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On generators of the Chow group of 0-cycles on diagonal cubic surfaces over 3-adic fields

by

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1. Introduction. Let k be a p-adic field, and let X be a diagonal cubic surface over k, that is, a smooth projective surface in

$$\mathbb{P}^3_k = \operatorname{Proj}(k[T_0, T_1, T_2, T_3])$$

defined by an equation of the form

$$X: \ T_0^3 + aT_1^3 + bT_2^3 + cT_3^3 = 0 \quad (a, b, c \in k^*).$$

The main subject of this paper is the Chow group $\mathrm{CH}_0(X)$ of 0-cycles on X modulo rational equivalence, which has been computed in many cases [B], [CT2], [CT3], [CTS], [CTSn], [Da1], [Da2], [P], [SS], [U]. An interesting problem is to investigate the group structure of $\mathrm{CH}_0(X)$ and to find an explicit generating set. Our fundamental tool to study $\mathrm{CH}_0(X)$ is the canonical pairing introduced by Manin [M1], called the *Brauer–Manin pairing*:

$$\mathrm{CH}_0(X) \times \mathrm{Br}(X) \to \mathbb{Q}/\mathbb{Z},$$

where $\mathrm{Br}(X)$ denotes the Grothendieck–Brauer group $H^2(X_{\text{ét}}, \mathbb{G}_{\mathrm{m}})$. Let

$$A_0(X) = \operatorname{Ker}(\deg : \mathrm{CH}_0(X) \to \mathbb{Z})$$

be the Chow group of 0-cycles of degree 0. In this paper, we will construct an explicit generating set of $A_0(X)$ by calculating the Brauer–Manin pairing directly using the Hilbert symbol.

If $a, b, c \in (k^*)^3$, then $A_0(X) = 0$ because X is isomorphic to the blow-up of $\mathbb{P}^2_k$ at six k-valued points in general position. In this paper, we are mainly concerned with the case $a = b = 1$ and $c \notin (k^*)^3$. We have the following fact due to Uematsu:

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Theorem 1.1 (Uematsu, [U, Theorem 4]). *Assume that $p \neq 3$, k contains a primitive cube root of unity and $a = b = 1$ and $c \notin (k^*)^3$. Then $A_0(X)$ is generated by the classes of rational points.*

The following theorem is the main result of this paper.

Theorem 1.2. *Assume that $p = 3$, $a = b = 1$ and* $\mathrm{ord}_k(c) \not\equiv 0 \pmod 3$. *Then $A_0(X)$ is generated by the classes of rational points, and it is isomorphic to*

$$\begin{cases} \mathbb{Z}/3\mathbb{Z} & \text{if } \zeta_3 \notin k, \\ (\mathbb{Z}/3\mathbb{Z})^{\oplus 2} & \text{if } \zeta_3 \in k. \end{cases}$$

To prove Theorem 1.2, we construct an explicit generating set of $A_0(X_L)$, where $X_L = X \otimes_k L$ (L is the unramified cubic extension over k) and we push forward these cycles to $A_0(X)$. The construction of explicit generators consisting of k-rational points under the assumption $p = 3$ seems to be more difficult than in the case $p \neq 3$ because we cannot use Hensel's lemma when $p = 3$. In this construction, we compute the Brauer–Manin pairing explicitly using Hilbert symbols.

Notation.

- For a scheme X, we define the Brauer group $\mathrm{Br}(X)$ as the étale cohomology $H^2(X_{\text{ét}}, \mathbb{G}_{\mathrm{m}})$.
- For a scheme X over a field k, we define the residue field $k(P)$ at P as the residue field of $\mathscr{O}_{X,P}$.
- An algebraic cycle on a scheme X is a formal finite sum of closed integral subschemes of X. The group $Z_0(X)$ of 0-cycles on X is the free abelian group $\mathbb{Z}[X_{(0)}]$ generated by the set $X_{(0)}$ of closed points of X. The Chow group of 0-cycles modulo rational equivalence is the quotient group
$$\mathrm{CH}_0(X) := Z_0(X)/Z_0(X)_{\mathrm{rat}}.$$
Here $Z_0(X)_{\mathrm{rat}}$ is the subgroup of $Z_0(X)$ generated by divisors of rational functions on one-dimensional closed subschemes on X.

2. Preliminaries

2.1. Hilbert symbol and its computation. Let k be a p-adic field containing a primitive pth root of unity. The Hilbert symbol is a non-degenerate skew-symmetric bilinear mapping defined by the local classfield theory:

$$(2.1) \qquad (\ ,\ )_k : k^*/(k^*)^p \times k^*/(k^*)^p \to \mu_p, \qquad (x, y) \mapsto \frac{\rho(y)(\sqrt[p]{x})}{\sqrt[p]{x}}.$$

Here μ_p is the group of pth roots of unity and ρ is the reciprocity map $\rho : k^* \to \mathrm{Gal}(k^{\mathrm{ab}}/k)$, where k^{ab} is the maximal abelian extension of k. The

Hilbert symbol has the following properties:

(1) $(x,y)_k = (y,x)_k^{-1}$ (skew-symmetry).
(2) $(xy,z)_k = (x,z)_k \cdot (y,z)_k$.
(3) $(x^p,y)_k = 1$.
(4) $(x,1-x)_k = 1$ for $x \neq 1$.
(5) Let L be a finite separable extension of k. If $x \in L^*$ and $y \in k^*$, then $(x,y)_L = (\mathrm{Nr}_{L/k}(x),y)_k$.
(6) Let L be a finite separable extension of k. If $x \in k^*$ and $y \in k^*$, then $(x,y)_L = (x,y)_k^{[L:k]}$.

We now explain an explicit formula concerning the Hilbert symbol described in [Se, p. 237, Proposition 6 and Exercice 3] (see also [Da3, pp. 43–44]).

Let π be a uniformizer of k. We denote by e the absolute ramification index of k, and define $e' := pe/(p-1)$. We set $B := k^*/(k^*)^p$, and define the descending filtration $\{B^j\}_{j\geq 0}$ by $B^0 := B$, $B^j := \mathrm{Im}(\{1+\pi^j x \mid x \in \mathscr{O}_k\} \to B)$. Then $B^{e'+1} = 0$ and

$$B^{e'} \cong \mathbb{F}_k/(1-\phi)\mathbb{F}_k, \qquad 1-p(1-\zeta_p)x \mapsto \overline{x}, \tag{2.2}$$

where $\mathbb{F}_k$ is the residue field of k, $\overline{x}$ is the residue class of $x \in \mathscr{O}_k$ in $\mathbb{F}_k$, and ϕ is the Frobenius map which sends $a \in \mathbb{F}_k$ to a^p. We calculate the Hilbert symbol $(\ ,\ )_k$ by using this descending filtration. The pairing (2.1) induces a bilinear map

$$B^j/B^{j+1} \times B^{e'-j}/B^{e'-j+1} \to \mu_p \tag{2.3}$$

because (2.1) vanishes on $B^j \times B^{e'-j+1}$ and $B^{j+1} \times B^{e'-j}$. Let u be the residue class in $\mathbb{F}_k^*$ of the unit $(1-\zeta_p)^{-p}\pi^{e'} \in \mathscr{O}_k^*$. For $a \in \mathbb{Z}/p\mathbb{Z}$ and $\xi \in \mathbb{F}_k$, when $j = e'$, the pairing (2.3) is described as

$$(\xi,\ \pi^a) \mapsto \zeta_p^{-a\,\mathrm{Tr}_{\mathbb{F}_k/\mathbb{F}_p}(u\xi)}$$

under the isomorphism (2.2).

2.2. Brauer group of diagonal cubic surfaces. Let X be a diagonal cubic surface defined by an equation of the form

$$X : T_0^3 + T_1^3 + T_2^3 + cT_3^3 = 0 \qquad (c \in k^*).$$

The structure of the Brauer group $\mathrm{Br}(X)$ has been studied by Manin and Uematsu, which we are going to review briefly.

Suppose that k contains a primitive cube root of unity ζ_3. For a and b in $k(X)^*$, we write $\{a,b\}_{\zeta_3}$ for the inverse image of $\{a,b\}$ in $H^2(k(X),\mu_3^{\otimes 2})$ under the isomorphism

$${}_3\mathrm{Br}(k(X)) \overset{\cong}{\to} H^2(k(X),\mu_3^{\otimes 2}), \qquad x \mapsto x \otimes \zeta_3.$$

Concerning the structure and generators of the Brauer group of X, we have the following facts:

Theorem 2.1 ([M2, Example 45.3], [CTKS, Proposition 1]). *Assume that* $c \notin (k^*)^3$ *and* $\zeta_3 \in k$. *Set*

$$f = \frac{T_0 + \zeta_3 T_1}{T_0 + T_1}, \qquad g = \frac{T_0 + T_2}{T_0 + T_1}.$$

Then $\mathbf{e}_1 = \{c, f\}_{\zeta_3}$ *and* $\mathbf{e}_2 = \{c, g\}_{\zeta_3}$ *belong to* $\mathrm{Br}(X)$, *and* $\mathrm{Br}(X)/\mathrm{Br}(k)$ *is isomorphic to* $(\mathbb{Z}/3\mathbb{Z})^{\oplus 2}$ *generated by* $\mathbf{e}_1$ *and* $\mathbf{e}_2$.

When k does not contain primitive cubic roots of unity, we have the following fact:

Proposition 2.2 ([CTW, Proposition 2.1], [SS, Proposition 4.2.6]). *Assume that* k *does not contain a primitive cube root of unity. Set* $L = k(\zeta_3)$. *Let* $\mathbf{e}_1 \in \mathrm{Br}(X_L)$ *be as in Theorem* 2.1. *Let* $\mathrm{cor}_{X_L/X} : \mathrm{Br}(X_L) \to \mathrm{Br}(X)$ *be the corestriction map. If* $c \notin (k^*)^3$, *then* $\mathrm{Br}(X)/\mathrm{Br}(k)$ *is isomorphic to* $\mathbb{Z}/3\mathbb{Z}$ *and generated by* $\mathrm{cor}_{X_L/X}(\mathbf{e}_1)$.

3. Proof of the main result.

In this section, we prove Theorem 1.2. Let k be a 3-adic field and let X be a diagonal cubic surface over k defined by an equation of the form

$$X : {T_0}^3 + {T_1}^3 + {T_2}^3 + c{T_3}^3 = 0 \quad (c \in k^*).$$

Let $\langle\ ,\ \rangle$ be the Brauer–Manin pairing [M1]:

$$\langle\ ,\ \rangle : \mathrm{CH}_0(X) \times \mathrm{Br}(X) \to \mathbb{Q}/\mathbb{Z}.$$

Our important tool to study the structure of $A_0(X)$ is the following map induced by the Brauer–Manin pairing:

$$\Phi_X : A_0(X) \to \mathrm{Hom}(\mathrm{Br}(X)/\mathrm{Br}(k), \mathbb{Q}/\mathbb{Z}),$$

where $\mathrm{Br}(X)/\mathrm{Br}(k) := \mathrm{Br}(X)/\mathrm{Im}(\mathrm{Br}(X) \to \mathrm{Br}(k))$. Concerning the injectivity of Φ_X, we have the following fact due to Colliot-Thélène, which plays a fundamental role in the proof of our main result.

Theorem 3.1 (Colliot-Thélène, [CT1, Propositions 5 and 7]). Φ_X *is injective.*

We have the following lemma.

Lemma 3.2. *Let* E *be an elliptic curve* $T_0^3 + T_1^3 + T_2^3 = 0$ *over* k, *and let* O *be the point* $(1 : 0 : -1)$ *of* E. *If there exists a point* P *on* E *of degree* m, *then* $P - (m-1)O$ *is rational equivalent to a* k*-rational point on* E.

Proof. The assertion immediately follows from the Riemann–Roch theorem for E. ∎

Proof of Theorem 1.2. We consider the following two cases:

(1) $\zeta_3 \in k$ and $\mathrm{ord}_k(c) \not\equiv 0 \bmod 3$,
(2) $\zeta_3 \notin k$ and $\mathrm{ord}_k(c) \not\equiv 0 \bmod 3$.

In case (1), $\mathrm{Br}(X)/\mathrm{Br}(k)$ is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^{\oplus 2}$ and generated by $\mathbf{e}_1$ and $\mathbf{e}_2$ in Theorem 2.1. Let L be a cubic unramified extension of k and let $\phi : X_L \to X$ be the natural map. We will construct four L-valued points P_{ζ_3}, Q_l $(l = 1, 2)$ on X_L which are contained in E_L and such that

$$\det\begin{pmatrix} \langle \phi_* Q_2 - \phi_* P_{\zeta_3}, \mathbf{e}_1\rangle_{X_L} & \langle \phi_* Q_2 - \phi_* P_{\zeta_3}, \mathbf{e}_2\rangle_{X_L} \\ \langle \phi_* Q_1 - \phi_* P_{\zeta_3}, \mathbf{e}_1\rangle_{X_L} & \langle \phi_* Q_1 - \phi_* P_{\zeta_3}, \mathbf{e}_2\rangle_{X_L} \end{pmatrix} \neq 0,$$

in Proposition 3.4 below. These formulas imply that $\phi_* Q_2 - \phi_* P_{\zeta_3}$ and $\phi_* Q_1 - \phi_* P_{\zeta_3}$ are linearly independent over $\mathbb{Z}/3\mathbb{Z}$ and that $A_0(X)$ is generated by the cycles $\phi_* Q_2 - \phi_* P_{\zeta_3}$ and $\phi_* Q_1 - \phi_* P_{\zeta_3}$ on E. Moreover, there exist three k-valued points q_1, q_2, p on E such that $\phi_* Q_i - \phi_* P_{\zeta_3}$ is rationally equivalent to $q_i - p$ $(i = 1, 2)$ (we use Lemma 3.2 for $m = 3$). Hence $A_0(X)$ is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^{\oplus 2}$ and generated by the classes of k-rational points.

To explain case (2) we suppose that $\zeta_3 \notin k$ and set $M := k(\zeta_3)$. In this case, $\mathrm{Br}(X)/\mathrm{Br}(k)$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$ and generated by $\mathrm{cor}_{X_M/X}(\mathbf{e}_1)$ by Proposition 2.2. Let $q : X_M \to X$ be the natural map. By Proposition 3.4, there exists a 0-cycle C $(= \phi_* Q_2 - \phi_* P_{\zeta_3})$ on X_M such that

$$\langle C, \mathbf{e}_1\rangle_{X_M} \neq 0.$$

We will check, in Proposition 3.5 below, that

$$\langle q_* C, \mathrm{cor}_{X_M/X}(\mathbf{e}_1)\rangle_X = 2\langle C, \mathbf{e}_1\rangle_{X_M}.$$

Thus the 0-cycle $q_* C$ is the generator of $A_0(X)$. Moreover, there exist k-valued points q_2', p' on X such that $q_* C$ is rationally equivalent to $q_2' - p'$ (we use Lemma 3.2 for $m = 6$). Hence $A_0(X)$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$ and generated by the classes of k-rational points. ∎

3.1. The case $\zeta_3 \in k$. Fix a uniformizer π of k. We denote by e the absolute ramification index of k. We write $\mathbb{F}_k$ for the residue field of k. We set $e' := 3e/2$ and define

$$\delta := (1 - \zeta_3)\pi^{-e/2}, \quad \epsilon := 3\pi^{-e},$$

which belong to $\mathscr{O}_k^*$. Let L be a cubic unramified extension of k, which is unique up to isomorphism over k. We write $\mathbb{F}_L$ for the residue field of L. We prove the following lemma.

Lemma 3.3. *For a closed point P of X such that $f(P) \neq 0$, we have*

$$\zeta_3^{\langle [P],\, \mathbf{e}\rangle} = (c, f(P))_{k(P)} \quad (i = 1, 2).$$

Here we set $\mathbf{e} := \{c, f\}_{\zeta_3} \in \mathrm{Br}(X)$, where f is a rational function in $k(X)^*$.

Proof. For a closed point P of X and $\omega \in \mathrm{Br}(X)$, we have $\langle [P], \omega\rangle = \mathrm{inv}_{k(P)}(\omega(P))$. If $\omega = \mathbf{e}$ $(= \{c, f\}_{\zeta_3})$ and $f \in k(X)^*$, then $\mathbf{e}(P) = \{c, f(P)\}_{\zeta_3}$. We define $\chi_c \in \mathrm{Hom}(G_L, \mathbb{Z}/3\mathbb{Z})$ by setting $\chi_c(\sigma) := j$ for $\sigma \in G_L$, where j

satisfies $\sigma(\sqrt[3]{c})/\sqrt[3]{c} = \zeta_3^j$. We have

$$\begin{aligned} \zeta_3^{\langle [P],\mathbf{e}\rangle} &= \zeta_3^{\mathrm{inv}_{k(P)}(\{c,f(P)\}_{\zeta_3})} = \zeta_3^{\mathrm{inv}_{k(P)}(\chi_c \cup f(P))} \\ &= \zeta_3^{\chi_c(\rho(f(P)))} \quad \text{([Se, Chap. XIV, Proposition 3])} \\ &= \frac{\rho(f(P))(\sqrt[3]{c})}{\sqrt[3]{c}} \\ &= (c, f(P))_{k(P)} \quad ((2.1)). \end{aligned}$$

Here $\rho : k^* \to \mathrm{Gal}(k^{\mathrm{ab}}/k)$ is the reciprocity map. ∎

PROPOSITION 3.4. *Let $\phi : X_L \to X$ be the natural projection. Consider the following L-valued points on X_L:*

$$P_{\zeta_3} := \left(1 : \pi^e u : -\sqrt[3]{1+\pi^{3e}u^3} : 0\right),$$
$$Q_l := \left(1 : \pi^{e/2}\delta\sqrt[3]{1+\pi^{e'}w_l} : -\sqrt[3]{1+\pi^{e'}\delta^3(1+\pi^{e'}w_l)} : 0\right) \quad (l = 1, 2),$$

where u is a unit of $\mathscr{O}_L$ such that $\mathrm{Tr}_{\mathbb{F}_L/\mathbb{F}_3}(\overline{(1-\zeta_3)^{-3}\pi^{e'}\delta u}) = 2$ and w_l $(l = 1, 2)$ are units of $\mathscr{O}_k$ such that $\mathrm{ord}_k(c)\,\mathrm{Tr}_{\mathbb{F}_k/\mathbb{F}_3}(\overline{(1-\zeta_3)^{-3}\pi^{e'}w_l}) - [\mathbb{F}_k : \mathbb{F}_3] \equiv l \bmod 3$. Then $A_0(X)$ is generated by the following two 0-cycles of degree zero which are linearly independent and have order 3:

$$C_1 = \phi_*[Q_2] - \phi_*[P_{\zeta_3}], \quad C_2 = \phi_*[Q_1] - \phi_*[P_{\zeta_3}].$$

In the above proposition, the cube roots $\sqrt[3]{1+\pi^{3e}u^3}$ belongs to L by Hensel's lemma.

Proof of Proposition 3.4. The map

$$\varPhi_X : A_0(X) \to \mathrm{Hom}(\mathrm{Br}(X)/\mathrm{Br}(k), \mathbb{Q}/\mathbb{Z})$$

is injective by Theorem 3.1. On the other hand,

$$\mathrm{Hom}(\mathrm{Br}(X)/\mathrm{Br}(k), \mathbb{Q}/\mathbb{Z}) \cong (\mathbb{Z}/3\mathbb{Z})^{\oplus 2}$$

by Theorem 2.1. If we construct C_1, C_2 in $A_0(X)$ for which $\varPhi_X(C_1)$ and $\varPhi_X(C_2)$ are linearly independent, then $\varPhi_X$ is surjective, we have $A_0(X) \cong (\mathbb{Z}/3\mathbb{Z})^{\oplus 2}$ and C_1, C_2 are generators of $A_0(X)$.

The following equations hold for a closed point P on X_L and $\mathbf{e}_1 = \{c, f\}_{\zeta_3}$, $\mathbf{e}_2 = \{c, g\}_{\zeta_3} \in \mathrm{Br}(X)$ (see Theorem 2.1) by Lemma 3.3 (here $f = \frac{T_0+\zeta_3 T_1}{T_0+T_1}$, $g = \frac{T_0+T_2}{T_0+T_1}$):

$$\zeta_3^{\langle \phi_*[P],\mathbf{e}_1\rangle_X} = \zeta_3^{\langle [P],\mathbf{e}_1\rangle_{X_L}} = (c, f(P))_L,$$
$$\zeta_3^{\langle \phi_*[P],\mathbf{e}_2\rangle_X} = \zeta_3^{\langle [P],\mathbf{e}_2\rangle_{X_L}} = (c, g(P))_L.$$

We suppose that cycles C_1', C_2' satisfy

$$(3.1) \qquad \det M(C_1', C_2') \neq 0,$$

where

$$M(C_1', C_2') := \begin{pmatrix} \langle C_1', \{c, f\}_{\zeta_3} \rangle_{X_L} & \langle C_1', \{c, g\}_{\zeta_3} \rangle_{X_L} \\ \langle C_2', \{c, f\}_{\zeta_3} \rangle_{X_L} & \langle C_2', \{c, g\}_{\zeta_3} \rangle_{X_L} \end{pmatrix}.$$

Hence C_1', C_2' are linearly independent. Then $C_1 := \phi_* C_1'$ and $C_2 := \phi_* C_2'$ are linearly independent by the projection formula. We construct the 0-cycles $C_1' = [Q_2] - [P_{\zeta_3}]$ and $C_2' = [Q_1] - [P_{\zeta_3}]$ on X_L as follows.

In the following steps (I) and (II), we will construct the L-valued points P_ζ, Q_l $(l = 1, 2)$.

(I) *Computations of* P_{ζ_3}. We suppose that P_{ζ_3} is of the form $P_{\zeta_3} = (1 : t_1 : t_2 : 0)$. We have

$$f(P_{\zeta_3}) = \frac{1 + \zeta_3 t_1}{1 + t_1}, \qquad g(P_{\zeta_3}) = \frac{1 + t_2}{1 + t_1}.$$

Note that $1 + t_1^3 + t_2^3 = 0$. We further suppose that

$$t_1 = \pi^e u.$$

Here u is an element of $\mathscr{O}_L$ such that

$$\mathrm{Tr}_{\mathbb{F}_L/\mathbb{F}_3}\left(\overline{(1 - \zeta_3)^{-3} \pi^{e'} \delta u}\right) = 2.$$

We have $t_2 = -\sqrt[3]{1 + \pi^{3e} u}$. Then

$$P_{\zeta_3} = \left(1 : \pi^e u : -\sqrt[3]{1 + \pi^{3e} u^3} : 0\right).$$

We have

$$\begin{aligned} (c, f(P_{\zeta_3}))_L &= \left(c, \frac{1 + \zeta_3 t_1}{1 + t_1}\right)_L = \left(c, 1 + (\zeta_3 - 1)\frac{t_1}{1 + t_1}\right)_L \\ &= \left(c, 1 - \pi^{e/2} \delta \frac{\pi^e u}{1 + \pi^e u}\right)_L = \zeta_3^{-\operatorname{ord}_k(c) \operatorname{Tr}_{\mathbb{F}_L/\mathbb{F}_3}(\overline{(1 - \zeta_3)^{-3} \pi^{e'} \delta u})} \\ &= \begin{cases} \zeta_3 & \text{if } \operatorname{ord}_k(c) \equiv 1 \bmod 3, \\ \zeta_3^2 & \text{if } \operatorname{ord}_k(c) \equiv 2 \bmod 3. \end{cases} \end{aligned}$$

Let q be the integer with $0 \le q \le 1$ such that $(c, g(P_{\zeta_3}))_L = \zeta_3^q$.

(II) *Computations of* Q_l $(l = 1, 2)$. We set

$$Q_l := \left(1 : \pi^{e/2} \delta \sqrt[3]{1 + \pi^{e'} w_l} : -\sqrt[3]{1 + \pi^{e'} \delta^3 (1 + \pi^{e'} w_l)} : 0\right)$$

$(l = 1, 2)$. Here w_l $(l = 1, 2)$ are units of $\mathscr{O}_k$ such that

$$\mathrm{Tr}_{\mathbb{F}_k/\mathbb{F}_3}(\overline{(1 - \zeta_3)^{-3} \pi^{e'} w_l}) = l.$$

We have

$$(c, f(Q_l))_L = \left(c, \frac{1+\zeta_3 t_1}{1+t_1}\right)_L = \frac{(c, 1+\pi^{e/2}\delta\zeta_3\sqrt[3]{1+\pi^{e'}w_l})_L}{(c, 1+\pi^{e/2}\delta\sqrt[3]{1+\pi^{e'}w_l})_L}$$
$$= \frac{(c, 1+\pi^{e'}\delta^3(1+\pi^{e'}w_l))_k}{(c, 1+\pi^{e'}\delta^3(1+\pi^{e'}w_l))_k} = 1$$

and

$$(c, g(Q_l))_L = \frac{\left(c, 1-\sqrt[3]{1+\pi^{e'}\delta^3(1+\pi^{e'}w_l)}\right)_L}{\left(c, 1+\pi^{e/2}\delta\sqrt[3]{1+\pi^{e'}w_l}\right)_L} = \frac{(c, -\pi^{e'}\delta^3(1+\pi^{e'}w_l))_k}{(c, 1+\pi^{e'}\delta^3(1+\pi^{e'}w_l))_k}$$
$$= (c, 1+\pi^{e'}w_l)_k \cdot \zeta_3^{-\operatorname{Tr}_{\mathbb{F}_k/\mathbb{F}_3}(\overline{(1-\zeta_3)^{-3}\pi^{e'}\delta^3})}$$
$$= (c, 1+\pi^{e'}w_l)_k \cdot \zeta_3^{-\operatorname{Tr}_{\mathbb{F}_k/\mathbb{F}_3}(\overline{1})}$$
$$= \zeta_3^{\operatorname{ord}_k(c)\operatorname{Tr}_{\mathbb{F}_k/\mathbb{F}_3}(\overline{(1-\zeta_3)^{-3}\pi^{e'}w_l})-[\mathbb{F}_k:\mathbb{F}_3]} = \zeta_3^l.$$

We will check that the 0-cycles $C_1' = [Q_2] - [P_{\zeta_3}]$ and $C_2' = [Q_1] - [P_{\zeta_3}]$ satisfy (3.1).

(i) $\operatorname{ord}_k(c) \equiv 1 \bmod 3$: We have

$$M(C_1', C_2') = \begin{pmatrix} 2 & 2-q \\ 2 & 1-q \end{pmatrix}$$

and $\det M(C_1', C_2') = -2 \neq 0$.

(ii) $\operatorname{ord}_k(c) \equiv 2 \bmod 3$: We have

$$M(C_1', C_2') = \begin{pmatrix} 1 & 2-q \\ 1 & 1-q \end{pmatrix}$$

and $\det M(C_1', C_2') = -1 \neq 0$.

Therefore $A_0(X)$ is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^{\oplus 2}$ and generated by $C_1 = \phi_*[Q_2] - \phi_*[P_{\zeta_3}]$ and $C_2 = \phi_*[Q_1] - \phi_*[P_{\zeta_3}]$. This completes the proof. ∎

3.2. The case $\zeta_3 \notin k$. In this section, we assume that $\zeta_3 \notin k$.

Proposition 3.5. *Set $M = k(\zeta_3)$ and $q : X_M \to X$. Let L be a cubic unramified extension over M. Then $A_0(X)$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$ and generated by q_*C with*

$$C := \phi_*[Q_2] - \phi_*[P'_{\zeta_3}],$$

where Q_2 and P_{ζ_3} are defined in Proposition 3.4. Here u is a unit of $\mathscr{O}_L$ such that $\operatorname{Tr}_{\mathbb{F}_L/\mathbb{F}_3}(\overline{(1-\zeta_3)^{-3}\pi^{e'}\delta u}) = 2$ and w_2 is unit of $\mathscr{O}_k$ such that

$$\operatorname{ord}_k(c)\operatorname{Tr}_{\mathbb{F}_k/\mathbb{F}_3}(\overline{(1-\zeta_3)^{-3}\pi^{e'}w_2}) - [\mathbb{F}_k : \mathbb{F}_3] \equiv 2 \bmod 3.$$

Proof. We recall that k is a 3-adic field which does not contain a primitive cube root ζ_3 of unity. The map $\Phi_X : A_0(X) \to \mathrm{Hom}(\mathrm{Br}(X)/\mathrm{Br}(k), \mathbb{Q}/\mathbb{Z})$ is injective by Theorem 3.1. On the other hand,

$$\mathrm{Hom}(\mathrm{Br}(X)/\mathrm{Br}(k), \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z}$$

by Proposition 2.2. Thus $A_0(X) \subseteq \mathbb{Z}/3\mathbb{Z}$. For $C'' \in A_0(X_M)$, we have the following calculations for the Brauer–Manin pairing:

$$\begin{aligned}\langle q_* C'', \mathrm{cor}_{X_M/X}(\mathbf{e}_1)\rangle_X &= \langle C'', q^* q_*(\mathbf{e}_1)\rangle_{X_M} && \text{(projection formula)}\\ &= \langle C'', (1+\sigma)\mathbf{e}_1\rangle_{X_M} && (\sigma \in \mathrm{Gal}(M/k))\\ &= 2\langle C'', \mathbf{e}_1\rangle_{X_M} && (\sigma(\mathbf{e}_1) = \mathbf{e}_1).\end{aligned}$$

Therefore if we construct a cycle C'' of $A_0(X_M)$ such that $\langle C'', \mathbf{e}_1\rangle \neq 0$, then $q_* C''$ generates $A_0(X)$. By Proposition 3.4 we have $\langle C, \mathbf{e}_1\rangle_{X_M} \neq 0$, where $C := \phi_*[Q_2] - \phi_*[P'_{\zeta_3}]$. Thus we can take $C'' = C$. This completes the proof. ∎

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