definition of r, there are $t \in (0,1)$ and $u \in V$ such that $y = tu + (1-t)x$. From (6) and the fact that V is convex, it follows that $tu + (1-t)v_0 \in V$ and

(7) $$\begin{aligned}\mu_U(y,V) &\le \mu_U((tu + (1-t)v_0 - tu - (1-t)x) \\ &= (1-t)\mu_U(v_0 - x) \le (1-t)(1+t)\mu_U(x,V) \le \mu_U(x,V).\end{aligned}$$

Since $x_0 s x_{i_k}$ for all $k = 1, 2, \ldots$ and $\varphi \circ g$ is upper semicontinuous on $\overline{\Omega \setminus V}$, by (7) we get

$$\begin{aligned}\varphi \circ g(x_0) - \varepsilon\mu_U(x_0, V) &\le \varphi \circ g(x_{i_k}) - \varepsilon\mu_U(x_{i_k}, V) \\ &\le \varphi \circ g(x) - \varepsilon\mu_U(x,V)\end{aligned}$$

whenever $x \notin V$. Otherwise $0 = \mu_U(x,V) \le \mu_U(x_{i_k}, V)$, and the above inequality holds as well. This shows that $x \in \Omega_s(x_0)$ and the condition (i) of Theorem 2.2 is satisfied.

Now we examine the condition (ii). Consider any $x \in \Omega_s(x_0) \setminus V$ and observe that there are, by (3), an $x' \in \partial(\varphi \circ g)(V,x)$ and an $v \in V$ such that

(8) $$\varphi \circ g(x_t) - \varphi \circ g(x) \ge \langle x', x_t - x\rangle \quad \text{for some } t \in (0,1]$$

where $x_t = tv + (1-t)x$. Simultaneously, $x' \in G(V,x)^*$ and since $x_t - x = t(v-x) \in G(V,x)$ we get

(9) $$\langle x', x_t - x\rangle \ge 0.$$

From (8) and (9), it follows that

(10) $$\varphi \circ g(x_t) - \varphi \circ g(x) \ge 0 \ge -t^2\varepsilon\mu_U(x,V).$$

It is easy to check (cf. (7)) that

(11) $$\mu_U(x_t, V) \le (1-t^2)\mu_U(x,V).$$

From (10) and (11), we get

$$\varphi \circ g(x_t) - \varphi \circ g(x) \ge -t^2\varepsilon\mu_U(x,V) \ge \varepsilon\mu_U(x_t,V) - \varepsilon\mu_U(x,V).$$

Consequently,

$$\varphi \circ g(x_t) - \varepsilon\mu_U(x_t,V) \ge \varphi \circ g(x) - \varepsilon\mu_U(x,V).$$

Thus xrx_t, $x \ne x_t$ and $x_t \in \Omega_s(x_0)$ (recall that s is transitive), which shows that (ii) holds. Therefore Theorem 2.2 implies that

$$\forall x \in \Omega \setminus V\ \exists v \in \Omega \cap V: \quad xsv.$$

So we have

$\forall x \in \Omega \setminus V \ \exists v \in V$:

$$\varphi \circ g(x) - \varepsilon \mu_U(x, V) \leq \varphi \circ g(v) - \varepsilon \mu_U(v, V) = \varphi \circ g(v) \leq \sup_{v \in V} \varphi \circ g(v).$$

Since $\varepsilon > 0$ is arbitrary, the first assertion of the theorem holds.

Assume now that $\varphi \circ g$ attains its maximum over V at $v_0 \in V$. Then it follows from the first part of Theorem 3.1 that

$$\forall x \in \Omega : \quad \varphi \circ g(x) - \varphi \circ g(v_0) \leq 0. \tag{12}$$

If v_0 were not a weak P-maximum of g over Ω, there would exist $y \in \Omega$ such that

$$g(y) - g(v_0) \in K^0.$$

Since φ is strictly monotone with respect to K^0, this and (12) would imply

$$0 < \varphi \circ g(y) - \varphi \circ g(v_0) \leq 0,$$

which is a contradiction. Therefore g has a weak P-maximum over Ω at $v_0 \in V$. ∎

REMARK 3.2. If we assume that $\varphi \circ g$ is upper semicontinuous on V, then $\varphi \circ g$ attains its maximum over V at some $v_0 \in V$. Moreover, one can only assume increasing semicontinuity of $\varphi \circ g$ (instead of upper semicontinuity) (see [5]) and the same assertion holds. Indeed, this is a consequence of the following lemma.

LEMMA 3.3 [4]. *Let X be an abstract set and $f : X \to \mathbb{R}$. The function f attains its supremum at some point of X if and only if for every sequence $\{x_i\} \subseteq X$ such that for every $i = 1, 2, \ldots$,*

$$f(x_i) \leq f(x_{i+1})$$

there is an $x \in X$ such that $f(x_i) \leq f(x)$ for all $i = 1, 2, \ldots$

The following example shows that Theorem 3.1 does not give necessary conditions for a weak P-maximum of g to be in V when Z is more than one-dimensional and φ is linear and strictly monotone with respect to K^0.

EXAMPLE 3.4. Let K^0 be the cone of vectors in $\mathbb{R}^2$ with positive coordinates. Define $g : \Omega \equiv [1, 2] \to \mathbb{R}^2$ by $g(x) = (x, 1/x)$ and $\varphi_\varepsilon : \mathbb{R}^2 \to \mathbb{R}$ by $\varphi_\varepsilon(x, y) = \varepsilon x + (1 - \varepsilon) y$ with $\varepsilon \in [0, 1]$. Set $V = \{3/2\}$. It is easy to see that $x_0 = 3/2$ is a weak P-maximum. Now we show that for every φ_ε, $\varepsilon \in [0, 1]$, condition (3) is not satisfied for some $y \in \Omega \setminus V$.

1° If $\varepsilon \in [0, 4/13)$, then set $y = 1$. It is easy to see that $G(V, y) = \{u \in \mathbb{R} : u > 0\}$, so $G(V, y)^* = \{x' \in \mathbb{R} : x' \geq 0\}$. On the other hand,

$$\varphi_\varepsilon \circ g(t x_0 + (1 - t) y) - \varphi_\varepsilon \circ g(y) < 0 \quad \text{for all } t \in (0, 1],$$

so $\partial(\varphi_\varepsilon \circ g)(V, y) \subset \{x' \in \mathbb{R} : x' < 0\}$, hence $\partial(\varphi_\varepsilon \circ g)(V, y) \cap G(V, y)^* = \emptyset$.

2° If $\varepsilon \in [4/13, 1]$, then set $y = 2$. Since $G(V, y) = \{u \in \mathbb{R} : u < 0\}$ we have $G(V, y)^* = \{x' \in \mathbb{R} : x' \leq 0\}$. On the other hand,

$$\varphi_\varepsilon \circ g(tx_0 + (1-t)y) - \varphi \circ g(y) < 0 \quad \text{for all } t \in (0, 1],$$

so $\partial(\varphi_\varepsilon \circ g)(V, y) \subset \{x' \in \mathbb{R} : x' > 0\}$, hence $\partial(\varphi_\varepsilon \circ g)(V, y) \cap G(V, y)^* = \emptyset$.

REMARK 3.5. When Z is one-dimensional a stronger result than Theorem 3.1 is available. For simplicity we assume a continuity condition slightly stronger than necessary.

PROPOSITION 3.6. *Let X be a locally convex topological vector space and Ω be a nonempty subset of X. Assume that $g : \Omega \to \mathbb{R}$ is upper semicontinuous on Ω. Assume that V is a sequentially compact convex subset of Ω such that $\overline{\Omega \setminus V} \subseteq \Omega$. Then g attains its maximum over Ω at some point of V if and only if*

$$\emptyset \neq \partial g(V, y) \cap G(V, y)^* \quad \textit{for every } y \in \Omega \setminus V.$$

Proof. First, suppose that g attains a global maximum over Ω at some $v \in V$. Then

$$g(v) - g(y) \geq 0 \quad \text{for all } y \in \Omega.$$

Thus $x'_0 = 0 \in X'$ satisfies

$$g(v) - g(y) \geq \langle x'_0, v - y \rangle.$$

By (1), we have $x'_0 \in \partial g(V, y) \cap G(V, y)^*$. The sufficiency follows from Theorem 3.1 and Remark 3.2. ∎

In order to present a necessary condition for a vector-valued function g to attain its weak Pareto maximum in a given convex subset V of Ω, we need some definitions.

Recall that

$$K^* = \{z' \in Z' : \langle z', k \rangle \geq 0 \text{ for all } k \in K\}$$

is the dual cone of K and Z' is the dual space of Z. Define

$$K_0^* = \{k' \in K^* : \langle k', k_0 \rangle > 0 \text{ for all } k_0 \in K^0\}.$$

It is easy to show that $K_0^* = K^* \setminus \{0\}$. Clearly, if $k' \in K_0^*$, then $\langle k', \cdot \rangle$ is strictly monotone with respect to K^0.

PROPOSITION 3.7. *Assume that X and Z are locally convex topological vector spaces and Ω is a convex subset of X. Let $g : \Omega \to Z$ be a K-concave function, and V be a convex subset of X. If g attains its weak P-maximum over Ω in V, then there exists $k' \in K_0^*$ such that*

$$\partial(k'g)(V, y) \cap G(V, y)^* \neq \emptyset \quad \textit{for every } y \in \Omega \setminus V.$$

Proof. Suppose that g has a weak P-maximum at some $v \in V$. Then there is no $y \in \Omega$ such that

$$-(g(v) - g(y)) \in K^0.$$

By Craven's alternative theorem (see [1, p. 31]), there exists $k' \in K_0^*$ such that

$$\langle k', g(v) - g(y)\rangle \geq 0 \quad \text{ for all } y \in \Omega.$$

Thus, $v \in V$ maximizes the linear scalar function $k'g$ over Ω. An application of Proposition 3.6 completes the proof. ■

The following examples show that the main assumptions in Theorem 3.1 and Proposition 3.6 cannot be dropped.

Example 3.8. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{for } x \in [0,1) \cup (4,5], \\ x & \text{for } x \in [1,2], \\ 2 & \text{for } x \in (2,3], \\ -x+5 & \text{for } x \in [3,4]. \end{cases}$$

Let $V = [0,1] \cup [4,5]$. Then V is a sequentially compact non-convex set and f is upper semicontinuous on $\Omega = [0,5]$. Moreover, for all $y \in \Omega \setminus V$ we have $\partial f(V,y) \cap G(V,y)^* \neq \emptyset$, since

$$\begin{array}{lll} \partial f(V,y) = \mathbb{R} & \text{and} \quad G(V,y)^* = \{0\} & \forall y \in (1,2), \\ \partial f(V,y) = \mathbb{R} & \text{and} \quad G(V,y)^* = \{0\} & \forall y \in (2,3), \\ \partial f(V,y) = \mathbb{R} & \text{and} \quad G(V,y)^* = \{0\} & \forall y \in (3,4), \\ \partial f(V,y) = \mathbb{R} \setminus (0,1) & \text{and} \quad G(V,y)^* = \{0\} & \text{for } y = 2, \\ \partial f(V,y) = \mathbb{R} \setminus (-1,0) & \text{and} \quad G(V,y)^* = \{0\} & \text{for } y = 3. \end{array}$$

However, f does not have a maximum over Ω in V.

Example 3.9. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{for } x \in (-\infty,3), \\ 2 & \text{for } x \in [3,+\infty). \end{cases}$$

(i) Let $\Omega = (-\infty,\infty)$ and $V = [3,4]$. Then V is convex and compact, but f is not upper semicontinuous on $\overline{\Omega \setminus V}$. It is easy to see that $\partial f(V,y) \cap G(V,y)^* \neq \emptyset$ for all $y \in \Omega \setminus V$, since

$$\begin{array}{lll} \partial f(V,y) = (-\infty,1] & \text{and} \quad G(V,y)^* = \mathbb{R}_+ \cup \{0\} & \forall y \in (-\infty,3), \\ \partial f(V,y) = [0,\infty) & \text{and} \quad G(V,y)^* = \mathbb{R}_- \cup \{0\} & \forall y \in (4,\infty). \end{array}$$

However, f does not have a maximum over Ω in V.

(ii) Let $\Omega = (-\infty,3) \cup \left[3\frac{1}{2},4\right]$ and $V = \left[3\frac{1}{2},4\right]$. Then V is convex and sequentially compact and f is upper semicontinuous on Ω, but $\overline{\Omega \setminus V} \not\subseteq \Omega$. It is easy to see that $\partial f(V,y) \cap G(V,y)^* \neq \emptyset$ for every $y \in \Omega \setminus V$, since

$\partial f(V,y) = (-\infty,1]$ and $G(V,y)^* = [0,\infty)$ for all $y \in (-\infty,3)$. However, f does not have a maximum over Ω in V.

(iii) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{for } x \in (-\infty,3], \\ 2 & \text{for } x \in (3,\infty). \end{cases}$$

Let $\Omega = (-\infty,\infty)$ and $V = [5,6]$. Then V is convex and sequentially compact, f is upper semicontinuous on Ω, $\partial f(V,3) = (-\infty,-1/3)$ and $G(V,3)^* = \mathbb{R}_+ \cup \{0\}$, i.e. $\partial f(V,3) \cap G(V,3)^* = \emptyset$. Observe that f does not have a maximum over Ω in V.

Example 3.10. Let $\Omega = \{x = (t_i) : \sum_{i=1}^{\infty} |t_i| < \infty\}$ and $\|x\| = \sum_{i=1}^{\infty} |t_i|$, for $x = (t_i) \in \Omega$. Define $V = \{x = (t_i) \in \Omega : \|x\| \le 1 \text{ and } t_i \ge 0 \text{ for all } i = 1,2,\ldots\}$. It is easy to see that V is convex but is not sequentially compact. Let $x_0 = (-2,-1,-1/2,-1/4,\ldots)$. Consider the sequence $\{x_k\} \subseteq \Omega$, $x_k = (t_i^k)$, such that

$$\begin{aligned} x_1 &= \alpha_1 x_0 + (1-\alpha_1)e_1, \\ x_2 &= \alpha_2 x_1 + (1-\alpha_2)e_2, \\ &\vdots \\ x_n &= \alpha_n x_{n-1} + (1-\alpha_n)e_n, \end{aligned}$$

where $\alpha_i = 2^{-i}$ for $1,2,\ldots$ and $e_1 = (1,0,0,\ldots)$, $e_2 = (0,1,0,\ldots)$ and so on. Clearly, $e_i \in V$ for all i, and consequently, $x_{n+1} \in \operatorname{conv}(x_n, V)$. It is easy to see that $\{x_n\} \subset \Omega \setminus V$ because

$$t_1^n = -(1/2)^{(n-1)(n+2)/2} < 0$$

and obviously $\|x_n\| < \infty$ for all $n \in \mathbb{N}$. Define $f : \Omega \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0, & x \ne x_i, \\ i, & x = x_i. \end{cases}$$

Now we show that the sequence $\{x_k\}$ has no cluster points. Indeed, for fixed $i \in \mathbb{N}$, $t_i^k \to 0$ as k tends (no matter how) to infinity. So the only cluster point of $\{x_k\}$ might be $e_0 = (0,0,\ldots)$. On the other hand,

$$\|x_k\| = \sum_{i=1}^{\infty} |t_i^k| \ge |t_k^k| \xrightarrow[k\to\infty]{} 1.$$

This shows that $\{x_k\}$ has no cluster points and, consequently, f is upper semicontinuous on Ω. Finally, we verify condition (3). For $y \notin \{x_k\}$, (3) is satisfied because $e_0' = (0,0,\ldots) \in G(V,y)^* \cap \partial f(V,y)$. The same is true when $y = x_n$ for some $n \in \mathbb{N}$. Indeed, $e_0' \in G(V,x_n)^*$ by definition and

$$f(x_{n+1}) - f(x_n) = 1 > \langle e_0', x_{n+1} - x_n \rangle,$$

which together with the decomposition

$$x_{n+1} = \alpha_{n+1}x_n + (1 - \alpha_{n+1})e_{n+1}$$

implies that $e'_0 \in \partial f(V, x_n)$. Hence (3) is satisfied for all $y \in \Omega \backslash V$. However, the maximum of f over Ω is not attained in V.

REMARK 3.11. Example 3.10 is related to the drop property. Since V is a closed unit ball, the sequence $\{x_n\}$ has a convergent subsequence if and only if the norm $\|\cdot\|$ has the drop property (see [7], Proposition 2). It is well known that the space Ω is not reflexive, so Ω does not have the drop property, and consequently, $\{x_n\}$ may not have a convergent subsequence (cf. [7]).

References

[1] B. D. Craven, *Mathematical Programming and Control Theory*, Halsted, London, 1982.
[2] R. Engelking, *General Topology*, Monograf. Mat. 60, PWN, Warszawa, 1977.
[3] L. Gajek and D. Zagrodny, *Existence of maximal points with respect to ordered bipreference relations*, J. Optim. Theory Appl. 70 (1991), 355–364.
[4] —, —, *Countably orderable sets and their applications in optimization*, Optimization 26 (1992), 287–301.
[5] —, —, *Weierstrass theorem for monotonically semicontinuous functions*, ibid. 29 (1994), 199–203.
[6] B. Pshenichnyĭ, *Necessary Conditions for an Extremum*, Nauka, Moscow, 1982 (in Russian).
[7] S. Rolewicz, *On drop property*, Studia Math. 85 (1987), 27–35.
[8] W. Rudin, *Functional Analysis*, Moscow, Mir, 1975 (in Russian); English original: McGraw-Hill, New York, 1973.
[9] C. Swartz, *Pshenichnyĭ's theorem for vector minimization*, J. Optim. Theory Appl. 53 (1987), 309–317.

Agnieszka Drwalewska and Lesław Gajek
Institute of Mathematics
Technical University of Łódź
Al. Politechniki 11
90–924 Łódź, Poland
E-mail: gal@ck-sg.p.lodz.pl

Received on 19.6.1998;
revised version on 31.5.1999