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On a certain additive divisor problem

by

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1. Introduction. Additive divisor problems have a rich history in analytic number theory. A classical example is the binary additive divisor problem, which is concerned with finding asymptotic estimates for the sum

$$\sum_{n\le x} d(n)d(n+h), \qquad h\ge 1, \tag{1.1}$$

where $d(n)$ is the usual divisor function. There has been a lot of effort in studying this problem, one reason being its intimate link to the fourth power moment of the Riemann zeta function, which has been of great interest since the classical work of Ingham [13]. We refer to [23] for an extensive discussion of the history of this problem.

In other applications to L-functions (see [4] and [11]), variations of the binary additive divisor problem have come up, usually concerning sums of the form

$$D(x_1,x_2) := \sum_{\substack{n_1,n_2\\ r_1n_2-r_2n_1=h}} w_1\left(\frac{n_1}{x_1}\right)w_2\left(\frac{n_2}{x_2}\right)d(n_1)d(n_2). \tag{1.2}$$

Here r_1 and r_2 are positive coprime integers, h is non-zero, and w_1 and w_2 are smooth weight functions, which we assume to be compactly supported in $[1/2,1]$ (the assumption that r_1 and r_2 be coprime is not restrictive—otherwise h has to be divisible by their greatest common divisor, and we can divide both sides of the equation by that number).

The case $r_1=r_2=1$ is just a smoothed version of (1.1) and has probably received most of the attention, but a few nice results are available for general r_1 and r_2 as well. Besides the implicit treatment in [4], there is the work of

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Duke, Friedlander and Iwaniec [10], who showed that

$$(1.3)\qquad D(x_1,x_2) = (\text{main term}) + \mathcal{O}\big((r_2x_1 + r_1x_2)^{1/4}(r_1r_2x_1x_2)^{1/4+\varepsilon}\big).$$

The size of the error term in this result is inferior compared to what can be achieved in the classical case $r_1 = r_2 = 1$, which is due to the fact that they did not make use of spectral theory. Nevertheless, the range of uniformity in h for which this asymptotic formula is non-trivial is quite impressive. In the case $r_2 = 1$, this result has been improved recently by Aryan [1].

With applications in mind to be considered in a separate paper [28], we consider the following generalized sums:

$$D(x_1,x_2) := \sum_n w_1\bigg(\frac{r_1n+f_1}{x_1}\bigg)w_2\bigg(\frac{r_2n+f_2}{x_2}\bigg)d(r_1n+f_1)d(r_2n+f_2).$$

Note that for $(r_1,r_2) = 1$ and the choice $h = r_1f_2 - r_2f_1$, this is the sum (1.2). For r_1 and r_2 not coprime, however, this leads to a new and more difficult problem. We have not been able to find any results in the literature concerning this sum for general r_1 and r_2, and the following asymptotic formula seems to be new.

THEOREM 1.1. *Set*

$$(1.4)\qquad r_0 := \min\{(r_1, r_2^\infty), (r_2, r_1^\infty)\}, \qquad h := r_1f_2 - r_2f_1.$$

Then, for $h \neq 0$ *and* $f_1 \ll x_1^{1-\varepsilon}$, $f_2 \ll x_2^{1-\varepsilon}$,

$$D(x_1,x_2) = M(x_1,x_2) + \mathcal{O}\big(r_0(r_2x_1)^{1/2+\theta+\varepsilon}\big),$$

where the main term is given by

$$M(x_1,x_2) := \int w_1\bigg(\frac{r_1\xi+f_1}{x_1}\bigg)w_2\bigg(\frac{r_2\xi+f_1}{x_2}\bigg)P(\log(r_1\xi+f_1),\log(r_2\xi+f_2))\,d\xi$$

with $P(\xi_1,\xi_2)$ *a quadratic polynomial depending on* r_1, r_2, f_1 *and* f_2. *The implicit constants depend only on* w_1, w_2 *and* ε.

Here we denote by θ the bound in the Ramanujan–Petersson conjecture (see Section 2.3 for a precise definition—in any case $\theta = 7/64$ is admissible, as we know by the work of Kim and Sarnak [17]). The polynomial appearing in the main term can be stated fairly explicitly (see (3.20)).

Our aim was to make use of the powerful tools of spectral theory to get an asymptotic formula for $D(x_1,x_2)$ for r_1 and r_2 as large as possible. In particular when r_1 and r_2 are not coprime, this turned out to be much more difficult to achieve than one might expect at first glance (we will say more about the arising difficulties below). On the other hand, we have not aimed at the largest possible range of uniformity in f_1 and f_2. In fact, with some work it should be possible to extend our result to include f_1 and f_2 in a larger range than required above. It also seems likely that the dependence on r_0 is not optimal, but here it is not immediately clear how an improvement might be achieved. Our result improves on (1.3), although the latter estimate is valid for much larger h than ours. In the special case $r_2 = 1$, our result is the same as [1, Theorem 0.1].

We also want to state the following result, which deals with an analogue of $D(x_1, x_2)$ with sharp cut-off.

THEOREM 1.2. *Let r_0 and $h \neq 0$ be defined as above. Assume that*

$$f_1 \ll (r_1 x)^{1-\varepsilon}, \qquad f_2 \ll (r_2 x)^{1-\varepsilon}, \qquad (r_0 r_1 r_2, h) h \ll r_0^{1/3} (r_1 r_2)^{5/3} x^{1/3-\varepsilon}.$$

Then

$$\sum_{x/2<n\leq x} d(r_1 n + f_1) d(r_2 n + f_2) = x P(\log x) + \mathcal{O}\big((r_0 r_1 r_2, h)^{\theta} r_0^{2/3+\theta} (r_1 r_2)^{1/3} x^{2/3+\varepsilon}\big),$$

where $P(\xi)$ is a quadratic polynomial depending on r_1, r_2, f_1 and f_2, and where the implicit constants only depend on ε.

Correlations of a much more general type have been investigated by Matthiesen [20], but the methods used there do not apply to our case and do not give power savings in the error term. Similar problems, where the divisor functions are replaced by Fourier coefficients of automorphic forms, have been studied as well (see e.g. [2]). In particular, for Fourier coefficients of holomorphic cusp forms, Pitt [25, Theorem 1.4] was able to prove an analogue of our Theorem 1.1 for r_1, r_2 squarefree and $f_1 = f_2 = -1$. Unfortunately, his method relies on Jutila's variant of the circle method, which becomes ineffective when a main term is present, as is the case in our problem.

The proof of Theorems 1.1 and 1.2 initially follows standard lines: We split one of the divisor functions and use the Voronoi summation formula to deal with the divisor sums in arithmetic progressions. Serious difficulties arise when the sums of Kloosterman sums take the stage at this point. In a simplified form, we are faced with sums roughly of the form

$$\sum_{\substack{c \\ (c, r_2)=1}} \frac{S(1 - r_1 \overline{r_2}, 1; r_1 c)}{r_1 c} F(r_1 c),$$

where F is some weight function, and where $\overline{r_2}$ is understood to be mod c. We could bound the Kloosterman sums individually using Weil's bound, and the resulting error terms in our theorems would be of a size comparable to (1.3). But as already mentioned above, our aim is to use spectral methods to get results beyond that.

If r_1 and r_2 are coprime, we can use the Kuznetsov formula with an appropriate choice of cusps to do that. Otherwise, it is not directly clear how the Kuznetsov formula might be put into use here. The novelty of our paper is to show how this can nevertheless be accomplished. We solve the problem by splitting the variable $r_1 = tv$ into a factor t which is coprime to r_2, and a factor v which contains only the same prime factors as r_2. By twisted multiplicativity of Kloosterman sums we have

$$\frac{S(1 - r_1\overline{r_2}, 1; r_1c)}{r_1c} = \frac{S(\overline{tc}, \overline{tc}; v)}{v} \frac{S(r_2 - r_1, \overline{v^2r_2}; tc)}{tc},$$

where now all the inverses are understood to be modulo the respective modulus of the Kloosterman sum. Following an idea of Blomer and Milićević [6], we separate the variable c occurring in the first factor by exploiting the orthogonality of Dirichlet characters, as follows:

$$\frac{S(\overline{tc}, \overline{tc}; v)}{v} = \frac{1}{\varphi(v)} \sum_{\chi \bmod v} \chi(tc)\hat{S}_v(\chi) \text{ with } \hat{S}_v(\chi) := \sum_{\substack{y\,(v) \\ (y,v)=1}} \overline{\chi}(y) \frac{S(\overline{y}, \overline{y}; v)}{v},$$

where the left sum runs over all Dirichlet characters mod v. This way we are led to sums of Kloosterman sums twisted by a Dirichlet character, which we can indeed treat by spectral methods.

2. Preliminaries. In the following, ε always stands for some positive real number, which can be chosen arbitrarily small. However, it need not be the same at each occurrence, even if it appears in the same equation.

To avoid confusion, we recall some of the notation to be used in this paper. When we write $A \asymp B$, this means $A \ll B \ll A$. Given a function $f : \mathbb{R} \to \mathbb{C}$, we will occasionally write

$$\operatorname{supp} f \asymp X$$

to mean that there exist constants $c_1, c_2 > 0$ such that $\operatorname{supp} f \subseteq [c_1X, c_2X]$.

The expression (a, b) denotes the greatest common divisor of a and b. The summation

$$\sum_{a\,(c)} (\dots) := \sum_{a \bmod c} (\dots)$$

means that the variable a runs over some residue system mod c. Analogously, we will frequently write $n \equiv h\,(c)$ instead of $n \equiv h \bmod c$.

As usual, $e(q) := e^{2\pi iq}$ and

$$S(m, n; c) := \sum_{\substack{a\,(c) \\ (a,c)=1}} e\left(\frac{ma + n\overline{a}}{c}\right), \qquad c_q(n) := \sum_{\substack{a\,(q) \\ (a,q)=1}} e\left(\frac{na}{q}\right),$$

which are the usual notations for Kloosterman sums and Ramanujan sums (here $\overline{a}$ indicates a solution to $a\overline{a} \equiv 1 \bmod c$).

2.1. The Voronoi summation formula and Bessel functions. Using the well-known Voronoi formula for the divisor function (see [15, Chapter 4.5] or [16, Theorem 1.6]) and the identity

$$\sum_{\substack{n=1\\ n\equiv h\,(c)}}^{\infty} d(n)f(n)=\frac{1}{c}\sum_{d\mid c}\sum_{\substack{b\,(d)\\ (b,d)=1}} e\left(\frac{-bh}{d}\right)\sum_{n=1}^{\infty} d(n)f(n)e\left(\frac{bn}{d}\right),$$

it is not hard to show the following summation formula for the divisor function in arithmetic progressions:

THEOREM 2.1. *Let h and $c\ge 1$ be integers. Let $f:(0,\infty)\to\mathbb{R}$ be smooth and compactly supported. Then*

$$\begin{aligned}\sum_{n\equiv h\,(c)} d(n)f(n)=\frac{1}{c}\int\lambda_{h,c}(\xi)f(\xi)\,d\xi\\ -\frac{2\pi}{c}\sum_{d\mid c}\sum_{n=1}^{\infty} d(n)\frac{S(h,n;d)}{d}\int Y_0\left(4\pi\frac{\sqrt{n\xi}}{d}\right)f(\xi)\,d\xi\\ +\frac{4}{c}\sum_{d\mid c}\sum_{n=1}^{\infty} d(n)\frac{S(h,-n;d)}{d}\int K_0\left(4\pi\frac{\sqrt{n\xi}}{d}\right)f(\xi)\,d\xi\end{aligned}$$

with

$$\lambda_{h,c}(\xi):=\sum_{d\mid c}\frac{c_d(h)}{d}(\log\xi+2\gamma-2\log d).$$

We want to sum up some well-known facts concerning the Bessel functions $K_0(\xi)$ and $Y_\nu(\xi)$ for $\nu\in\mathbb{Z}$ (see e.g. [14, Appendix B.4]). Regarding $K_0(\xi)$, it is known that, for $\xi\gg 1$,

$$K_0^{(\mu)}(\xi)\ll\frac{1}{e^{\xi}\sqrt{\xi}}\quad\text{for }\mu\ge 0,$$

and for $\xi\ll 1$,

$$K_0(\xi)\ll|\log\xi|\quad\text{and}\quad K_0^{(\mu)}(\xi)\ll 1/\xi^{\mu}\quad\text{for }\mu\ge 1.$$

Regarding the other Bessel function $Y_\nu(\xi)$, we know that, for $\xi\gg 1$,

$$Y_\nu^{(\mu)}(\xi)\ll 1/\sqrt{\xi}\quad\text{for }\nu,\mu\ge 0.$$

For $\xi\ll 1$, we have the following bounds:

$$\begin{aligned}Y_0(\xi)\ll|\log\xi|,\quad Y_0^{(\mu)}(\xi)\ll 1/\xi^{\mu}\quad\text{for }\mu\ge 1,\\ Y_\nu^{(\mu)}(\xi)\ll 1/\xi^{\nu+\mu}\quad\text{for }\nu\ge 1,\ \mu\ge 0.\end{aligned}$$

From the recurrence relation $(\xi^\nu Y_\nu(\xi))' = \xi^\nu Y_{\nu-1}(\xi)$, we get the identity

$$\int Y_0\left(4\pi\frac{\sqrt{h\xi}}{c}\right) f(\xi)\,d\xi = \left(\frac{-2c}{4\pi\sqrt{h}}\right)^\nu \int \xi^{\nu/2} Y_\nu\left(4\pi\frac{\sqrt{h\xi}}{c}\right) f^{(\nu)}(\xi)\,d\xi, \tag{2.1}$$

which is useful when estimating the sizes of the Bessel transforms appearing in the Voronoi summation formula.

The Bessel function $Y_\nu(\xi)$ oscillates for large values, and to make use of this behaviour we state the following lemma (see [27, Lemma 2.3]).

LEMMA 2.2. *For any $\nu \geq 0$, there is a smooth function $v_Y : (0,\infty) \to \mathbb{C}$ such that*

$$Y_\nu(\xi) = 2\,\mathrm{Re}\left(e\left(\frac{\xi}{2\pi}\right) v_Y\left(\frac{\xi}{\pi}\right)\right)$$

and, for any $\mu \geq 0$,

$$v_Y^{(\mu)}(\xi) \ll 1/\xi^{\mu+1/2} \qquad \text{for } \xi \gg 1,$$

where the implied constants depend on ν and μ.

2.2. The Hecke congruence subgroup and Kloosterman sums. Here and in the following sections we will go through some results from the theory of automorphic forms. A general account of the spectral theory of automorphic forms can be found for instance in [14] or [15, Chapters 14–16], while [12] gives a nice introduction to Maaß forms of higher weight with arbitrary nebentypus.

Besides the Kuznetsov formula, our main tools are the large sieve inequalities, which were proven by Deshouillers and Iwaniec [8] with respect to Hecke congruence subgroups. Their results can be extended to our specific setting, the details of which have been worked out by Drappeau [9]. We also want to cite [4] as a reference, from which we borrow parts of the notation.

Let q be some positive integer, let $\kappa \in \{0,1\}$, and let χ be a character mod q_0, with $q_0 \mid q$, such that

$$\chi(-1) = (-1)^\kappa.$$

Let $\Gamma := \Gamma_0(q)$ be the Hecke congruence subgroup of level q. The character χ naturally extends to Γ by setting

$$\chi(\gamma) := \chi(d) \qquad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

Every cusp $\mathfrak{a}$ of Γ is equivalent to some u/w with $(u,w) = 1$ and $w \mid q$. It is called *singular* if

$$\chi(\gamma) = 1 \qquad \text{for all } \gamma \in \Gamma_{\mathfrak{a}},$$

where $\Gamma_{\mathfrak{a}}$ is the stabilizer of $\mathfrak{a}$.

For any cusp $\mathfrak{a}$ of Γ we can choose $\sigma_{\mathfrak{a}} \in \mathrm{SL}_2(\mathbb{R})$ such that

$$\sigma_{\mathfrak{a}}\infty = \mathfrak{a} \quad \text{and} \quad \sigma_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}} = \Gamma_\infty.$$

Given two singular cusps $\mathfrak{a}$, $\mathfrak{b}$, we define for $n, m \in \mathbb{Z}$ the Kloosterman sum

$$S_{\mathfrak{a}\mathfrak{b}}(m,n;\gamma) := \sum_{\delta \bmod \gamma\mathbb{Z}} \overline{\chi}\left(\sigma_{\mathfrak{a}}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\sigma_{\mathfrak{b}}^{-1}\right) e\left(m\frac{\alpha}{\gamma} + n\frac{\delta}{\gamma}\right),$$

where the sum runs over all $\delta \bmod \gamma\mathbb{Z}$ for which there exist α and β such that

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}.$$

Note that this definition depends on the chosen scaling matrices $\sigma_{\mathfrak{a}}$ and $\sigma_{\mathfrak{b}}$.

As an example, for $\mathfrak{a} = \mathfrak{b} = \infty$ and the choice $\sigma_\infty = 1$, the sum is non-empty exactly when $q \mid c$, and in this case it reduces to the usual twisted Kloosterman sum

$$S_{\infty\infty}(m,n;c) = S_\chi(m,n;c) := \sum_{\substack{a\,(c)\\ (a,c)=1}} \chi(a) e\left(\frac{ma + n\overline{a}}{c}\right).$$

It is well-known that, for any prime p, this sum can be bounded by

$$S_\chi(m,n;p) \le 2(m,n,p)^{1/2}p^{1/2}.$$

However, for general modulus we have to account for the conductor of χ as well, and in this case we can use the following bound (see [18, Theorem 9.2]):

$$S_\chi(m,n;c) \ll (m,n,c)^{1/2}q_0^{1/2}c^{1/2+\varepsilon}.$$

Another important example is for $q = rs$ with $(r,s) = 1$ and $q_0 \mid r$. Consider the two singular cusps ∞ and $1/s$, together with the choice

$$\sigma_{1/s} = \begin{pmatrix}\sqrt{r} & 1\\ s\sqrt{r} & \sqrt{r}^{\,-1}\end{pmatrix}.$$

Now the sum $S_{\infty(1/s)}(m,n;\gamma)$ is non-empty exactly when γ may be written as

$$\gamma = \sqrt{r}\, sc \quad \text{with } c \in \mathbb{Z}\setminus\{0\}, \quad (c,r) = 1,$$

and in this case we have

$$S_{\infty(1/s)}(m,n;\gamma) = e(n\overline{s}/r)\overline{\chi}(c)S(m,n\overline{r};sc).$$

2.3. Automorphic forms and their Fourier expansions. We denote by $\mathcal{S}_k(q,\chi)$ the finite-dimensional Hilbert space of holomorphic cusp forms of weight $k \equiv \kappa \bmod 2$ with respect to $\Gamma_0(q)$ and with nebentypus χ. Let $\theta_k(q,\chi)$ be its dimension. For each k, we choose an orthonormal Hecke eigenbasis $f_{j,k}$, $1 \leq j \leq \theta_k(q,\chi)$. Then the Fourier expansion of $f_{j,k}$ around a singular cusp $\mathfrak{a}$ (with associated scaling matrix $\sigma_{\mathfrak{a}}$) is given by

$$i(\sigma_{\mathfrak{a}}, z)^{-k} f_{j,k}(\sigma_{\mathfrak{a}} z) = \sum_{n=1}^{\infty} \psi_{j,k}(n,\mathfrak{a})(4\pi n)^{k/2} e(nz),$$

where we have set

$$i(\gamma, z) := cz + d \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Next, let $\mathcal{L}^2(q,\chi)$ be the space of Maaß forms of weight κ with respect to $\Gamma_0(q)$ and with nebentypus χ, and let $\mathcal{L}_0^2(q,\chi) \subset \mathcal{L}^2(q,\chi)$ be its subspace of Maaß cusp forms. Let u_j, $j \geq 1$, run over an orthonormal Hecke eigenbasis of $\mathcal{L}_0^2(q,\chi)$, with the corresponding real eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots$. We can assume that each u_j is either even or odd. We set $t_j^2 = \lambda_j - 1/4$, where we choose the sign of t_j so that $it_j \geq 0$ if $\lambda_j < 1/4$, and $t_j \geq 0$ if $\lambda_j \geq 1/4$. Then the Fourier expansions of these functions around a singular cusp $\mathfrak{a}$ are given by

$$j(\sigma_{\mathfrak{a}}, z)^{-\kappa} u_j(\sigma_{\mathfrak{a}} z) = \sum_{n \neq 0} \rho_j(n,\mathfrak{a}) W_{\frac{n}{|n|}\frac{\kappa}{2}, it_j}(4\pi|n|y) e(nx),$$

where

$$j(\gamma, z) := \frac{cz+d}{|cz+d|} \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The Selberg eigenvalue conjecture says that $\lambda_1 \geq 1/4$, which would imply that all t_j are real and non-negative. While for $\kappa = 1$ this is known to be true, it remains an open question for $\kappa = 0$. The eigenvalues with $0 < \lambda_j < 1/4$, together with the corresponding values t_j, are called *exceptional*, and lower bounds for these exceptional λ_j imply upper bounds for the corresponding it_j. Let $\theta \in [0,\infty)$ be such that $it_j \leq \theta$ for all exceptional t_j uniformly for all levels q and any nebentypus; by the work of Kim and Sarnak [17], we know that we can choose

$$\theta = 7/64.$$

The orthogonal complement to $\mathcal{L}_0^2(q,\chi)$ in $\mathcal{L}^2(q,\chi)$ is the Eisenstein spectrum $\mathcal{E}(q,\chi)$ (plus possibly the space of constant functions if χ is trivial). It can be described explicitly by means of the Eisenstein series $E_{\mathfrak{c}}(z; 1/2 + it)$, where $\mathfrak{c}$ is a singular cusp and $t \in \mathbb{R}$. The Fourier expansions of these Eisenstein series around the cusp $\mathfrak{a}$ are given by

$$j(\sigma_{\mathfrak{a}}, z)^{-\kappa} E_{\mathfrak{c}}(\sigma_{\mathfrak{a}} z; 1/2+it) = c_{\mathfrak{c},1}(t) y^{1/2+it} + c_{\mathfrak{c},2}(t) y^{1/2-it} + \sum_{n \neq 0} \varphi_{\mathfrak{c},t}(n,\mathfrak{a}) W_{\frac{n}{|n|}\frac{\kappa}{2}, it}(4\pi|n|y) e(nx).$$

Note that by the choice of our basis, we have

$$|\rho_j(-n,\infty)| = |t_j|^{\kappa}|\rho_j(n,\infty)| \quad \text{for } n \ge 1.$$

Furthermore, since all Eisenstein series are even, the same is true for their Fourier coefficients:

$$|\varphi_{\mathfrak{c},t}(-n,\infty)| = |t|^{\kappa}|\varphi_{\mathfrak{c},t}(n,\infty)| \quad \text{for } n \ge 1.$$

2.4. The Kuznetsov formula. With the whole notation set up, we can now formulate the famous Kuznetsov formula, which in our case reads as follows.

THEOREM 2.3. *Let $f : (0,\infty) \to \mathbb{C}$ be smooth with compact support, let $\mathfrak{a}$, $\mathfrak{b}$ be singular cusps, and let m, n be positive integers. Then*

$$\sum_{\gamma} \frac{S_{\mathfrak{a}\mathfrak{b}}(m,n;\gamma)}{\gamma} f\left(4\pi\frac{\sqrt{mn}}{\gamma}\right) = \sum_{j=1}^{\infty} \overline{\rho_j}(m,\mathfrak{a})\rho_j(n,\mathfrak{b})\frac{\sqrt{mn}}{\cosh(\pi t_j)}\tilde{f}(t_j)$$
$$+ \sum_{\mathfrak{c}\ sing.} \frac{1}{4\pi}\int_{-\infty}^{\infty} \overline{\varphi_{\mathfrak{c},t}}(m,\mathfrak{a})\varphi_{\mathfrak{c},t}(n,\mathfrak{b})\frac{\sqrt{mn}}{\cosh(\pi t)}\tilde{f}(t)\,dt$$
$$+ \sum_{\substack{k\equiv\kappa\,(2),\,k>\kappa\\ 1\le j\le\theta_k(q,\chi)}} (k-1)!\,\overline{\psi_{j,k}}(m,\mathfrak{a})\psi_{j,k}(n,\mathfrak{b})\sqrt{mn}\,\dot{f}(k),$$

and

$$\sum_{\gamma} \frac{S_{\mathfrak{a}\mathfrak{b}}(m,-n;\gamma)}{\gamma} f\left(4\pi\frac{\sqrt{mn}}{\gamma}\right) = \sum_{j=1}^{\infty} \overline{\rho_j}(m,\mathfrak{a})\rho_j(-n,\mathfrak{b})\frac{\sqrt{mn}}{\cosh(\pi t_j)}\check{f}(t_j)$$
$$+ \sum_{\mathfrak{c}\ sing.} \frac{1}{4\pi}\int_{-\infty}^{\infty} \overline{\varphi_{\mathfrak{c},t}}(m,\mathfrak{a})\varphi_{\mathfrak{c},t}(-n,\mathfrak{b})\frac{\sqrt{mn}}{\cosh(\pi t)}\check{f}(t)\,dt,$$

where γ runs over all positive real numbers for which $S_{\mathfrak{a}\mathfrak{b}}(m,n;\gamma)$ is non-empty, and where the Bessel transforms are defined by

$$\tilde{f}(t) = \frac{2\pi i t^{\kappa}}{\sinh(\pi t)}\int_0^{\infty}\big(J_{2it}(\xi) - (-1)^{\kappa}J_{-2it}(\xi)\big)f(\xi)\,\frac{d\xi}{\xi},$$

$$\check{f}(t) = 8i^{-\kappa}\cosh(\pi t)\int_0^{\infty} K_{2it}(\xi)f(\xi)\,\frac{d\xi}{\xi},$$

$$\dot{f}(k) = 4i^{k}\int_0^{\infty} J_{k-1}(\xi)f(\xi)\,\frac{d\xi}{\xi}.$$

Proof. For $\mathfrak{a} = \mathfrak{b} = \infty$, the first formula was proven in [26], and the second in [5, Proposition 2]. The extension to our situation with general cusps is straightforward. ∎

For $\mathfrak{a} = \mathfrak{b} = \infty$, the sum of Kloosterman sums in the theorem above is

$$\sum_{\gamma} \frac{S_{\infty\infty}(m, \pm n; \gamma)}{\gamma} f\left(4\pi \frac{\sqrt{mn}}{\gamma}\right) = \sum_{c \equiv 0\,(q)} \frac{S_{\chi}(m, \pm n; c)}{c} f\left(4\pi \frac{\sqrt{mn}}{c}\right),$$

while in the case $q = rs$ with $(r, s) = 1$ and $q_0 \mid r$ mentioned above, we have

$$\sum_{\gamma} \frac{S_{\infty(1/s)}(m, \pm n; \gamma)}{\gamma} f\left(4\pi \frac{\sqrt{mn}}{\gamma}\right) = e\left(\frac{\pm n\overline{s}}{r}\right) \sum_{\substack{c \\ (c,r)=1}} \overline{\chi}(c) \frac{S(m, \pm n\overline{r}; sc)}{\sqrt{r}\, sc} f\left(4\pi \frac{\sqrt{mn}}{\sqrt{r}\, sc}\right). \tag{2.2}$$

To get some first estimates for the Bessel transforms appearing above, we refer to [3, Lemma 2.1], where the case $\kappa = 0$ is covered. The proofs carry over to the case $\kappa = 1$ with minimal changes.

LEMMA 2.4. *Let $f : (0, \infty) \to \mathbb{C}$ be a smooth and compactly supported function such that*

$$\operatorname{supp} f \asymp X, \qquad f^{(\nu)}(\xi) \ll 1/Y^{\nu} \quad \textit{for } \nu = 0, 1, 2,$$

for positive X and Y with $X \gg Y$. Then

$$\tilde{f}(it), \check{f}(it) \ll \frac{1 + Y^{-2t}}{1 + Y} \qquad \textit{for } 0 \le t < 1/4,$$

$$\frac{\tilde{f}(t)}{(1+t)^{\kappa}}, \check{f}(t), \dot{f}(t) \ll \frac{1 + |\log Y|}{1 + Y} \qquad \textit{for } t \ge 0,$$

$$\frac{\tilde{f}(t)}{(1+t)^{\kappa}}, \check{f}(t), \dot{f}(t) \ll \left(\frac{X}{Y}\right)^{2} \left(\frac{1}{t^{5/2}} + \frac{X}{t^{3}}\right) \quad \textit{for } t \gg \max(X, 1).$$

For certain oscillating functions, we can do better. Assume that the function $w : (0, \infty) \to \mathbb{C}$ is smooth and compactly supported, and satisfies

$$\operatorname{supp} w \asymp X, \qquad w^{(\nu)}(\xi) \ll 1/X^{\nu} \quad \text{for } \nu \ge 0.$$

For $\alpha > 0$ define

$$f(\xi) := e\left(\xi \frac{\alpha}{2\pi}\right) w(\xi).$$

Then we have the following bounds for the Bessel transforms of f.

LEMMA 2.5. *Assume that $X \ll 1$ and $\alpha X \gg 1$. Then, for $\nu, \mu \ge 0$,*

$$\tilde{f}(it), \check{f}(it) \ll X^{-2t+\varepsilon} \left(X^{\mu} + \frac{1}{(\alpha X)^{\nu}}\right) \quad \textit{for } 0 < t \le 1/4, \tag{2.3}$$

$$\frac{\tilde{f}(t)}{(1+t)^{\kappa}}, \check{f}(t), \dot{f}(t) \ll \frac{\alpha^{\varepsilon}}{\alpha X} \left(\frac{\alpha X}{t}\right)^{\nu} \qquad \textit{for } t > 0. \tag{2.4}$$

Proof. For $\kappa = 0$, a proof of these bounds can be found in Lemma 2.6 in [27]. It carries over to the case $\kappa = 1$ without any difficulties. ∎

2.5. Large sieve inequalities and estimates for Fourier coefficients. Another important tool is the large sieve inequalities for Fourier coefficients of cusp forms and Eisenstein series. Let $\mathfrak{a}$ be a singular cusp of Γ written in the form $\mathfrak{a} = u/w$ with $(u,w)=1$. For a sequence a_n of complex numbers we set

$$\Sigma^{(1)}_{j,\pm}(N) := \frac{(1+|t_j|)^{\pm\kappa/2}}{\sqrt{\cosh(\pi t_j)}} \sum_{N<n\le 2N} a_n \rho_j(\pm n,\mathfrak{a})\sqrt{n},$$

$$\Sigma^{(2)}_{\mathfrak{c},t,\pm}(N) := \frac{(1+|t|)^{\pm\kappa/2}}{\sqrt{\cosh(\pi t)}} \sum_{N<n\le 2N} a_n \varphi_{\mathfrak{c},t}(\pm n,\mathfrak{a})\sqrt{n},$$

$$\Sigma^{(3)}_{j,k}(N) := \sqrt{(k-1)!} \sum_{N<n\le 2N} a_n \psi_{j,k}(n,\mathfrak{a})\sqrt{n}.$$

Then the following bounds are known as the *large sieve inequalities*.

THEOREM 2.6. *Let $T\ge 1$, $N\ge 1/2$, and $\mathfrak{a}$ as above. Let a_n be a sequence of complex numbers. Then*

$$\sum_{|t_j|\le T} |\Sigma^{(1)}_{j,\pm}(N)|^2 \ll \left(T^2 + q_0^{1/2}\left(w,\frac{q}{w}\right)\frac{N^{1+\varepsilon}}{q}\right) \sum_{N<n\le 2N} |a_n|^2,$$

$$\sum_{\mathfrak{c}\ sing.} \int_{-T}^{T} |\Sigma^{(2)}_{\mathfrak{c},t,\pm}(N)|^2\,dt \ll \left(T^2 + q_0^{1/2}\left(w,\frac{q}{w}\right)\frac{N^{1+\varepsilon}}{q}\right) \sum_{N<n\le 2N} |a_n|^2,$$

$$\sum_{\substack{k\le T,\, k\equiv\kappa\,(2)\\ 1\le j\le \theta_k(q,\chi)}} |\Sigma^{(3)}_{k,j}(N)|^2 \ll \left(T^2 + q_0^{1/2}\left(w,\frac{q}{w}\right)\frac{N^{1+\varepsilon}}{q}\right) \sum_{N<n\le 2N} |a_n|^2,$$

where the implicit constants depend only on ε.

Proof. With the appropriate changes, these bounds can be deduced essentially in the same way as in [8, Section 5]. We refer to [9] for details. ∎

When there is no averaging over n, the following lemma gives useful bounds, especially when q or T is large.

LEMMA 2.7. *Let $T,n\ge 1$, and $\mathfrak{a}$ as above. Then*

$$\sum_{|t_j|\le T} \frac{(1+|t_j|)^{\pm\kappa}}{\cosh(\pi t_j)} |\rho_j(\pm n,\mathfrak{a})|^2 n \ll T^2 + (qnT)^{\varepsilon}\sqrt{(q,n)q_0}\left(w,\frac{q}{w}\right)\frac{\sqrt{n}}{q},$$

$$\sum_{\mathfrak{c}\ sing.} \int_{-T}^{T} \frac{(1+|t|)^{\pm\kappa}}{\cosh(\pi t)} |\varphi_{\mathfrak{c},t}(\pm n,\mathfrak{a})|^2 n\,dt \ll T^2 + (qnT)^{\varepsilon}\sqrt{(q,n)q_0}\left(w,\frac{q}{w}\right)\frac{\sqrt{n}}{q},$$

$$\sum_{\substack{k\le T,\, k\equiv\kappa\,(2)\\ 1\le j\le \theta_k(q,\chi)}} (k-1)!|\psi_{j,k}(n,\mathfrak{a})|^2 n \ll T^2 + (qnT)^{\varepsilon}\sqrt{(q,n)q_0}\left(w,\frac{q}{w}\right)\frac{\sqrt{n}}{q},$$

where the implicit constants depend only on ε.

Proof. For the full modular group and trivial nebentypus, a proof for the first two bounds can be found for example in [24, Lemma 2.4]. If we use an appropriate formula as a starting point (e.g. [12, Proposition 5.2]), the proof carries over easily to our case. Except for the same kind of modifications, the proof of the last bound is a simpler variant of [8, Proposition 4]. ∎

For n large, the following bounds are often better.

LEMMA 2.8. *Let $T, n \geq 1$. Then*

$$\sum_{|t_j|\leq T} \frac{(1+|t_j|)^{\pm\kappa}}{\cosh(\pi t_j)} |\rho_j(\pm n,\infty)|^2 n \ll (qnT)^{\varepsilon} T^2 n^{2\theta}, \tag{2.5}$$

$$\sum_{\mathfrak{c}\ sing.} \int_{-T}^{T} \frac{(1+|t|)^{\pm\kappa}}{\cosh(\pi t)} |\varphi_{\mathfrak{c},t}(\pm n,\infty)|^2 n\, dt \ll (qnT)^{\varepsilon} T, \tag{2.6}$$

$$\sum_{\substack{k\leq T,\, k\equiv\kappa\,(2)\\ 1\leq j\leq\theta_k(q,\chi)}} (k-1)!|\psi_{j,k}(n,\infty)|^2 n \ll (qnT)^{\varepsilon} T^2, \tag{2.7}$$

where the implicit constants depend only on ε.

Proof. The bounds (2.5) and (2.7) can be proven along the lines of [22, Proposition 2.3]. For (2.6) we refer to [6, Lemma 1]. ∎

Finally, in order to be able to handle exceptional eigenvalues, which occur in the case $\kappa = 0$, the following result will turn out to be useful.

LEMMA 2.9. *Let $X, n \geq 1$, and $\mathfrak{a}$ as above. Assume that*

$$X \gg X_0 \quad \textit{with} \quad X_0 := \frac{q}{(q,n)^{1/2} q_0^{1/2} (w, q/w) n^{1/2}}.$$

Then

$$\sum_{t_j\ exc.} \frac{|\rho_j(\pm n,\mathfrak{a})|^2 n}{\cosh(\pi t_j)} X^{4it_j} \ll (qnX)^{\varepsilon} \left(\frac{X}{X_0}\right)^{4\theta} \left(1+(q,n)^{1/2} q_0^{1/2} \left(w, \frac{q}{w}\right) \frac{n^{1/2}}{q}\right),$$

where the implicit constants only depend on ε.

Proof. We have

$$\sum_{t_j\ \text{exc.}} \frac{|\rho_j(\pm n,\mathfrak{a})|^2 n}{\cosh(\pi t_j)} X^{4it_j} \ll \left(\frac{X}{X_0}\right)^{4\theta} \sum_{t_j\ \text{exc.}} \frac{|\rho_j(\pm n,\mathfrak{a})|^2 n}{\cosh(\pi t_j)} (1+X_0)^{4it_j}.$$

Now we use the fact that, for any $Y \geq 1$,

$$\sum_{t_j\ \text{exc.}} \frac{|\rho_j(\pm n,\mathfrak{a})|^2 n}{\cosh(\pi t_j)} Y^{4it_j} \ll 1 + (qnY)^{\varepsilon} (q,n)^{1/2} q_0^{1/2} \left(w, \frac{q}{w}\right) \frac{n^{1/2} Y}{q},$$

which can be shown as in [15, Chapter 16.5], and the result follows. ∎

3. Proof of Theorems 1.1 and 1.2. Let $w_1, w_2 : (0,\infty) \to [0,\infty)$ be smooth functions which are compactly supported in $[1/2,1]$ and satisfy

$$w_i^{(\nu)}(\xi) \ll 1/\Omega^\nu \quad \text{for } \nu \ge 0, \qquad \int |w_i^{(\nu)}(\xi)|\,d\xi \ll 1/\Omega^{\nu-1} \quad \text{for } \nu \ge 1,$$

for some $\Omega \le 1$. We will look at the sum

$$D(x_1,x_2) := \sum_n w_1\left(\frac{r_1 n + f_1}{x_1}\right) w_2\left(\frac{r_2 n + f_2}{x_2}\right) d(r_1 n + f_1) d(r_2 n + f_2),$$

with the aim of showing that it can be written asymptotically as

$$D(x_1,x_2) = M(x_1,x_2) + R(x_1,x_2),$$

where $M(x_1,x_2)$ denotes the main term, which has the form

$$M(x_1,x_2) = \int w_1\left(\frac{r_1\xi + f_1}{x_1}\right) w_2\left(\frac{r_2\xi + f_2}{x_2}\right) \cdot P(\log(r_1\xi + f_1), \log(r_2\xi + f_2))\,d\xi \tag{3.1}$$

with a quadratic polynomial $P(\xi_1,\xi_2)$, and where $R(x_1,x_2)$ denotes the error term. The assumptions we need to make are

$$f_1 \ll x_1^{1-\varepsilon}, \qquad f_2 \ll x_2^{1-\varepsilon}, \qquad h \ll \Omega^2 r_2 x_1^{1-\varepsilon}. \tag{3.2}$$

We can also assume that

$$r_0^2 r_1^2 r_2 \ll x_1,$$

since otherwise our results become trivial. Note that from the first two bounds in (3.2) and the size of the supports of w_1 and w_2, it follows that

$$r_2 x_1 \asymp r_1 x_2.$$

We will prove the following three bounds for the error term:

$$R(x_1,x_2) \ll r_0 (r_2 x_1)^{1/2+\varepsilon}\left(\frac{|h|^\theta}{\Omega^{1/2}} + (r_2 x_1)^\theta\right), \tag{3.3}$$

$$R(x_1,x_2) \ll r_0 (r_2 x_1)^{1/2+\varepsilon} \cdot \left(\frac{1}{\Omega^{1/2}} + \left(\frac{(r_0 r_1 r_2, h) x_1}{r_0 r_1^2 r_2}\right)^\theta \left(1 + \frac{(r_0 r_1 r_2, h)^{1/4} |h|^{1/4}}{r_0^{1/4}(r_1 r_2)^{1/2}}\right)\right), \tag{3.4}$$

$$R(x_1,x_2) \ll r_0 (r_2 x_1)^{1/2+\varepsilon} \cdot \left(\frac{1}{\Omega^{1/2}} + \left(\frac{r_2 x_1}{|h|}\right)^\theta \left(1 + \frac{(r_0 r_1 r_2, h)^{1/4} |h|^{1/4}}{r_0^{1/4}(r_1 r_2)^{1/2}}\right)\right). \tag{3.5}$$

Recall that r_0 was defined in (1.4). From the first bound and with the choice $\Omega = 1$, we immediately get Theorem 1.1. In order to prove Theorem 1.2, we choose

$$\Omega = \frac{r_0^{2/3} r_1^{2/3} r_2^{1/3}}{x_1^{1/3}},$$

and use the second bound (3.4) to show that

$$R(x_1,x_2) \ll (r_0r_1r_2,h)^\theta r_0^{2/3+\theta}(r_1r_2)^{1/3}\left(\frac{x_1}{r_1}\right)^{2/3+\varepsilon} \tag{3.6}$$

for h satisfying

$$(r_0r_1r_2,h)h \ll \frac{(r_1r_2)^{4/3}}{r_0^{1/3}}\left(\frac{x_1}{r_1}\right)^{2/3}\left(\frac{r_0^2r_1^2r_2}{x_1}\right)^{4\theta}.$$

Unfortunately, due to the presence of θ, the possible range for h is weakened considerably in its size, even if we take the currently best value for this constant. We can improve this slightly, however, by making use of the third bound (3.5) to show that the bound (3.6) also holds in the range

$$\frac{(r_1r_2)^{4/3}}{r_0^{1/3}}\left(\frac{x_1}{r_1}\right)^{2/3}\left(\frac{r_0^2r_1^2r_2}{x_1}\right)^{4\theta} \ll (r_0r_1r_2,h)h \ll r_0^{1/3}r_1^{4/3}r_2^{5/3}x_1^{1/3-\varepsilon}.$$

Theorem 1.2 follows by setting $x_1 = r_1x$, $x_2 = r_2x$ and using suitable weight functions.

Before diving into the proof, we first describe a smooth decomposition of the divisor function which was used by Meurman [21] to treat the binary additive divisor problem (and which goes back to Heath-Brown). Let $v_0 : \mathbb{R} \to [0,\infty)$ be a smooth and compactly supported function such that

$$v_0(\xi) = 1 \quad \text{for } |\xi| \le 1, \qquad v_0(\xi) = 0 \quad \text{for } |\xi| \ge 2,$$

and set

$$v(\xi) := v_0\left(\frac{\xi}{\sqrt{x_2}}\right), \qquad h(a,b) := v(a)(2-v(b)).$$

For $ab \le x_2$, we have $(v(a)-1)(v(b)-1) = 0$, so that, for $n \le x_2$,

$$d(n) = \sum_{ab=n} v(a)(2-v(b)) = \sum_{ab=n} h(a,b).$$

It will furthermore be helpful to dyadically split the supports of the variables a and b. In order to do so, we choose smooth and compactly supported functions $u_X : (0,\infty) \to [0,\infty)$ such that

$$\operatorname{supp} u_X \subset [X/2, 2X], \qquad u_X^{(\nu)}(\xi) \ll 1/X^\nu \quad \text{for } \nu \ge 0, \qquad \sum_X u_X \equiv 1,$$

where the last sum runs over powers of 2. Then we set

$$h_{AB}(a,b) := h(a,b)u_A(a)u_B(b).$$

Back to our sum—we split the second divisor function and use the dyadic decomposition described just before, so that we can write

$$D(x_1,x_2) = \sum_{A,B} D_{AB}(x_1,x_2),$$

where

$$D_{AB}(x_1,x_2) := \sum_n w_1\Big(\frac{r_1n+f_1}{x_1}\Big)w_2\Big(\frac{r_2n+f_2}{x_2}\Big)d(r_1n+f_1)\sum_{\substack{a,b\\ ab=r_2n+f_2}} h_{AB}(a,b)$$

$$= \sum_{\substack{a,b\\ ab\equiv f_2\,(r_2)}} \tilde f(a,b)d\Big(\frac{r_1}{r_2}(ab-f_2)+f_1\Big),$$

$$\tilde f(a,b) := w_1\bigg(\frac{\frac{r_1}{r_2}(ab-f_2)+f_1}{x_1}\bigg)w_2\Big(\frac{ab}{x_2}\Big)h_{AB}(a,b).$$

Note that the variables A and B, which run over powers of 2, satisfy

$$AB \asymp x_2, \qquad A \ll B, \qquad A \ll x_2^{1/2}.$$

In the following, we have to pay a lot of attention to possible common divisors between the different parameters, and it will be helpful to define, for $i=1,2$,

$$u_i := (r_i,f_i), \qquad s_i := \frac{r_i}{u_i}, \qquad g_i := \frac{f_i}{u_i}, \qquad h := r_1f_2 - f_1r_2, \qquad h_0 := \frac{h}{u_1u_2}.$$

Now, since the product ab in the above sum must be divisible by u_2, we can write

$$D_{AB}(x_1,x_2) = \sum_{u_2^*|u_2}\ \sum_{\substack{a\\ (a,s_2u_2^*)=1}}\ \sum_{\substack{b\\ ab\equiv g_2\,(s_2)}} \tilde f\Big(\frac{u_2}{u_2^*}a, u_2^*b\Big)d\Big(\frac{r_1}{s_2}(ab-g_2)+f_1\Big).$$

Choose $\tilde a$ and $\tilde s_2$ such that $a\tilde a + s_2\tilde s_2 = 1$, so that b in the above sum has the form

$$b = \tilde a g_2 + s_2 n \quad \text{with } n\in\mathbb{Z},$$

and hence

$$D_{AB}(x_1,x_2) = \sum_{\substack{u_2^*|u_2\\ a,n\\ (a,s_2u_2^*)=1}} \tilde f\Big(\frac{u_2}{u_2^*}a, \frac{u_2^*}{u_2a}(r_2(an-g_2\tilde s_2)+f_2)\Big) \cdot d(r_1(an-g_2\tilde s_2)+f_1)$$

$$= \sum_{\substack{u_2^*|u_2\\ a\\ (a,s_2u_2^*)=1}}\ \sum_{n\equiv f_1-g_2r_1\overline{s_2}\,(r_1a)} d(n)f(n;a)$$

with

$$f(\xi;a) := w_1\Big(\frac{\xi}{x_1}\Big)w_2\bigg(\frac{\frac{r_2}{r_1}(\xi-f_1)+f_2}{x_2}\bigg)h_{AB}\Big(\frac{u_2}{u_2^*}a, \frac{u_2^*}{u_2a}\Big(\frac{r_2}{r_1}(\xi-f_1)+f_2\Big)\Big).$$

Note that the modular inverse $\overline{s_2}$, which occurs in the congruence condition, is understood to be mod a. Also note that the support of $f(\xi;a)$ is given by

$$\operatorname{supp} f(\bullet;a) \asymp x_1, \qquad \operatorname{supp} f(\xi;\bullet) \asymp \frac{u_2^*}{u_2}A,$$

and that its derivatives can be bounded by

$$\frac{\partial^{\nu_1+\nu_2}}{\partial\xi^{\nu_1}a^{\nu_2}} f(\xi;a) \ll \frac{1}{(x_1\Omega)^{\nu_1}}\left(\frac{u_2}{u_2^*A}\right)^{\nu_2} \qquad \text{for } \nu_1,\nu_2 \geq 0,$$

while also satisfying

$$\int \left| \frac{\partial^{\nu_1+\nu_2}}{\partial\xi^{\nu_1}a^{\nu_2}} f(\xi;a)\right| d\xi \ll \frac{1}{(x_1\Omega)^{\nu_1-1}}\left(\frac{u_2}{u_2^*A}\right)^{\nu_2} \quad \text{for } \nu_1 \geq 1,\ \nu_2 \geq 0.$$

3.1. Use of Voronoi summation. We use Voronoi summation in the form of Theorem 2.1 to treat the divisor sum in arithmetic progressions. This way we are led to

$$D_{AB}(x_1,x_2) = \Sigma_{AB}^0 - 2\pi\Sigma_{AB}^+ + 4\Sigma_{AB}^-$$

with

$$\Sigma_{AB}^0 := \frac{1}{r_1} \sum_{\substack{u_2^*\mid u_2 \\ a \\ (a,s_2u_2^*)=1}} \frac{1}{a}\int \lambda_{f_1-g_2r_1\overline{s_2},r_1a}(\xi) f(\xi;a)\,d\xi,$$

$$\Sigma_{AB}^\pm := \frac{1}{r_1} \sum_{\substack{u_2^*\mid u_2 \\ a \\ (a,s_2u_2^*)=1}} \sum_{c\mid r_1a} \frac{c}{a} \sum_{n=1}^{\infty} d(n) \frac{S(f_1-g_2r_1\overline{s_2},\pm n;c)}{c^2} \cdot \int B^\pm\left(4\pi\frac{\sqrt{n\xi}}{c}\right) f(\xi;a)\,d\xi,$$

and

$$B^+(\xi) := Y_0(\xi), \qquad B^-(\xi) := K_0(\xi).$$

The main term will be extracted from Σ_{AB}^0, but we will postpone this until the end and first take care of $\Sigma_{AB}^\pm$.

We reshape these sums a little:

$$\Sigma_{AB}^\pm = \frac{1}{r_1}\sum_{u_2^*\mid u_2} \sum_{\substack{a,c \\ c\mid r_1a \\ (a,s_2u_2^*)=1}} (\dots) = \frac{1}{r_1}\sum_{\substack{u_2^*\mid u_2 \\ r_1^*\mid r_1}} \sum_{\substack{d \\ (d,r_1)=r_1^*}} \sum_{\substack{a,c \\ dc=r_1a \\ (a,s_2u_2^*)=1}} (\dots)$$

$$= \frac{1}{r_1}\sum_{\substack{u_2^*\mid u_2 \\ r_1^*\mid r_1}} \sum_{\substack{d \\ (d,r_1^*s_2u_2^*)=1}} \sum_{\substack{c \\ (c,s_2u_2^*)=1}} (\dots),$$

where we have to replace c by r_1^*c and a by dc, so that

$$\Sigma_{AB}^{\pm} = \sum_{\substack{u_2^*|u_2 \\ r_1^*|r_1}} \sum_{\substack{d \\ (d,r_1^*s_2u_2^*)=1}} \frac{R_{AB}^{\pm}}{d}$$

with

$$R_{AB}^{\pm} := \sum_{\substack{c \\ (c,s_2u_2^*)=1}} \sum_{n=1}^{\infty} d(n) \frac{S(f_1 - g_2 r_1 \overline{s_2}, \pm n; r_1^* c)}{r_1^* c} F^{\pm}(r_1^* c; dc, n),$$

$$F^{\pm}(\eta; a, n) := \frac{r_1^*}{\eta r_1} \int B^{\pm}\left(4\pi \frac{\sqrt{n\xi}}{\eta}\right) f(\xi; a)\, d\xi.$$

As a reminder, the modular inverse $\overline{s_2}$ occurring in the Kloosterman sum is now understood to be mod dc.

Let

$$N_0^- := \frac{x_1^{\varepsilon}}{x_1} A^{*2}, \qquad N_0^+ := \frac{x_1^{\varepsilon}}{x_1 \Omega^2} A^{*2}, \qquad A^* := \frac{u_2^*}{u_2} \frac{r_1^* A}{d}.$$

Regarding $F^{\pm}(r_1^* c; dc, n)$, we have the bounds

$$F^+(r_1^* d; dc, n) \ll \frac{(x_1 \Omega)^{1/2}}{n^{1/2}} \left(\frac{A^*}{\sqrt{n x_1}\, \Omega}\right)^{\nu - 1/2},$$

$$F^-(r_1^* d; dc, n) \ll \frac{x_1^{1/2}}{n^{1/2}} \left(\frac{A^*}{\sqrt{n x_1}}\right)^{\nu - 1/2},$$

which can be shown using (2.1). With the help of these bounds, it is not hard to see that the sum over n in $R_{AB}^{\pm}$ can be cut at $N_0^{\pm}$. After dyadically dividing the remaining sum, we are left with

$$R_{AB}^{\pm}(N) := \sum_{\substack{c \\ (c,s_2u_2^*)=1}} \sum_{N<n\le 2N} d(n) \frac{S(f_1 - g_2 r_1 \overline{s_2}, \pm n; r_1^* c)}{r_1^* c} F^{\pm}(r_1^* c; dc, n).$$

3.2. Treatment of the Kloosterman sums. Not surprisingly, we would like to handle the sum of the Kloosterman sums occurring in $R_{AB}^{\pm}(N)$ using the Kuznetsov formula. However, in our situation this does not seem to be possible directly. To deal with this difficulty, we factor out the part of the variable r_1^* which has the same prime factors as $s_2 u_2^*$,

$$v := (r_1^*, (s_2 u_2^*)^{\infty}), \quad t_1 := r_1^*/v,$$

and use the twisted multiplicativity of Kloosterman sums,

$$\frac{S(f_1 - g_2 r_1 \overline{s_2}, \pm n; r_1^* c)}{r_1^* c} = \frac{S(f_1 \overline{c t_1}, \pm n \overline{c t_1}; v)}{v} \frac{S(h_0 u_1, \pm n \overline{v}^2 \overline{s_2}; c t_1)}{c t_1}.$$

Here, each modular inverse is finally understood to be modulo the respective modulus of the Kloosterman sum. Obviously the first factor still depends on c, but here we follow an idea of Blomer and Milićević [6] and use Dirichlet characters to separate this variable. We define

$$\hat{S}_v(\chi;n) := \sum_{\substack{y\,(v)\\(y,v)=1}} \overline{\chi}(y)\frac{S(f_1\overline{y},\pm n\overline{y};v)}{v},$$

where χ is a Dirichlet character modulo v, so that by the orthogonality of Dirichlet characters we have

$$\frac{S(f_1\overline{ct_1},\pm n\overline{ct_1};v)}{v} = \frac{1}{\varphi(v)}\sum_{\chi \bmod v}\overline{\chi}(ct_1)\hat{S}_v(\overline{\chi};n),$$

where the sum runs over all Dirichlet characters modulo v. Hence

$$R^{\pm}_{AB}(N) = \frac{1}{\varphi(v)}\sum_{\chi \bmod v}\overline{\chi}(t_1)R^{\pm}_{AB}(N;\chi)$$

with

$$R^{\pm}_{AB}(N;\chi) := \sum_{N<n\le 2N} d(n)\hat{S}_v(\overline{\chi};n)K^{\pm}_{AB}(\chi;n),$$

$$K^{\pm}_{AB}(n;\chi) := \sum_{\substack{c\\(c,s_2u_2^*)=1}} \frac{S(h_0u_1u_2^*,\pm n\overline{s_2u_2^*v^2};t_1c)}{t_1c}\overline{\chi}(c)F^{\pm}(r_1^*c;dc,n).$$

Of course it is important to have good bounds for $\hat{S}_v(\chi;n)$. Directly using Weil's bound for Kloosterman sums we get

$$\hat{S}_v(\chi;n) \ll (f_1,n,v)^{1/2}v^{1/2+\varepsilon},$$

but this can be improved with a little effort, and the remainder of this section will be concerned with proving the improved bound

$$\hat{S}_v(\chi;n) \ll (f_1,n,v/\mathrm{cond}(\chi))v^{\varepsilon}, \tag{3.7}$$

where $\mathrm{cond}(\chi)$ is the conductor of χ. The sum actually vanishes in a lot of cases, in particular when f_1, n and v have certain common factors, but this result will be sufficient for our purposes. At this point, we also mention that

$$\frac{1}{\varphi(v)}\sum_{\chi \bmod v}\frac{v}{\mathrm{cond}(\chi)} = \frac{v}{\varphi(v)}\sum_{v^*|v}\frac{1}{v^*}\sum_{\substack{\chi \bmod v\\ \mathrm{cond}(\chi)=v^*}} 1 \le \frac{v}{\varphi(v)}d(v) \ll v^{\varepsilon}, \tag{3.8}$$

which will be useful later.

In order to prove (3.7), note first that $\hat{S}_v(\chi;n)$ is quasi-multiplicative in the sense that if $v=v_1v_2$ with coprime v_1 and v_2, and $\chi=\chi_1\chi_2$ with the corresponding Dirichlet characters $\chi_1 \pmod{v_1}$ and $\chi_2 \pmod{v_2}$, then

$$\hat{S}_v(\chi;n) = \chi_1(v_2)\chi_2(v_1)\hat{S}_{v_1}(\chi_1;n)\hat{S}_{v_2}(\chi_2;n).$$

It is therefore enough to look at the case where v is a prime power, $v=p^k$.

Assume first that $\chi = \chi_0$ is the principal character. For $v = p$, we have

$$\hat{S}_p(\chi; n) = \frac{1}{p} \sum_{\substack{x,y\,(p)\\(x,p)=1}} e\left(\frac{y(f_1 x \pm n\overline{x})}{p}\right) - \frac{\varphi(p)}{p}$$

$$= \sum_{\substack{x\,(p)\\ f_1 x \pm n\overline{x} \equiv 0\,(p)}} 1 - \frac{\varphi(p)}{p} \ll (f_1, n, p),$$

and for prime powers $v = p^k$, $k \geq 2$, we have

$$\hat{S}_{p^k}(\chi; n) = \frac{1}{p^k} \sum_{\substack{x,y\,(p^k)\\(x,p)=1}} e\left(\frac{y(f_1 x \pm n\overline{x})}{p^k}\right) - \frac{1}{p^k} \sum_{\substack{x\,(p^k),\, y\,(p^{k-1})\\(x,p)=1}} e\left(\frac{y(f_1 x \pm n\overline{x})}{p^{k-1}}\right)$$

$$= \#\{x\,(p^k) \mid f_1 x \pm n\overline{x} \equiv 0\,(p^k)\} - \frac{1}{p}\#\{x\,(p^k) \mid f_1 x \pm n\overline{x} \equiv 0\,(p^{k-1})\}$$

$$\ll (f_1, n, p^k).$$

We can now assume that χ is non-principal. For $v = p$ prime this means that χ is primitive, so that we can transform $\hat{S}_p(\chi; n)$ as follows:

$$\hat{S}_p(\chi; n) = \frac{1}{p} \sum_{\substack{x,y\,(p)\\(xy,p)=1}} \chi(y) e\left(\frac{y(f_1 x \pm n\overline{x})}{p}\right)$$

$$= \frac{1}{p} \sum_{\substack{x,y\,(p)\\ f_1 x \pm n\overline{x} \not\equiv 0\,(p)}} \chi(y)\overline{\chi}(f_1 x \pm n\overline{x}) e\left(\frac{y}{p}\right) - \frac{1}{p} \sum_{\substack{x,y\,(p)\\ f_1 x \pm n\overline{x} \equiv 0\,(p)}} \chi(y)$$

$$= \frac{\tau(\chi)}{p}\left(\sum_{\substack{x\,(p)\\(x,p)=1}} \overline{\chi}(f_1 x \pm n\overline{x})\right).$$

Both the Gauß sum $\tau(\chi)$ on the left and the character sum on the right are bounded by $\mathcal{O}(\sqrt{p})$, which is well-known for the former and follows from Weil's work for the latter (see e.g. [15, Theorem 11.23] or [19, Chapter 6, Theorem 3]). As a consequence, we get the bound

$$\hat{S}_p(\chi; n) \ll 1.$$

It remains to look at the case of χ having modulus $v = p^k$, $k \geq 2$, which is slightly more complicated. Let χ be induced by the primitive character χ^* of modulus $v^* = p^{k^*}$, and set $v^\circ := p^{k-k^*}$. In our sum

$$\hat{S}_{p^k}(\chi; n) = \frac{1}{p^k} \sum_{\substack{x\,(p^k)\\(x,p^k)=1}} \sum_{y\,(p^k)} \chi(y) e\left(\frac{y(f_1 x \pm n\overline{x})}{p^k}\right)$$

we parametrize y by

$$y = y_1 + v^* y_2 \qquad \text{with } y_1 \bmod v^* \text{ and } y_2 \bmod v^\circ.$$

Then

$$\begin{aligned}
\hat{S}_{p^k}(\chi; n) &= \frac{1}{v} \sum_{\substack{x\,(v)\\(x,v)=1}} \sum_{y_1\,(v^*)} \chi^*(y_1) e\left(\frac{y_1(f_1 x \pm n\overline{x})}{v}\right) \sum_{y_2\,(v^\circ)} e\left(\frac{y_2(f_1 x \pm n\overline{x})}{v^\circ}\right) \\
&= \frac{1}{v^*} \sum_{\substack{x\,(v)\\(x,v)=1\\ f_1 x \pm n\overline{x} \equiv 0\,(v^\circ)}} \sum_{y_1\,(v^*)} \chi^*(y_1) e\left(\frac{y_1(f_1 x \pm n\overline{x})}{v}\right) \\
&= \frac{\tau(\chi^*)}{v^*} \sum_{\substack{x\,(v)\\(x,v)=1\\ f_1 x \pm n\overline{x} \equiv 0\,(v^\circ)}} \overline{\chi^*}\left(\frac{f_1 x \pm n\overline{x}}{v^\circ}\right).
\end{aligned}$$

We set

$$\tilde{v}^\circ := \frac{v^\circ}{(f_1, n, v^\circ)}, \qquad \tilde{v} := v^* \tilde{v}^\circ, \qquad \tilde{f}_1 := \frac{f_1}{(f_1, n, v^\circ)}, \qquad \tilde{n} := \frac{n}{(f_1, n, v^\circ)},$$

and the sum becomes

$$\hat{S}_{p^k}(\chi; n) = (f_1, n, v^\circ) \frac{\tau(\chi^*)}{v^*} \sum_{\substack{x\,(\tilde{v})\\ \tilde{f}_1 x \pm \tilde{n}\overline{x} \equiv 0\,(\tilde{v}^\circ)}} \overline{\chi^*}\left(\frac{\tilde{f}_1 x \pm \tilde{n}\overline{x}}{\tilde{v}^\circ}\right).$$

If $\tilde{v}^\circ = 1$, we have square-root cancellation for the character sum on the right (see [29, Theorem 2]), so that $\hat{S}_{p^k}(\chi; n) \ll (f_1, n, v^\circ)$.

Otherwise we can assume that both $\tilde{f}_1$ and $\tilde{n}$ are coprime to p, or else the sum is empty. We parametrize x by

$$x = x_1(1 + \tilde{v}^\circ x_2) \qquad \text{with } x_1 \bmod \tilde{v}^\circ,\ (x_1, \tilde{v}^\circ) = 1 \text{ and } x_2 \bmod v^*.$$

In this case, we can write $\overline{x} \bmod \tilde{v}$ in the following way:

$$\overline{x} \equiv \overline{x_1}\big(1 - \tilde{v}^\circ x_2 \overline{(1 + \tilde{v}^\circ x_2)}\big) \bmod \tilde{v},$$

and after putting this in our sum, we have

$$\hat{S}_{p^k}(\chi; n) = (f_1, n, v^\circ) \frac{\tau(\chi^*)}{v^*} \sum_{\substack{x_1\,(\tilde{v}^\circ)\\ \tilde{f}_1 x_1 \pm \tilde{n}\overline{x_1} \equiv 0\,(\tilde{v}^\circ)}} \sum_{x_2\,(v^*)} \overline{\chi^*}(P(x_2)),$$

where $P(X)$ is the rational function

$$P(X) := \frac{\tilde{f}_1 x_1 \tilde{v}^\circ X^2 + 2\tilde{f}_1 x_1 X + \frac{\tilde{f}_1 x_1 \pm \tilde{n}\overline{x_1}}{\tilde{v}^\circ}}{\tilde{v}^\circ X + 1}.$$

If $p \geq 3$, we can use [7, Theorem 1.1] to get

$$\sum_{x_2\,(v^*)} \overline{\chi^*}(P(x_2)) \ll 1.$$

If $p = 2$ and $\tilde{v}^\circ \geq 8$, we rewrite this sum as

$$\sum_{x_2\,(v^*)} \overline{\chi^*}(P(x_2)) = \sum_{x_2\,(2v^*)} \overline{\chi^*}\Big(P\Big(\frac{x_2}{2}\Big)\Big) = 2\sum_{x_2\,(v^*)} \overline{\chi^*}\Big(P\Big(\frac{x_2}{2}\Big)\Big),$$

so that we can again apply the cited theorem to show that this sum is $\mathcal{O}(1)$. Finally, for the remaining cases $\tilde{v}^\circ = 2$ and $\tilde{v}^\circ = 4$, we can use [7, Theorem 2.1] to show square-root cancellation. This concludes the proof of (3.7).

3.3. Auxiliary estimates. We want to use the Kuznetsov formula in the form (2.2) with

$$\tilde{q} := t_1 s_2 u_2^* v^2, \quad \tilde{r} := s_2 u_2^* v^2, \quad \tilde{s} := t_1 \quad \tilde{q}_0 := v, \quad \tilde{m} := h_0 u_1 u_2^*, \quad \tilde{n} := n.$$

First, some technical arrangements have to be made. Set

$$\tilde{F}^\pm(c;n) := h(n)\frac{r_1^*}{r_1}\,\frac{v\sqrt{s_2u_2^*}}{4\pi}\sqrt{\frac{r_2}{n|h|}}\int cB^\pm\Big(c\sqrt{\xi\frac{r_2}{|h|}}\Big) f\Big(\xi; 4\pi\frac{d\sqrt{n}}{r_1^*c}\sqrt{\frac{|h|}{r_2}}\Big)\,d\xi,$$

where h is a smooth and compactly supported bump function such that

$$h(n) = 1 \quad \text{for } n \in [N, 2N], \qquad \operatorname{supp} h \asymp N,$$

and

$$h^{(\nu)}(n) \ll 1/N^\nu \quad \text{for } \nu \geq 0.$$

We have defined this function in such a way that

$$F^\pm(r_1^*c; dc, n) = \frac{1}{\sqrt{\tilde{r}}}\tilde{F}^\pm\Big(4\pi\frac{\sqrt{|\tilde{m}\tilde{n}|}}{\sqrt{\tilde{r}}\,\tilde{s}c}; n\Big) \qquad \text{for } n \in [N, 2N].$$

Note that

$$\operatorname{supp}\tilde{F}^\pm(\bullet; n) \asymp C := \frac{1}{A^*}\sqrt{\frac{N|h|}{r_2}}, \qquad \tilde{F}^\pm(c;n) \ll v\sqrt{s_2u_2^*}\,\frac{r_1^*}{A^*r_1}x_1^{1+\varepsilon}.$$

We need to separate the variable n to be able to use the large sieve inequalities later, and to this end we make use of Fourier inversion,

$$\tilde{F}^\pm(c;n) = \int G_0(\lambda)G_\lambda^\pm(c)e(\lambda n)\,d\lambda, \quad G_\lambda^\pm(c) := \frac{1}{G_0(\lambda)}\int \tilde{F}^\pm(c;n)e(-\lambda n)\,dn,$$

where

$$G_0(\lambda) := v\sqrt{s_2u_2^*}\,\frac{r_1^*}{A^*r_1}x_1^{1+\varepsilon}\min\Big(N, \frac{1}{N\lambda^2}\Big).$$

Eventually, our sum of Kloosterman sums looks like

$$K_{AB}^{\pm}(\chi;n) := \int G_0(\lambda)e(\lambda n) \sum_{\substack{c \\ (c,\tilde{r})=1}} \overline{\chi}(c) \frac{S(\tilde{m}, \pm\tilde{n}\bar{\tilde{r}}; \tilde{s}c)}{c\tilde{s}\sqrt{\tilde{r}}} \tilde{G}_{\lambda}^{\pm}\bigg(4\pi \frac{\sqrt{|\tilde{m}\tilde{n}|}}{c\tilde{s}\sqrt{\tilde{r}}}\bigg)\, d\lambda.$$

Next, we need to find good estimates for the Bessel transforms occurring in the Kuznetsov formula. For convenience set

$$C := \frac{1}{A^*}\sqrt{\frac{|h|N}{r_2}}, \qquad Z := \frac{1}{A^*}\sqrt{x_1 N}.$$

Note that due to the assumptions made in (3.2), we have $C \ll 1$.

Lemma 3.1. *If $N \ll N_0^-$, then*

$$\tilde{G}_{\lambda}^{\pm}(it), \check{G}_{\lambda}^{\pm}(it) \ll C^{-2t} \qquad \textit{for } 0 \le t < 1/4, \tag{3.9}$$

$$\frac{\tilde{G}_{\lambda}^{\pm}(t)}{(1+t)^{\kappa}}, \check{G}_{\lambda}^{\pm}(t), \dot{G}_{\lambda}^{\pm}(t) \ll \frac{x_1^{\varepsilon}}{1+t^{5/2}} \qquad \textit{for } t \ge 0. \tag{3.10}$$

If $N_0^- \ll N \ll N_0^+$, then, for any $\nu \ge 0$,

$$\tilde{G}_{\lambda}^{\pm}(it), \check{G}_{\lambda}^{\pm}(it) \ll x_1^{-\nu} \qquad \textit{for } 0 \le t < 1/4, \tag{3.11}$$

$$\frac{\tilde{G}_{\lambda}^{\pm}(t)}{(1+t)^{\kappa}}, \check{G}_{\lambda}^{\pm}(t), \dot{G}_{\lambda}^{\pm}(t) \ll \frac{x_1^{\varepsilon}}{Z^{5/2}}\bigg(\frac{Z}{t}\bigg)^{\nu} \qquad \textit{for } t \ge 0. \tag{3.12}$$

Proof. We will closely follow the proof of [27, Lemma 3.1], where a similar situation was handled. As all the occurring integrals can be interchanged, we can look directly at the Bessel transforms inside $\tilde{F}^{\pm}(c,n)$ and their partial derivatives in n. We will confine ourselves to the treatment of $\tilde{F}^{\pm}(c,n)$, since the estimates for the derivatives can be shown the same way.

First, we will look at the situation when $N \ll N_0^-$. Here again, we can look directly at the function inside the integral over ξ, given by

$$H_1(c) := cB^{\pm}\bigg(c\sqrt{\frac{\xi r_2}{|h|}}\bigg) f\bigg(\xi; 4\pi \frac{d}{r_1^* c}\sqrt{\frac{n|h|}{r_2}}\bigg).$$

We have

$$\operatorname{supp} H_1 \asymp C, \qquad H_1^{(\nu)}(c) \ll x_1^{\varepsilon} C (x_1^{\varepsilon}/C)^{\nu} \quad \text{for } \nu \ge 0,$$

so that by Lemma 2.4,

$$\tilde{H}_{\lambda}^{\pm}(it), \check{H}_{\lambda}^{\pm}(it) \ll C^{1-2t} \qquad \text{for } 0 \le t < 1/4,$$

$$\frac{\tilde{H}_{\lambda}^{\pm}(t)}{(1+t)^{\kappa}}, \check{H}_{\lambda}^{\pm}(t), \dot{H}_{\lambda}^{\pm}(t) \ll \frac{x_1^{\varepsilon} C}{1+t^{5/2}} \quad \text{for } t \ge 0,$$

from which we get (3.9) and (3.10).

Now assume $N_0^- \ll N \ll N_0^+$. By using Lemma 2.2 and integrating by parts once over ξ, we get

$$\tilde{F}^+(c) = \frac{1}{\pi}\,\frac{h(n)}{\sqrt{n}}\,\frac{r_1^* v\sqrt{s_2 u_2^*}}{r_1}\,\mathrm{Im}\Big(\int e\Big(\frac{c}{2\pi}\sqrt{\frac{\xi r_2}{|h|}}\Big)\tilde{w}(c,\xi)\,d\xi\Big)$$

with

$$\tilde{w}(c,\xi) := \frac{\partial}{\partial\xi}\Big(\sqrt{\xi}\,v_Y\Big(\frac{c}{\pi}\sqrt{\frac{\xi r_2}{|h|}}\Big) f\Big(\xi; 4\pi\frac{d}{r_2^* c}\sqrt{\frac{n|h|}{r_2}}\Big)\Big).$$

It is hence enough to look at

$$H_2(c) := e\Big(\frac{c}{2\pi}\sqrt{\frac{\xi r_2}{|h|}}\Big)\tilde{w}(c,\xi).$$

Note that

$$\operatorname{supp}\tilde{w}(\bullet,\xi) \asymp C, \qquad \frac{\partial^\nu}{\partial c^\nu}\tilde{w}(c,\xi) \ll \frac{\omega(\xi)}{x_1^{1/2}Z^{1/2}}\,\frac{1}{C^\nu} \quad \text{for } \nu \ge 0,$$

where

$$\omega(\xi) := 1 + \Big|w_1'\Big(\frac{\xi}{x_1}\Big)\Big| + \Big|w_2'\Big(\frac{\frac{r_2}{r_1}(\xi - f_1) + f_2}{x_2}\Big)\Big|.$$

Here we use Lemma 2.5 with $\alpha = \sqrt{\xi r_2/|h|}$ and $X = C$, which is possible since $\alpha X \gg (x_1 N_0^-)^{1/2}/A^* \gg x_1^\varepsilon$. We get, for any $\nu \ge 0$,

$$\begin{aligned} &\tilde{H}_\lambda^\pm(it), \check{H}_\lambda^\pm(it) \ll x_1^{-\nu} && \text{for } 0 \le t < 1/4,\\ &\frac{\tilde{H}_\lambda^\pm(t)}{(1+t)^\kappa}, \check{H}_\lambda^\pm(t), \dot{H}_\lambda^\pm(t) \ll \frac{x_1^\varepsilon}{x_1^{1/2}Z^{3/2}}\Big(\frac{Z}{t}\Big)^\nu && \text{for } t \ge 0, \end{aligned}$$

and (3.11) and (3.12) follow immediately. ∎

3.4. Use of the Kuznetsov formula. Here we will only look at $K_{AB}^+(\chi; n)$ and we will assume that $h > 0$, since all other cases can be treated similarly.

A use of Theorem 2.3 gives

$$R_{AB}^+(N;\chi) = \int G_0(\lambda)(\Xi_1(N) + \Xi_2(N) + \Xi_3(N))\,d\lambda,$$

where

$$\Xi_1(N) := \sum_{j=1}^\infty \frac{\tilde{G}_\lambda^+(t_j)}{(1+|t_j|)^\kappa}\Big(\frac{(1+|t_j|)^{\kappa/2}}{\sqrt{\cosh(\pi t_j)}}\overline{\rho_j}(\tilde{m},\infty)\sqrt{\tilde{m}}\Big)\Sigma_j^{(1)}(N),$$

$$\Xi_2(N) := \sum_{\mathfrak{c}\text{ sing.}} \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{\tilde{G}_\lambda^+(t)}{(1+|t|)^\kappa}\Big(\frac{(1+|t|)^{\kappa/2}}{\sqrt{\cosh(\pi t)}}\overline{\varphi_{\mathfrak{c},r}}(\tilde{m},\infty)\sqrt{\tilde{m}}\Big)\Sigma_{\mathfrak{c},r}^{(2)}(N)\,dt,$$

$$\Xi_3(N) := \sum_{\substack{k\equiv\kappa\,(2),\,k>\kappa\\ 1\le j\le\theta_k(q,\chi)}} \dot{G}_\lambda^+(k)\big(\sqrt{(k-1)!}\,\overline{\psi_{j,k}}(\tilde{m},\infty)\sqrt{\tilde{m}}\big)\Sigma_{j,k}^{(3)}(N),$$

with

$$\Sigma_j^{(1)}(N) := \frac{(1+|t_j|)^{\kappa/2}}{\sqrt{\cosh(\pi t_j)}} \sum_{N<n\le 2N} d(n)\hat{S}_v(\overline{\chi};n)e\bigg(\lambda n - n\frac{\bar{\tilde{s}}}{\tilde{r}}\bigg)\rho_j\bigg(n,\frac{1}{\tilde{s}}\bigg)\sqrt{n},$$

$$\Sigma_{\mathfrak{c},r}^{(2)}(N) := \frac{(1+|t|)^{\kappa/2}}{\sqrt{\cosh(\pi t)}} \sum_{N<n\le 2N} d(n)\hat{S}_v(\overline{\chi};n)e\bigg(\lambda n - n\frac{\bar{\tilde{s}}}{\tilde{r}}\bigg)\varphi_{\mathfrak{c},t}\bigg(n,\frac{1}{\tilde{s}}\bigg)\sqrt{n},$$

$$\Sigma_{j,k}^{(3)}(N) := \sqrt{(k-1)!} \sum_{N<n\le 2N} d(n)\hat{S}_v(\overline{\chi};n)e\bigg(\lambda n - n\frac{\bar{\tilde{s}}}{\tilde{r}}\bigg)\psi_{j,k}\bigg(n,\frac{1}{\tilde{s}}\bigg)\sqrt{n}.$$

Assume first that $N \ll N_0^-$. We divide $\Xi_1(N)$ into three parts:

$$\Xi_1(N) = \sum_{t_j\le 1}(\dots) + \sum_{t_j>1}(\dots) + \sum_{t_j\text{ exc.}}(\dots) =: \Xi_{1\mathrm{a}}(N) + \Xi_{1\mathrm{b}}(N) + \Xi_{1\mathrm{c}}(N).$$

We use Cauchy–Schwarz on $\Xi_{1\mathrm{a}}(N)$, and then Lemma 3.1, Theorem 2.6 and Lemma 2.8 to bound the different factors, which leads to

$$\begin{aligned}\Xi_{1\mathrm{a}}(N) &\le \max_{t_j\le 1}\bigg|\frac{\tilde{G}_\lambda^+(t_j)}{(1+t_j)^\kappa}\bigg| \\ &\qquad\cdot\bigg(\sum_{t_j\le 1}\frac{(1+t_j)^\kappa}{\cosh(\pi t_j)}|\rho_j(\tilde{m},\infty)|^2\tilde{m}\bigg)^{1/2}\bigg(\sum_{t_j\le 1}|\Sigma_j^{(1)}(N)|^2\bigg)^{1/2} \\ &\ll x_1^\varepsilon\tilde{m}^\theta\bigg(1+\tilde{q}_0^{1/2}\frac{N}{\tilde{q}}\bigg)^{1/2}\bigg(N^\varepsilon v^\varepsilon\sum_{N<n\le 2N}(f_1,n,v^\circ)^2\bigg)^{1/2} \\ &\ll (v^\circ)^{1/2}x_1^\varepsilon h^\theta\frac{A^*}{x_1}\bigg(x_1^{1/2}+\frac{A^*}{v^{1/4}(r_1^*s_2u_2^*)^{1/2}}\bigg),\end{aligned}$$

where we have set

$$v^\circ := v/\mathrm{cond}(\overline{\chi}).$$

We split up $\Xi_{1\mathrm{b}}(N)$ into dyadic segments

$$\Xi_{1\mathrm{b}}(N,T) := \sum_{T<t_j\le 2T}\tilde{G}_\lambda^+(t_j)\frac{\overline{\rho_j}(\tilde{m},\infty)\sqrt{\tilde{m}}}{\sqrt{\cosh(\pi t_j)}}\Sigma_j^{(1)}(N),$$

and in the same way as above we can show that

$$\Xi_{1\mathrm{b}}(N,T) \ll (v^\circ)^{1/2}x_1^\varepsilon h^\theta\frac{A^*}{T^{1/2}x_1}\bigg(x_1^{1/2}+\frac{A^*}{v^{1/4}(r_1^*s_2u_2^*)^{1/2}}\bigg),$$

which gives the same bound for $\Xi_{1\mathrm{b}}(N)$ as for $\Xi_{1\mathrm{a}}(N)$. Finally,

$$\Xi_{1\mathrm{c}}(N) \ll (v^\circ)^{1/2} x_1^{\varepsilon} (r_2 x_1)^{\theta} \frac{A^*}{x_1} \left(x_1^{1/2} + \frac{A^*}{v^{1/4}(r_1^* s_2 u_2^*)^{1/2}} \right),$$

and all in all this leads to

$$\int G_0(\lambda) \Xi_1(N)\, d\lambda \ll (v^\circ)^{1/2} v (r_2 x_1)^{1/2+\theta+\varepsilon}. \tag{3.13}$$

In exactly the same manner, but using Lemma 2.7 instead of Lemma 2.8, we can also get the bounds

$$\Xi_{1\mathrm{a}}(N),\ \Xi_{1\mathrm{b}}(N) \ll (v^\circ)^{1/2} x_1^{\varepsilon} \frac{A^*}{x_1} \left(x_1^{1/2} + \frac{A^*}{v^{1/4}(r_1^* s_2 u_2^*)^{1/2}} \right) (1 + \Psi(r_1^*)),$$

$$\Xi_{1\mathrm{c}}(N) \ll (v^\circ)^{1/2} x_1^{\varepsilon} \left(\frac{r_2 x_1}{h} \right)^{\theta} \frac{A^*}{x_1} \left(x_1^{1/2} + \frac{A^*}{v^{1/4}(r_1^* s_2 u_2^*)^{1/2}} \right) (1 + \Psi(r_1^*))$$

with

$$\Psi(y) = \frac{(y r_2 v, h)^{1/4} h^{1/4}}{(y r_2)^{1/2} v^{1/4}},$$

so that

$$\int G_0(\lambda) \Xi_1(N)\, d\lambda \ll (v^\circ)^{1/2} v (r_2 x_1)^{1/2+\varepsilon} \left(\frac{r_2 x_1}{h} \right)^{\theta} (1 + \Psi(r_1)). \tag{3.14}$$

Furthermore since

$$\frac{x_1^{\varepsilon}}{C} \gg \frac{x_1^{1/2} r_2^{1/2}}{h^{1/2}} \gg \frac{r_1^* r_2 v^{1/2}}{(r_1^* r_2 v, h)^{1/2} h^{1/2}} = \frac{\tilde{q}}{(\tilde{q}, \tilde{m})^{1/2} \tilde{q}_0^{1/2} \tilde{m}^{1/2}},$$

we can also make use of Lemma 2.9 here, so that

$$\begin{aligned} \Xi_{1\mathrm{c}}(N) &\ll \left(\sum_{t_j \le 1} \frac{|\rho_j(\tilde{m}, \infty)|^2 \tilde{m}}{\cosh(\pi t_j)} \left(\frac{x_1^{\varepsilon}}{C} \right)^{4 i t_j} \right)^{1/2} \left(\sum_{t_j \le 1} |\Sigma_j^{(1)}(N)|^2 \right)^{1/2} \\ &\ll (v^\circ)^{1/2} x_1^{\theta+\varepsilon} \frac{(r_1^* r_2 v, h)^{\theta}}{(r_1^{*2} r_2 v)^{\theta}} \frac{A^*}{x_1} \left(x_1^{1/2} + \frac{A^*}{v^{1/4}(r_1^* s_2 u_2^*)^{1/2}} \right) (1 + \Psi(r_1^*)), \end{aligned}$$

and hence

$$\int G_0(\lambda) \Xi_1(N)\, d\lambda \ll \sqrt{v^\circ}\, v (r_2 x_1)^{1/2+\varepsilon} \left(x_1 \frac{(r_1 r_2 v, h)}{r_1^2 r_2 v} \right)^{\theta} (1 + \Psi(r_1)). \tag{3.15}$$

Now assume $N_0^- \ll N \ll N_0^+$. We split $\Xi_1(N)$ again into three parts,

$$\Xi_1(N) = \sum_{t_j \le Z} (\ldots) + \sum_{t_j > Z} (\ldots) + \sum_{t_j \text{ exc.}} (\ldots).$$

The sum over the exceptional eigenvalues causes no problems in this case, as the respective Bessel transforms are very small. The rest can be treated in the same way as above, and we get the bounds

$$\int G_0(\lambda)\Xi_1(N)\,d\lambda \ll (v^\circ)^{1/2} v(r_2x_1)^{1/2+\varepsilon}\frac{h^\theta}{\Omega^{1/2}}, \tag{3.16}$$

$$\int G_0(\lambda)\Xi_1(N)\,d\lambda \ll (v^\circ)^{1/2} v(r_2x_1)^{1/2+\varepsilon}\frac{1}{\Omega^{1/2}}(1+\Omega^{1/2}\Psi(r_1)). \tag{3.17}$$

The same reasoning applies similarly to $\Xi_2(N)$ and $\Xi_3(N)$, the main difference being that we do not have to worry about exceptional eigenvalues at all. In the end, we get: from (3.13) and (3.16),

$$R^+_{AB}(N;\chi) \ll (v^\circ)^{1/2} v(r_2x_1)^{1/2+\varepsilon}\bigg(\frac{h^\theta}{\Omega^{1/2}} + (r_2x_1)^\theta\bigg);$$

from (3.15) and (3.17),

$$R^+_{AB}(N;\chi) \ll (v^\circ)^{1/2} v(r_2x_1)^{1/2+\varepsilon}\bigg(\frac{1}{\Omega^{1/2}} + \bigg(\frac{x_1(r_1r_2v,h)}{r_1^2r_2v}\bigg)^\theta(1+\Psi(r_1))\bigg);$$

and from (3.14) and (3.17),

$$R^+_{AB}(N;\chi) \ll (v^\circ)^{1/2} v(r_2x_1)^{1/2+\varepsilon}\bigg(\frac{1}{\Omega^{1/2}} + \bigg(\frac{r_2x_1}{h}\bigg)^\theta(1+\Psi(r_1))\bigg).$$

On taking account of (3.8), these bounds finally lead to (3.3)–(3.5).

3.5. The main term. It remains to evaluate the main term. After summing over all A and B, it has the form

$$\begin{aligned}\Sigma^0 &:= \frac{1}{r_1}\sum_{u_2^*\mid u_2}\ \sum_{\substack{a\\(a,s_2u_2^*)=1}}\frac{1}{a}\int \lambda_{f_1-r_1g_2\overline{s_1},r_1a}(\xi)f(\xi;a)\,d\xi \\ &= \int w_1\bigg(\frac{r_1\xi+f_1}{x_1}\bigg)w_2\bigg(\frac{r_2\xi+f_2}{x_2}\bigg)\bigg(\sum_{u_2^*\mid u_2}\tilde\Sigma^0(\xi,u_2^*)\bigg)\,d\xi\end{aligned} \tag{3.18}$$

with

$$\begin{aligned}\tilde\Sigma^0(\xi,u_2^*) &:= \sum_{\substack{a\\(a,s_2u_2^*)=1}}\frac{\lambda_{f_1-r_1g_2\overline{s_2},r_1a}(r_1\xi+f_1)}{a}h\bigg(\frac{u_2a}{u_2^*},\frac{u_2^*}{u_2a}(r_2\xi+f_2)\bigg) \\ &= \frac{1}{2\pi i}\int_{(\sigma)}\hat h(s;\xi)Z(s;\xi)\,ds,\end{aligned} \tag{3.19}$$

where $\hat h(s;\xi)$ is the Mellin transform

$$\hat h(s;\xi) := \int_0^\infty h\bigg(\frac{u_2a}{u_2^*},\frac{u_2^*}{u_2a}(r_2\xi+f_2)\bigg)a^{s-1}\,da, \quad \mathrm{Re}(s)>0,$$

and the function $Z(s;\xi)$ is defined as the Dirichlet series

$$Z(s;\xi) := \sum_{\substack{a \\ (a,s_2u_2^*)=1}} \frac{\lambda_{f_1-r_1g_2\overline{s_2},r_1a}(r_1\xi+f_1)}{a^{1+s}}, \quad \mathrm{Re}(s)>0.$$

The integral in (3.19) is initially defined for $\sigma>0$. Our plan is to move the line of integration to $\sigma=-1+\varepsilon$, so that we can use the residue theorem to extract a main term. A meromorphic continuation of $\hat{h}(s;\xi)$ can easily be found by using partial integration. For $Z(s;\xi)$ the situation is not quite as obvious.

Define the operator

$$\varDelta_\delta(\xi) := \left(\log\xi+2\gamma+2\frac{\partial}{\partial\delta}\right)\bigg|_{\delta=0},$$

so that we can write

$$\lambda_{f_1-r_1g_2\overline{s_2},r_1a}(r_1\xi+f_1) = \varDelta_\delta(r_1\xi+f_1)\sum_{d|r_1a}\frac{c_d(f_1-r_1g_2\overline{s_2})}{d^{1+\delta}}.$$

Now we separate the part of r_1 which shares common factors with $s_2u_2^*$ from the rest by setting

$$v := (r_1,(s_2u_2^*)^\infty), \quad t_1 := r_1/v,$$

so that

$$\sum_{d|r_1a}\frac{c_d(f_1-r_1g_2\overline{s_2})}{d^{1+\delta}} = \left(\sum_{d|v}\frac{c_d(f_1)}{d^{1+\delta}}\right)\left(\sum_{d|t_1a}\frac{c_d(h_0u_1)}{d^{1+\delta}}\right),$$

and hence

$$Z(s;\xi) = \varDelta_\delta(r_1\xi+f_1)\left(\sum_{d|v}\frac{c_d(f_1)}{d^{1+\delta}}\right)\sum_{\substack{a \\ (a,s_2u_2^*)=1}}\frac{1}{a^{1+s}}\sum_{d|t_1a}\frac{c_d(h_0u_1)}{d^{1+\delta}}.$$

The two outer sums can be transformed to

$$\sum_{\substack{a \\ (a,s_2u_2^*)=1}}\frac{1}{a^{1+s}}\sum_{d|t_1a}\frac{c_d(h_0u_1)}{d^{1+\delta}} = \sum_{\substack{d \\ (d,s_2u_2^*)=1}}\frac{c_d(h_0u_1)(d,t_1)}{d^{2+\delta}}\tilde{Z}(s;d)$$

with

$$\tilde{Z}(s;d) := \zeta(1+s)\frac{(d,t_1)^s}{d^s}\prod_{p|s_2u_2^*}\left(1-\frac{1}{p^{1+s}}\right).$$

This is a meromorphic function, defined on the whole complex plane, which means that the desired meromorphic continuation for $Z(s;\xi)$ can be given by

$$Z(s;\xi) = \varDelta_\delta(r_1\xi+f_1)\left(\sum_{d|v}\frac{c_d(f_1)}{d^{1+\delta}}\right)\left(\sum_{\substack{d \\ (d,s_2u_2^*)=1}}\frac{c_d(h_0u_1)(d,t_1)}{d^{2+\delta}}\tilde{Z}(s;d)\right).$$

Hence

$$\tilde{\Sigma}^0(\xi, u_2^*) = \Delta_\delta(r_1\xi + f_1)\Bigg(\sum_{d|v} \frac{c_d(f_1)}{d^{1+\delta}}\Bigg)\Bigg(\sum_{\substack{d \\ (d, s_2 u_2^*)=1}} \frac{c_d(h_0 u_1)(d, t_1)}{d^{2+\delta}} \tilde{I}^0(\xi, d)\Bigg)$$

with

$$\tilde{I}^0(\xi, d) := \frac{1}{2\pi i} \int_{(\sigma)} \hat{h}(s;\xi)\tilde{Z}(s;d)\, ds.$$

The Mellin transform $\hat{h}(s;\xi)$ has at $s=0$ the Taylor expansion

$$\hat{h}(s;\xi) = \frac{2}{s} + \log(r_2\xi + f_2) + 2\log\frac{u_2^*}{u_2} + \mathcal{O}(s),$$

while that of $\tilde{Z}(s;d)$ is given by

$$\tilde{Z}(s;d) = \left(\frac{1}{s} + \gamma + \frac{\partial}{\partial\rho}\right)\bigg|_{\rho=0} \frac{(d, t_1)^\rho}{d^\rho} \prod_{p|s_2 u_2^*}\left(1 - \frac{1}{p^{1+\rho}}\right) + \mathcal{O}(s).$$

All in all, the residue of their product at $s=0$ is

$$\operatorname*{Res}_{s=0}\big(\hat{h}(s;\xi)\tilde{Z}(s;d)\big) = \Delta_\rho(r_2\xi + f_2)\left(\frac{u_2^*}{u_2}\right)^\rho \frac{(d, t_1)^\rho}{d^\rho} \prod_{p|s_2 u_2^*}\left(1 - \frac{1}{p^{1+\rho}}\right).$$

We now move the line of integration to $\sigma = -1+\varepsilon$,

$$\tilde{I}^0(\xi, d) = \Delta_\rho(r_2\xi + f_2)\left(\frac{u_2^*}{u_2}\right)^\rho \frac{(d, t_1)^\rho}{d^\rho} \prod_{p|s_2 u_2^*}\left(1 - \frac{1}{p^{1+\rho}}\right) + \mathcal{O}\left(\frac{d^{1-\varepsilon}}{x_2^{1/2-\varepsilon}}\right),$$

and hence

$$\tilde{\Sigma}^0(\xi, u_2^*) = \Delta_\delta(r_1\xi + f_1)\Delta_\rho(r_2\xi + f_2)\tilde{M}^0_{\delta,\rho}(\xi, u_2^*) + \mathcal{O}\big(x_2^\varepsilon / x_2^{1/2}\big)$$

with

$$\tilde{M}^0_{\delta,\rho}(\xi, u_2^*) := \left(\frac{u_2^*}{u_2}\right)^\rho \Bigg(\sum_{d|v} \frac{c_d(f_1)}{d^{1+\delta}}\Bigg) \prod_{p|s_2 u_2^*}\left(1 - \frac{1}{p^{1+\rho}}\right) \cdot \Bigg(\sum_{\substack{d \\ (d, s_2 u_2^*)=1}} \frac{c_d(h_0 u_1)(d, t_1)^{1+\rho}}{d^{2+\delta+\rho}}\Bigg).$$

An elementary but quite tedious calculation shows that

$$\sum_{u_2^*|u_2} \tilde{M}^0_{\delta,\rho}(\xi, u_2^*) = C_{\delta,\rho}(r_1, r_2, f_1, f_2),$$

where

$$C_{\delta,\rho}(r_1, r_2, f_1, f_2) := \sum_{\substack{u_1^*|u_1 \\ u_2^*|u_2}} \left(\frac{u_1^*}{u_1}\right)^\delta \left(\frac{u_2^*}{u_2}\right)^\rho \psi_\delta(s_1 u_1^*)\psi_\rho(s_2 u_2^*)\gamma_{\delta+\rho}(s_1 u_1^* s_2 u_2^*),$$

$$\psi_\alpha(n) := \prod_{p|n}\left(1 - \frac{1}{p^{1+\alpha}}\right), \qquad \gamma_\alpha(n) := \sum_{(d,n)=1} \frac{c_d(h_0)}{d^{2+\alpha}}.$$

After a look back at (3.18), we see that our main term has the form

$$\Sigma^0 = M(x_1, x_2) + \mathcal{O}(x_2^{1/2+\varepsilon}/r_2)$$

with

$$M(x_1, x_2) := \int w_1\left(\frac{r_1\xi + f_1}{x_1}\right) w_2\left(\frac{r_2\xi + f_2}{x_2}\right) P(\log(r_1\xi + f_1), \log(r_2\xi + f_2))\, d\xi,$$

where $P(\xi_1, \xi_2)$ is the quadratic polynomial given by

$$P(\log\xi_1, \log\xi_2) := \Delta_\delta(\xi_1)\Delta_\rho(\xi_2) C_{\delta,\rho}(r_1, r_2, f_1, f_2). \tag{3.20}$$

This concludes the proof of (3.1).

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