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Method of lines for pseudoparabolic equations

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Abstract. We consider initial-boundary-value problems for a class of pseudoparabolic partial differential equations. Local-in-time existence results and nonexistence of global-in-time solutions are proved. We apply the method of lines to approximate pseudoparabolic equations by systems of ordinary differential equations. We present a complete convergence analysis for this semidiscretization method. Blow-up for approximate solutions is showed. Numerical experiments confirm the theoretical results.

1. Introduction. Sobolev type equations, sometimes called Sobolev–Galpern type, are evolution equations such that the partial derivative in time of the highest order is given in an implicit form [27]. An important subclass of Sobolev type equations are pseudoparabolic equations where only a first derivative in time appears [25, 26, 29]. We will investigate initial-boundary-value problems for this type of equations.

Let $T, \varepsilon > 0$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain whose boundary $\partial\Omega$ is sufficiently smooth. We consider the pseudoparabolic equation

$$(1.1)\qquad \frac{\partial}{\partial t}(u - \varepsilon \Delta u) = F(x,t,u) \quad \text{on } \Omega \times [0,T]$$

with the initial and boundary conditions

$$(1.2)\qquad u(x,0) = \varphi(x) \quad \text{for } x \in \Omega, \qquad u(x,t) = 0 \quad \text{on } \partial\Omega \times [0,T],$$

where $F : \overline{\Omega} \times [0,T] \times \mathbb{R} \to \mathbb{R}$ and $\varphi : \overline{\Omega} \to \mathbb{R}$ are given continuous functions and Δ is the Laplace operator with respect to x.

Sobolev type equations, in particular pseudoparabolic equations, describe many physical processes such as homogeneous fluid flows through a fissured rock [3], unidirectional propagation of nonlinear long waves [4, 28], aggre-

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gation of population [22], two-temperature theory for heat conduction in nonsimple materials [7, 30] or nonstationary processes in semiconductors in the presence of sources [2, 15, 16]. There are applicable problems which can be obtained from the Dirichlet problem (1.1), (1.2) by specializing the given function F. Though the left-hand side of (1.1) does not contain any additional linear term with the Laplace operator, i.e. without the time derivative operator acting on it, [6, 25, 30], our formulation is not less general. The equation

$$\frac{\partial}{\partial t}(u - \varepsilon \Delta u) - \eta \Delta u = F(x, t, u) \quad \text{on } \Omega \times [0, T] \tag{1.3}$$

with $\eta \geq 0$ can be reduced to (1.1) by the substitution $v = ue^{\eta t/\varepsilon}$.

Serious investigations of blow-up problems for parabolic equations with sources begin with a paper of Fujita [12]. These investigations have been continued by many authors (see for instance [9, 19, 20] and the references therein). The same subject for Sobolev type equations is relatively less investigated: see [2, 6, 21] and the references therein.

In the cases when analytical considerations are not sufficiently effective, numerical methods allow one to investigate solutions of PDE problems in detail. Such investigations, also for pseudoparabolic type equations, are carried out by various numerical procedures. We are interested in a space discretization method called the method of lines (MOL) [1, 24]. The Rothe method discretizes partial differential problems with respect to time [5, 23]. The methods mentioned above are suitable for both theoretical and numerical analysis of PDE problems. Each of these methods follows a semianalytical approach and can be an intermediate step in the derivation of a fully discrete scheme. For finite difference schemes with full discretization in space and in time we refer the reader to [8, 11, 14] and the bibliography therein. According to our knowledge, the literature concerning convergence analysis of MOL for pseudoparabolic equations is rather poor, especially if we are looking for a rigorous mathematical treatment of these problems. For instance, in [1] the method of lines is employed to detect the blow-up time of solutions, but no theorem on the convergence of the method is given.

In the first part of the paper we give auxiliary existence results for the PDE problem (1.1), (1.2). We prove local-in-time existence and global-in-time nonexistence theorems for classical solutions under general assumptions on data. An estimate of the blow-up time is given. In the second part of the paper our main interest is to approximate solutions of the problem (1.1), (1.2) by the method of lines. The space discretization of MOL transforms PDEs into systems of ODEs which can be solved with the help of advanced ODE solver packages. We show the local existence of solutions to ODE problems obtained by MOL. We prove the convergence of our semidiscretization method. A result on blow-up for approximate solutions is also proved. We present numerical experiments which confirm the theoretical results, especially concerning blow-up times.

In this paper we focus on the mathematical analysis of MOL. We are interested in two aspects of this method. First, we are interested in giving a convergence analysis of MOL with a general function $F = F(x, t, u)$ and with an arbitrary $\varepsilon > 0$, the coefficient of Δu. The second aspect is the estimation of numerical blow-up points for such general F and comparison with the continuous case. The aim of our numerical experiments is to illustrate the theoretical results and to show consistency with the theory of Sections 2 and 3. To get estimates of the blow-up time we significantly generalize the ideas of [16].

2. Local solvability and blow-up of solutions. We consider classical solutions of the problem (1.1), (1.2). Let $C_0(\overline{\Omega})$ stand for the space of all continuous scalar functions on $\overline{\Omega}$ vanishing on $\partial\Omega$. We denote by $C^1([0,T]; C^2(\Omega)\cap C_0(\overline{\Omega}))$ the class of all $u : [0,T]\times\overline{\Omega} \to \mathbb{R}$ such that $u(t,\cdot)$ and $\frac{\partial u}{\partial t}(t,\cdot)$ are in $C^2(\Omega)\cap C_0(\overline{\Omega})$ for each t. We say that u is a *solution* of (1.1), (1.2) if $u \in C^1([0,T]; C^2(\Omega)\cap C_0(\overline{\Omega}))$ and (1.1), (1.2) hold true.

The problem (1.1), (1.2) can be reduced to the integro-differential equation

$$u(x,t) - \varepsilon\Delta u(x,t) = \varphi(x) - \varepsilon\Delta\varphi(x) + \int_0^t F(x,s,u(x,s))\,ds \tag{2.1}$$

in the class of functions $u \in C^1([0,T]; C^2(\Omega)\cap C_0(\overline{\Omega}))$. Note that by the uniqueness result for the Dirichlet problem, the initial condition $u(\cdot,0) = \varphi$ follows from (2.1).

For any real number b we define $b^+ = \max\{b,0\}$ and $b^- = -\min\{b,0\}$. Let $|\cdot|_\infty$ denote the maximum norm in $C_0(\overline{\Omega})$. In the following we assume that $\gamma \in C^2(\Omega)\cap C_0(\overline{\Omega})$ is a solution of the equation

$$\gamma(x) - \varepsilon\Delta\gamma(x) = 1, \qquad x \in \Omega.$$

Set

$$c_\gamma = |\gamma|_\infty. \tag{2.2}$$

By a simple comparison argument we have $0 < \gamma(x) \le 1$ in Ω, $0 < c_\gamma \le 1$.

In order to obtain a local existence result for (1.1), (1.2) we need the following lemma.

Lemma 2.1 (A priori estimates for PDE). *Suppose that*

(1) $\varphi \in C^2(\Omega)\cap C_0(\overline{\Omega})$ *and the function* $F = F(x,t,u)$ *is nondecreasing and locally Lipschitz continuous in* u,

(2) *$\alpha \in C^1([0,T_\alpha])$ is a solution of the ODE problem*
$$\frac{dr}{dt} = -c_\gamma\, g^-(t,r), \qquad r(0)=0,$$
and $\beta \in C^1([0,T_\beta])$ is a solution of the problem
$$\frac{dr}{dt} = c_\gamma\, G^+(t,r), \qquad r(0)=0,$$
where c_γ is given by (2.2) *and*

$$g(t,r) = \inf_{x\in\Omega} F(x,t,r+\varphi(x)), \qquad G(t,r) = \sup_{x\in\Omega} F(x,t,r+\varphi(x)). \tag{2.3}$$

If u is a solution of (1.1), (1.2), *then*
$$u(x,t) \ge \alpha(t) + \varphi(x), \qquad t\in[0, T\wedge T_\alpha],$$
and
$$u(x,t) \le \beta(t) + \varphi(x), \qquad t\in[0, T\wedge T_\beta],$$
with $x \in \overline{\Omega}$, where $\wedge$ stands for minimum.

Proof. Set $w = u - \varphi$. It follows from (2.1) that
$$w(x,t) - \varepsilon\Delta w(x,t) \le \int_0^t F(x,s,\overline{w}(s)+\varphi(x))\,ds \le \int_0^t G(s,\overline{w}(s))\,ds$$
on $\Omega\times[0,T]$, where $\overline{w}(t) = \max_{x\in\overline{\Omega}} w(x,t)$ is a continuous function. Observe that G is nondecreasing in r and in view of (2.1) we have $\overline{w}(0)=0$. If we fix $t\in[0,T]$ then $z(\cdot,t) = B(t)\gamma(\cdot)$ is a solution of the boundary value problem

$$z - \varepsilon\Delta z = B(t) \quad \text{on } \Omega, \qquad z|_{\partial\Omega} = 0, \tag{2.4}$$

with $B(t) = \int_0^t G(s,\overline{w}(s))\,ds$. By the maximum principle for (2.4) with the function $w(\cdot,t) - z(\cdot,t)$ [17, Lemma 1.6, p. 160] we have
$$w(x,t) \le \gamma(x)\,B(t).$$
Therefore
$$\overline{w}(t) \le c_\gamma\, B^+(t), \qquad \overline{w}(0)=0,$$
and finally
$$\overline{w}(t) \le c_\gamma \int_0^t G^+(s,\overline{w}(s))\,ds, \qquad \overline{w}(0)=0.$$
By a standard argument we obtain $\overline{w}(t)\le\beta(t)$, hence
$$w(x,t) \le \beta(t), \qquad x\in\overline{\Omega},\ t\in[0,T\wedge T_\beta].$$
Similarly setting $\underline{w}(t) = \min_{x\in\overline{\Omega}} w(x,t)$ we have
$$\underline{w}(t) \ge -c_\gamma \int_0^t g^-(s,\underline{w}(s))\,ds, \qquad \underline{w}(0)=0,$$
which completes the proof . ∎

Remark 2.2. Clearly, each of the comparison problems in assumption (2) of Lemma 2.1 has exactly one solution in some interval $[0, T_\alpha]$ or in $[0, T_\beta]$. Indeed, by the property of F assumed in (1) and by its continuity, both $g^- = g^-(t, r)$ and $G^+ = G^+(t, r)$ are continuous functions that are nondecreasing and locally Lipschitz continuous in r.

We will show that the solution of (1.1), (1.2) exists in a set generated by a priori estimates.

Theorem 2.3 (Existence for PDE). *Suppose that the assumptions of Lemma 2.1 are satisfied, $\partial\Omega$ is of class $C^{2+\mu}$ with $\mu \in (0, 1)$, and $\hat{T} = T_\alpha \wedge T_\beta$. Moreover, in the case $n > 1$ the function F is assumed to be Hölder continuous in x. Then there exists a unique solution of* (1.1), (1.2) *defined on $\overline{\Omega} \times [0, \hat{T}]$. This solution satisfies*

$$\alpha(t) + \varphi(x) \le u(x, t) \le \beta(t) + \varphi(x) \quad \text{in } \overline{\Omega} \times [0, \hat{T}].$$

Proof. We only need to prove the existence of solutions to problem (2.1) with $\varphi \equiv 0$. Recall that α, β are given in Lemma 2.1. Define

$$[\alpha, \beta] = \{v \in C([0, \hat{T}]; C_0(\overline{\Omega})) : \alpha(t) \le v(x, t) \le \beta(t) \text{ on } \overline{\Omega} \times [0, \hat{T}]\}.$$

Fix $v \in [\alpha, \beta]$ and $t \in [0, \hat{T}]$. Let $V(\cdot, t)$ be a weak solution of the problem

$$V(x, t) - \varepsilon \Delta V(x, t) = \int_0^t F(x, s, v(x, s))\, ds, \qquad V|_{\partial\Omega} = 0,$$

The function $V(\cdot, t)$ belongs to the class $C^1(\bar{\Omega})$ [17, Theorem 15.1, p. 251]. We have

$$V(x, t) - \varepsilon \Delta V(x, t) \le \int_0^t G(s, \beta(s))\, ds$$

in the weak sense, where G is given by (2.3) with $\varphi \equiv 0$. By a similar argument as in the proof of Lemma 2.1, with the aid of the weak maximum principle [13, Theorem 8.1, p. 179] we obtain

$$V(x, t) \le \gamma(x) \int_0^t G(s, \beta(s))\, ds \le c_\gamma \int_0^t G^+(s, \beta(s))\, ds = \beta(t).$$

One can show in a similar way that $V(x, t) \ge \alpha(t)$ on $\overline{\Omega} \times [0, \hat{T}]$. Thus the operator S given by $S[v] = V$ maps $[\alpha, \beta]$ into itself. It is also contractive with respect to the norm $\|\cdot\|_{\hat{T}}$ where

$$\|v\|_t = \sup_{(x,s) \in \overline{\Omega} \times [0, t]} |v(x, s)|\, e^{-\int_0^s 2K(\tau)\, d\tau}, \qquad t \in [0, \hat{T}],$$

and $K : [0, \hat{T}] \to \mathbb{R}_+$ is an integrable function such that

$$|F(x, t, u) - F(x, t, \tilde{u})| \le K(t)|u - \tilde{u}|$$

for all $x \in \Omega$, $u, \tilde{u} \in [\alpha(t), \beta(t)]$, $t \in [0, \hat{T}]$. Indeed, if $v, \tilde{v} \in [\alpha, \beta]$ then applying the theorem on weak inequalities we obtain

$$|S[v](x,t) - S[\tilde{v}](x,t)| \le \tfrac{1}{2}\gamma(x)\|v - \tilde{v}\|_t \, e^{\int_0^t 2K(\tau)\,d\tau}$$

and

$$\|S[v] - S[\tilde{v}]\|_{\hat{T}} \le \tfrac{1}{2}c_\gamma \|v - \tilde{v}\|_{\hat{T}}.$$

Since $\frac{1}{2}c_\gamma < 1$, one can apply the Banach contraction principle to get the unique fixed point $\tilde{v} \in [\alpha, \beta]$ such that $\tilde{v}(t, \cdot) \in C^1(\bar{\Omega})$ and

$$\tilde{v}(x,t) - \varepsilon \Delta \tilde{v}(x,t) = \int_0^t F(x, s, \tilde{v}(x,s))\,ds, \qquad \tilde{v}|_{\partial\Omega} = 0,$$

in a weak sense. In the case $n > 1$ the right-hand side is Hölder continuous in x, hence $\tilde{v}(t, \cdot)$ is a classical solution [17, Theorem 1.1, p. 145]. Note that if $n = 1$, then $S[v]$ is a classical solution. Thus in this case we do not need a weak formulation in the proof. ∎

Remark 2.4. Under the assumption (1) of Lemma 2.1 with $\varphi \in C^2(\Omega) \cap C_0(\overline{\Omega}) \cap C^1(\overline{\Omega})$, the solution u of (1.1), (1.2) satisfies $u \in C^1([0,T]; C^2(\Omega) \cap C_0(\overline{\Omega}) \cap C^1(\overline{\Omega}))$ (see [1, 2]).

The solution u may become unbounded at a time $T^* \le T$ (it blows up). This property is very important from the mathematical and physical point of view. For example, a disruption of a semiconductor can be described as a blow-up of a solution for a suitable differential problem. We say that $T^* > 0$ is a *blow-up* time for the solution u of (1.1), (1.2) if

$$\lim_{t \nearrow T^*} |u(\cdot, t)|_\infty = \infty.$$

The theorem on blow-up for the solution of (1.1), (1.2) is a consequence of the following result.

Lemma 2.5. *Suppose that*

$$\inf_{x \in \Omega} F(x,t,u) \ge F_*(t,u) \tag{2.5}$$

for some continuous function $F_* : [0,T] \times \mathbb{R} \to \mathbb{R}$ *which is convex and non-decreasing in* u. *Let* $\lambda > 1$ *be the first eigenvalue of the operator* $I - \varepsilon\Delta$ *and let* $\omega \ge 0$ *be the corresponding normalized eigenfunction, i.e.* $\int_\Omega \omega(y)\,dy = 1$ *[10, Theorem 1, p. 335]. Assume that* $\varrho \in C^1([0, T_\varrho])$ *is a minimal solution of the problem*

$$\frac{dr}{dt} = \frac{1}{\lambda} F_*(t, r), \qquad r(0) = \varrho_0 := \int_\Omega \varphi(y)\omega(y)\,dy. \tag{2.6}$$

If u is the solution of (1.1), (1.2) *for $t \in [0,T]$, then*

$$\int_\Omega u(y,t)\omega(y)\,dy \ge \varrho(t) \quad \text{for } t \in [0, T \wedge T_\varrho].$$

Proof. Multiplying (2.1) by ω and integrating by parts over Ω we obtain

$$\int_\Omega (u(y,t) - \varphi(y))(\omega(y) - \varepsilon\Delta\omega(y))\,dy = \int_0^t \int_\Omega \omega(y) F(y,s,u(y,s))\,dy\,ds$$

or

$$\begin{aligned}\lambda \int_\Omega (u(y,t) - \varphi(y))\omega(y)\,dy &= \int_0^t \int_\Omega \omega(y) F(y,s,u(y,s))\,dy\,ds \\ &\ge \int_0^t \int_\Omega \omega(y) F_*(s,u(y,s))\,dy\,ds.\end{aligned}$$

By the Jensen inequality we get

$$\lambda \int_\Omega (u(y,t) - \varphi(y))\omega(y)\,dy \ge \int_0^t F_*\Big(s, \int_\Omega \omega(y)u(y,s)\,dy\Big)\,ds.$$

We set

$$W(t) := \int_\Omega u(y,t)\omega(y)\,dy.$$

Then

$$W(t) \ge \varrho_0 + \frac{1}{\lambda}\int_0^t F_*(s, W(s))\,ds.$$

Since $W(0) = \varrho_0$, we have $W(t) \ge \varrho(t)$ for $t \in [0, T \wedge T_\varrho]$. This completes the proof. ∎

Remark 2.6. If F is nondecreasing in u and $F_*(t,u) = \inf_{x\in\Omega} F(x,t,u)$ is convex in u, then F satisfies the assumptions of Lemma 2.5 with this F_*. A simple example is $F(x,t,u) = \tilde{F}(t,u) + f(x,t)$, where $\tilde{F}(t,\cdot)$ is convex, nondecreasing and f is bounded from below.

It follows easily from Lemma 2.5 that if ϱ has a (positive) blow-up at some point $T_\varrho^* > T_\varrho > 0$, then the solution u of (1.1), (1.2) also blows up at some $T^* \in (0, T_\varrho^*]$. It is not difficult to prove the following result on blow-up for the problem (1.1), (1.2).

Corollary 2.7 (Blow-up for PDE). *Let $a : [0,T] \to \mathbb{R}_+$ be continuous, and let $H : \mathbb{R} \to \mathbb{R}_+$ be continuous and positive for $r > \varrho_0$ with ϱ_0 given in* (2.6). *Suppose that $F_* : [0,T] \times \mathbb{R} \to \mathbb{R}_+$ given by*

$$F_*(t,r) = a(t)H(r) \tag{2.7}$$

satisfies the assumptions of Lemma 2.5 *and*

$$\lambda \int_{\varrho_0}^{\infty} \frac{1}{H(r)}\,dr \le \int_0^T a(s)\,ds. \tag{2.8}$$

Then the solution u *of* (1.1), (1.2) *blows up at some* $T^* \in (0, T^*_\varrho]$, *where* $0 < T^*_\varrho \le T$ *is defined by*

$$T^*_\varrho = \min\Big\{ t \in (0,T] : \lambda \int_{\varrho_0}^{\infty} \frac{1}{H(r)}\,dr = \int_0^t a(s)\,ds \Big\}. \tag{2.9}$$

Example 2.8. Suppose that u is the solution of (1.1), (1.2) with

$$F(x,t,u) = e^{au}, \qquad a > 0.$$

Then $u(x,t) \le \beta(t) + \varphi(x)$ on $\Omega \times [0, T_\beta]$, where $T_\beta < \frac{1}{ac_\varphi c_\gamma}$ and

$$\beta(t) = \frac{1}{a} \ln \frac{1}{1 - ac_\varphi c_\gamma t}, \qquad t \in [0, T_\beta],$$

with $c_\varphi = e^{a\,|\varphi|_\infty}$ and c_γ given by (2.2). Thus

$$u(x,t) \le \frac{1}{a} \ln \frac{c_\varphi}{1 - ac_\varphi c_\gamma t} \qquad \text{on } \Omega \times [0, T_\beta].$$

By a simple argument we also have $u \ge \varphi$ in the domain of existence of u ($\hat{T} = T_\beta$). In view of (2.9), $T^*_\varrho = \lambda\,(ae^{a\varrho_0})^{-1}$ and for the blow-up time we have the upper estimate $T^* \le T^*_\varrho$.

Remark 2.9. The special case of F in Example 2.8 with $a = 1$ ($\varepsilon = 1$) was treated in [16, Theorem 1]. For $\varphi \equiv 0$ an upper bound for the blow-up time in [16] is $T_1 = \mu(\Omega)/\int_\Omega \gamma(x)\,dx$, where $\mu(\Omega)$ is the Lebesgue measure of Ω. We claim that our results imply a better estimation. Indeed, by Corollary 2.7 our upper bound is $T^*_\rho = \lambda$. By Rayleigh's formula [10, 6.5.1] we get

$$\frac{\int_\Omega |D\gamma(x)|^2\,dx}{\int_\Omega |\gamma(x)|^2\,dx} \ge \lambda - 1, \qquad \frac{\int_\Omega \gamma(x)\,dx - \int_\Omega |\gamma(x)|^2\,dx}{\int_\Omega |\gamma(x)|^2\,dx} \ge \lambda - 1,$$

hence

$$\frac{\int_\Omega \gamma(x)\,dx}{\int_\Omega |\gamma(x)|^2\,dx} \ge \lambda.$$

By the Schwarz inequality we have $(\int_\Omega \gamma(x)\,dx)^2 < \mu(\Omega) \int_\Omega |\gamma(x)|^2\,dx$. The inequality is strict since γ is not constant. This yields

$$T_1 = \frac{\mu(\Omega)}{\int_\Omega \gamma(x)\,dx} > \lambda = T^*_\rho.$$

For $n = 1$, $\Omega = (0,1)$ we can compute explicitly $T_1 = 13.199$, $T^*_\rho = 10.870$.

EXAMPLE 2.10. Let

$$F_*(t,r) = r^{1+\delta}e^{-bt}$$

for $t \in [0,\infty)$ with $\delta, b > 0$, and $\varphi \geq 0$, $\varrho_0 > 0$. We see that all the solutions of (1.1), (1.2) are nonnegative, thus we can restrict F_* to $r \geq 0$. Condition (2.8) and the number T^*_ϱ become

$$\frac{\lambda}{\delta\varrho_0^\delta} < \frac{1}{b}, \qquad T^*_\varrho = -\frac{1}{b}\ln\left(1 - \frac{b\lambda}{\delta\varrho_0^\delta}\right).$$

If $b = \delta\eta/\varepsilon$, then $\varrho_0 > (\lambda\eta/\varepsilon)^{1/\delta}$ which is a blow-up condition for (1.3) with

$$F(x,t,u) \geq u^{1+\delta}.$$

After reduction of (1.3) to (1.1) we obtain the right-hand side of the form $e^{\eta t/\varepsilon}F(x,t,e^{-\eta t/\varepsilon}u)$. We also have

$$T^*_\varrho = -\frac{\varepsilon}{\delta\eta}\ln\left(1 - \frac{\eta\lambda}{\varepsilon\varrho_0^\delta}\right) \quad \text{for } \eta > 0, \qquad T^*_\varrho = \frac{\lambda}{\delta\varrho_0^\delta} \quad \text{for } \eta = 0.$$

3. The method of lines. We formulate a discretization method in the case of one-dimensional spatial variable and we consider the problem (1.1), (1.2) with $\Omega = (0,\Lambda)$ where $\Lambda > 0$. Fix an integer $N > 0$. We define the uniform spatial grid on $\overline{\Omega} = [0,\Lambda]$ as follows. Let $h = \Lambda/N$ be the step of the mesh and

$$J = \{k : 0 \leq k \leq N\}, \qquad J_0 = \{k : 0 < k < N\}$$

be sets of indices. Set $x_k = kh$, $k \in J$. For $k \in J_0$ the point x_k is in Ω, while for $k \in J \setminus J_0$ it lies on $\partial\Omega$.

Denote by l^N the set of all $p = \{p_k\}_{k\in J}$ with $p_k \in \mathbb{R}$. We define the discrete Laplace operator on l^N by the standard approximation of the second derivative in x, i.e.

$$(\Delta_h\, p)_k = \frac{p_{k+1} - 2p_k + p_{k-1}}{h^2}, \qquad k \in J_0. \tag{3.1}$$

We will write $\Delta_h\, p_k = (\Delta_h\, p)_k$ hereafter. Obviously, we use Δ_h to approximate the Laplace operator Δ in (1.1) with $n = 1$. In this way we arrive at the system of ordinary differential equations

$$\frac{d}{dt}(z_k(t) - \varepsilon\Delta_h\, z_k(t)) = F(x_k, t, z_k(t)), \qquad k \in J_0, \tag{3.2}$$

where

$$z_k(t) = 0, \qquad k \in J \setminus J_0, \tag{3.3}$$

with the initial conditions

$$z_k(0) = \varphi(x_k), \qquad k \in J. \tag{3.4}$$

The method (3.2)-(3.4) is called *the method of lines* (MOL) for the PDE problem (1.1), (1.2).

The method of lines (3.2)–(3.4) is equivalent to the nonlocal problem

(3.5)
$$z_k(t) - \varepsilon\Delta_h\, z_k(t) = \varphi(x_k) - \varepsilon\Delta_h\varphi(x_k) + \int_0^t F(x_k, s, z_k(s))\, ds, \qquad k \in J_0,$$

(3.6) $$z_k(t) = 0, \quad k \in J \setminus J_0.$$

Note that the initial conditions (3.4) follow from (3.5).

We will need the following comparison lemma.

LEMMA 3.1. *If $p, q \in l^N$ satisfy the inequalities*

$$p_k - \varepsilon\Delta_h p_k \le q_k - \varepsilon\Delta_h q_k, \qquad k \in J_0,$$

then the relations $p_k \le q_k$ for $k \in J \setminus J_0$ imply

$$p_k \le q_k \quad \text{for } k \in J.$$

Proof. Set $r_k = p_k - q_k$ for $k \in J$. Then

$$(I - \varepsilon\Delta_h) r_k \le 0, \qquad k \in J_0.$$

Suppose that $M = \max_{k\in J}\{r_k\}$ is positive. We only need to prove that the equality $M = r_k$ is possible only for $k \in J \setminus J_0$. If there is $j \in J_0$ such that $r_j = M$, then

$$\begin{aligned}(I - \varepsilon\Delta_h) r_j &= -\varepsilon\frac{1}{h^2} r_{j+1} + \left(1 + \varepsilon\frac{2}{h^2}\right) M - \varepsilon\frac{1}{h^2} r_{j-1} \\ &\ge -\varepsilon\frac{1}{h^2} M + \left(1 + \varepsilon\frac{2}{h^2}\right) M - \varepsilon\frac{1}{h^2} M = M > 0,\end{aligned}$$

which contradicts the assumptions. ∎

Set $l_0^N = \{p \in l^N : p_k = 0 \text{ for } k \in J \setminus J_0\}$. Then l_0^N is a Hilbert space with the inner product given by $(p, q) := \sum_{k\in J} p_k q_k$. From now on we will consider Δ_h as the operator $\Delta_h : l_0^N \to l_0^N$ given by (3.1) for $k \in J_0$ and $(\Delta_h p)_k = 0$ for $k \in J \setminus J_0$.

We prove the following proposition.

PROPOSITION 3.2. *The operator $I - \varepsilon\Delta_h : l_0^N \to l_0^N$ is positive and self-adjoint. Moreover, $I - \varepsilon\Delta_h$ is invertible.*

Proof. We only need to prove that Δ_h is nonpositive and self-adjoint. Indeed, for $p \in l_0^N$ we have

$$(\Delta_h p, p) = \frac{1}{h^2}\sum_{k=1}^{N-1}(p_{k+1} - 2p_k + p_{k-1})p_k = \frac{1}{h^2}\Big(\sum_{k=2}^{N-1} p_k p_{k-1} - 2\sum_{k=1}^{N-1} p_k^2\Big)$$
$$\le \frac{1}{h^2}\Big(\sum_{k=2}^{N-1}(p_k^2 + p_{k-1}^2) - 2\sum_{k=1}^{N-1} p_k^2\Big) = -\frac{1}{h^2}(p_{N-1}^2 + p_1^2) \le 0,$$

and

$$(\Delta_h p, q) = \frac{1}{h^2}\Big(\sum_{k=1}^{N-1} p_k q_{k-1} - 2\sum_{k=1}^{N-1} p_k q_k + \sum_{k=1}^{N-1} p_k q_{k+1}\Big) = (p, \Delta_h q). \ \blacksquare$$

Suppose that $\eta = \eta(h) \in l_0^N$ is such that

$$\eta_k - \varepsilon\Delta_h\eta_k = 1, \qquad k \in J_0.$$

Define $c_\eta = c_\eta(h)$ by

$$c_\eta = \max_{k\in J}\eta_k. \tag{3.7}$$

It follows from Lemma 3.1 that $0 \le \eta_k \le 1$ for $k \in J$. Since η is nontrivial, we have $0 < c_\eta \le 1$.

We assume that condition (1) of Lemma 2.1 holds and we prove a lemma on a priori estimates for solutions to MOL.

LEMMA 3.3 (A priori estimates for MOL). *Suppose that* $\alpha_h \in C^1([0, T_{\alpha_h}])$ *is a solution of the problem*

$$\frac{dr}{dt} = -c_\eta g^-(t, r), \qquad r(0) = 0,$$

and $\beta_h \in C^1([0, T_{\beta_h}])$ *is a solution of the problem*

$$\frac{dr}{dt} = c_\eta G^+(t, r), \qquad r(0) = 0,$$

where g, G *are given by* (2.3) *and* $c_\eta = c_\eta(h)$ *is defined by* (3.7). *If* $z : [0, T] \to l^N$ *is a classical solution of the problem* (3.2)–(3.4) *then*

$$z_k(t) \ge \alpha_h(t) + \varphi(x_k), \qquad k \in J,\ t \in [0, T \wedge T_{\alpha_h}],$$
$$z_k(t) \le \beta_h(t) + \varphi(x_k), \qquad k \in J,\ t \in [0, T \wedge T_{\beta_h}].$$

Proof. We use the method of the proof of Lemma 2.1. Set $y_k = z_k - \varphi(x_k)$. By (3.5) we have

$$y_k(t) - \varepsilon\Delta_h\, y_k(t) \le \int_0^t F\big(x_k, s, \overline{y}(s) + \varphi(x_k)\big)\, ds \le \int_0^t G(s, \overline{y}(s))\, ds$$

where $\overline{y}(t) = \max_{k\in J} y_k(t)$. In view of Lemma 3.1, we obtain

$$y_k(t) \le \eta_k \int_0^t G(s, \overline{y}(s))\, ds, \qquad k \in J_0,$$

with $t \in [0,T]$. Therefore

$$\overline{y}(t) \le c_\eta \int_0^t G^+(s,\overline{y}(s))\,ds.$$

Moreover, $\overline{y}(0)=0$, and thus

$$\overline{y}(t) \le \beta_h(t) \quad \text{on } [0, T \wedge T_{\beta_h}].$$

In a similar way we obtain the other asserted inequality. ∎

The local existence theorem for the method (3.2)–(3.4) is the following.

Theorem 3.4 (Existence for MOL). *Suppose that the assumptions of Lemma 3.3 are satisfied and $\hat{T}_h = T_{\alpha_h} \wedge T_{\beta_h}$. Then there exists a unique classical solution of (3.2)–(3.4) defined on $[0,\hat{T}_h]$. This solution satisfies*

$$\alpha_h(t)+\varphi(x_k) \le z_k(t) \le \beta_h(t)+\varphi(x_k) \quad \textit{for } t \in [0,\hat{T}_h],\ k \in J.$$

Proof. As in the proof of Theorem 2.3, we consider only the case $\varphi \equiv 0$ in (3.5), (3.6). Define

$$\langle \alpha_h, \beta_h\rangle = \{v \in C([0,\hat{T}_h], l_0^N) : \alpha_h(t) \le v_k(t) \le \beta_h(t),\ k \in J_0,\ t \in [0,\hat{T}_h]\}.$$

Fix $v \in \langle \alpha_h,\beta_h\rangle$ and let $V = S_h[v]$ be a solution of the problem

$$V_k(t) - \varepsilon \Delta_h V_k(t) = \int_0^t F(x_k, s, v_k(s))\,ds,\ k \in J_0, \qquad V_k(t)=0,\ k \in J \setminus J_0$$

(see Proposition 3.2). It follows from the inequalities

$$V_k(t) - \varepsilon \Delta_h V_k(t) \le \int_0^t G(s,\beta_h(s))\,ds, \qquad k \in J_0,\ t \in [0,\hat{T}_h],$$

that

$$V_k(t) \le c_\eta \int_0^t G^+(s,\beta_h(s))\,ds = \beta_h(t).$$

Analogously, we prove that $V_k(t) \ge \alpha_h(t)$ for $k \in J_0$ on $[0,\hat{T}_h]$. Thus $S_h : \langle \alpha_h,\beta_h\rangle \to \langle \alpha_h,\beta_h\rangle$. Set

$$\|v\|_{h,t} = \max_{s\in[0,t],\, k\in J} |v_k(s)| e^{-\int_0^s 2K_h(\tau)\,d\tau}, \qquad t \in [0,\hat{T}_h],$$

where $K_h : [0,\hat{T}_h] \to \mathbb{R}$ is integrable and such that

$$|F(x,t,u) - F(x,t,\bar{u})| \le K_h(t)|u-\bar{u}|$$

for all $x \in [0,\Lambda]$ and $u,\bar{u} \in [\alpha_h(t),\beta_h(t)]$ and $t \in [0,\hat{T}_h]$. If $v,\bar{v} \in \langle \alpha_h,\beta_h\rangle$ then

$$|S_h[v](t) - S_h[\bar{v}](t)| \le \tfrac{1}{2}\eta_k \|v-\bar{v}\|_{h,t} e^{\int_0^t 2K_h(\tau)\,d\tau},$$

which implies

$$\|S_h[v] - S_h[\bar{v}]\|_{h,\hat{T}_h} \le \tfrac{1}{2} c_\eta \|v - \bar{v}\|_{h,\hat{T}_h}.$$

Applying the Banach fixed point theorem completes the proof. ∎

REMARK 3.5. For the function γ given in Section 2 (with $n = 1$) we have $\varepsilon\gamma'' = \gamma - 1 \le 0$. Therefore $\varepsilon\gamma^{IV} = \gamma'' \le 0$ and

$$\gamma(x_k) - \varepsilon \Delta_h \gamma(x_k) \ge \gamma(x_k) - \varepsilon\gamma''(x_k) = 1, \qquad k \in J_0.$$

Thus $\gamma(x_k) \ge \eta_k$, $k \in J$, and the constants c_γ and $c_\eta = c_\eta(h)$ defined by (2.2) and (3.7) satisfy the relation

$$c_\eta \le c_\gamma.$$

It follows that

$$\alpha_h(t) \ge \alpha(t) \quad \text{on } [0, T_\alpha], \qquad \beta_h(t) \le \beta(t) \quad \text{on } [0, T_\beta],$$
$$T_{\alpha_h} \ge T_\alpha, \qquad T_{\beta_h} \ge T_\beta, \qquad \hat{T}_h \ge \hat{T}.$$

It is not difficult to show that $c_\eta = c_\eta(h) \to c_\gamma$ as $h \to 0$, and consequently $\hat{T}_h \to \hat{T}$ provided the functions g, G given by (2.3) are sufficiently regular.

We state a result on convergence of approximate solutions.

THEOREM 3.6 (Convergence for MOL). *Assume that F satisfies the condition* (1) *of Lemma* 2.1. *If u is the solution of the problem* (1.1), (1.2) *on* $[0, T]$, *and $z : [0, T] \to l^N$ is the classical solution of* (3.2)–(3.4), *then*

$$\max_{k \in J} |u(x_k, t) - z_k(t)| \to 0 \qquad \text{as } h \to 0$$

uniformly on $[0, T]$.

Proof. It is easy to check that

$$\Delta_h u(x_k, t) = \int_0^1\!\!\int_0^1 \frac{\partial^2 u}{\partial x^2}(x_k + sh - \tau h, t)\, ds\, d\tau, \qquad k \in J_0,\ t \in [0, T].$$

Hence we immediately obtain

$$\Delta u(x_k, t) = \Delta_h u(x_k, t) + \xi_{h,k}(t)$$

and $|\xi_{h,k}(t)| \le \xi(h)$ where $\lim_{h\to 0} \xi(h) = 0$. Set

$$d_k(t) = u(x_k, t) - z_k(t) \qquad \text{for } k \in J \text{ on } [0, T].$$

It follows from (2.1) and (3.5) that

$$d_k(t) - \varepsilon \Delta_h d_k(t) = \varepsilon \xi_{h,k}(t) + \int_0^t \big(F(x_k, s, u(x_k, s)) - F(x_k, s, z_k(s))\big)\, ds$$

with $k \in J_0$. Let $K : [0, T] \to \mathbb{R}$ be integrable and

$$|F(x, t, u) - F(x, t, \tilde{u})| \le K(t)|u - \tilde{u}|$$

for $x \in [0, A]$, $u, \tilde{u} \in [\alpha(t), \beta(t)]$, and $t \in [0, T]$. The interval $[\alpha_h(t), \beta_h(t)]$ is contained in $[\alpha(t), \beta(t)]$ in view of Remark 3.5. Then we obtain

$$|d_k(t)| \le c_\eta\Big(\varepsilon\,\xi(h) + \int_0^t K(s)|d_k(s)|\,ds\Big), \quad k \in J_0.$$

By Gronwall's inequality we have

$$|d_k(t)| \le c_\eta \varepsilon \xi(h) e^{\int_0^t c_\eta K(s)\,ds}, \qquad k \in J_0,\ t \in [0, T], \tag{3.8}$$

and the assertion follows. ∎

Remark 3.7. It is obvious that

$$\Delta_h\, u(x_k, t) = \Delta\, u(x_k, t) + \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(\tau, t), \qquad k \in J_0,$$

with some $\tau \in (x_{k-1}, x_{k+1})$ and a sufficiently smooth solution u. Then the error estimate (3.8) holds true with $\xi(h) = \mathcal{O}(h^2)$.

We formulate a result on blow-up of approximate solutions. We suppose that F satisfies the condition (2.5).

Lemma 3.8. *Assume that $\lambda_h > 1$ is the first eigenvalue of the operator $I - \varepsilon\Delta_h : l_0^N \to l_0^N$ and $\omega_h = \{\omega_{h,k}\}_{k\in J}$ with $\omega_{h,k} \ge 0$, is the corresponding eigenfunction such that $\sum_{k\in J}\omega_{h,k} = 1$. Suppose that $\varrho_h \in C^1([0, T_{\varrho_h}])$ is a minimal solution of the problem*

$$\frac{dr}{dt} = \frac{1}{\lambda_h}F_*(t, r), \qquad r(0) = \varrho_{h,0} := \sum_{k\in J_0}\omega_{h,k}\varphi(x_k).$$

If $z : [0, T] \to l^N$ is the solution of (3.2)–(3.4) then the function

$$Z(t) := \sum_{k\in J_0}\omega_{h,k} z_k(t)$$

satisfies

$$Z(t) \ge \varrho_h(t) \qquad \text{on } [0, T \wedge T_{\varrho_h}].$$

Proof. It follows from (3.5) that

$$\sum_{k\in J_0}\omega_{h,k}(y_k(t) - \varepsilon\Delta_h y_k(t)) = \sum_{k\in J_0}\int_0^t \omega_{h,k}F(x_k, s, z_k(s))\,ds,$$

where $y_k(t) = z_k(t) - \varphi(x_k)$ for $k \in J$. Since $I - \varepsilon\Delta_h$ is a self-adjoint operator, we have

$$\sum_{k\in J_0} y_k(t)(\omega_{h,k} - \varepsilon\Delta_h\omega_{h,k}) = \sum_{k\in J_0}\int_0^t \omega_{h,k}F(x_k, s, z_k(s))\,ds.$$

By using the eigenvalue relation and convexity of F_* we obtain

$$\lambda_h \sum_{k\in J_0} \omega_{h,k} y_k(t) = \int_0^t \sum_{k\in J_0} \omega_{h,k} F(x_k, s, z_k(s))\, ds \geq \int_0^t \sum_{k\in J_0} \omega_{h,k} F_*(s, z_k(s))\, ds$$
$$\geq \int_0^t F_*\Big(s, \sum_{k\in J_0} \omega_{h,k} z_k(s)\Big)\, ds.$$

Therefore

$$\lambda_h Z(t) \geq \lambda_h \varrho_{h,0} + \int_0^t F_*(s, Z(s))\, ds$$

and $Z(0) = \varrho_{h,0}$. Thus $Z(t) \geq \varrho_h(t)$, which completes the proof. ■

An immediate consequence of Lemma 3.8 is the result on blow-up for the method (3.2)–(3.4).

COROLLARY 3.9 (Blow-up for MOL). *Suppose that F satisfies the condition* (2.5) *with F_* given by* (2.7). *If*

$$\lambda_h \int_{\varrho_{h,0}}^{\infty} \frac{1}{H(r)}\, dr \leq \int_0^T a(s)\, ds,$$

then the solution of (3.2)–(3.4) *blows up at some $T_h^* \in (0, T_{h,\varrho}^*]$, where $0 < T_{h,\varrho}^* \leq T$ is defined by*

$$T_{h,\varrho}^* = \min\Big\{ t \in (0,T] : \lambda_h \int_{\varrho_{h,0}}^{\infty} \frac{1}{H(r)}\, dr = \int_0^t a(s)\, ds \Big\}.$$

PROPOSITION 3.10. *The upper estimate for the numerical blow-up time tends to the upper estimate for the blow-up time in the continuous case, i.e. $T_{h,\varrho}^* \to T_\varrho^*$ as $h \to 0$.*

Proof. We have

$$\lambda = \varepsilon\Big(\frac{\pi}{\Lambda}\Big)^2 + 1, \qquad \omega(x) = \frac{\pi}{2\Lambda} \sin \frac{\pi}{\Lambda} x,$$

and provided $N \geq 4$ we have

$$\lambda_h = 2\varepsilon\Big(\frac{\pi}{\Lambda}\Big)^2 \frac{1}{1 + \sqrt{1 - (h\pi/\Lambda)^2}} + 1, \quad \omega_{h,k} = C^{-1}\Big(1 - \Big(\frac{h\pi}{\Lambda}\Big)^2\Big)^{k/2} \sin \frac{\pi h k}{\Lambda}$$

where

$$C = \sum_{j=1}^{N-1} \Big(1 - \Big(\frac{h\pi}{\Lambda}\Big)^2\Big)^{j/2} \sin \frac{\pi h j}{\Lambda}.$$

We see that λ_h tends to λ as $h \to 0$. We will show that $\varrho_{h,0} \to \varrho_0$ as $h \to 0$, hence $T_{h,\varrho}^* \to T_\varrho^*$ as $h \to 0$ (cf. Corollary 3.9). Since the integral

$\varrho_0 = \int_\Omega \varphi(y)\omega(y)\,dy$ can be approximated by the sum $h\sum_{k=1}^{N-1}\omega(x_k)\varphi(x_k)$, we only need to show that

$$\sum_{k=1}^{N-1}(h\omega(x_k)-\omega_{h,k})\varphi(x_k)\to 0 \quad \text{as } h\to 0\ (x_k=kh).$$

Since the functions $\varphi(\cdot)$ and $\sin(\cdot)$ are bounded, it is enough to prove that $S_h\to 0$, where

$$S_h := \sum_{k=1}^{N-1}\left|h\frac{\pi}{2\Lambda}-C^{-1}\left(1-\left(\frac{h\pi}{\Lambda}\right)^2\right)^{k/2}\right|.$$

Since

$$C=\frac{(1+\beta^N)\sin\alpha}{1-2\beta\cos\alpha+\beta^2} \quad \text{with} \quad \alpha=\frac{h\pi}{\Lambda},\ \beta=(1-\alpha^2)^{1/2},$$

we have

$$S_h=\sum_{k=1}^{N-1}\alpha\left|\frac{1}{2}-\frac{\beta^k(1-2\beta\cos\alpha+\beta^2)}{(1+\beta^N)\alpha\sin\alpha}\right|.$$

It follows from $\beta^N\le\beta^k\le 1$ $(k=1,\dots,N-1)$ and $\beta^N=\beta^{\pi/\alpha}\to 1$ as $h\to 0$, that

$$\lim_{h\to 0}\left(\frac{1}{2}-\frac{\beta^k(1-2\beta\cos\alpha+\beta^2)}{(1+\beta^N)\alpha\sin\alpha}\right)=0$$

uniformly in k. This and $\alpha(N-1)<\pi$ imply $S_h\to 0$. ∎

We formulate and analyse the method (3.2)–(3.4) in the case $n=1$. It is possible to generalize this method to a multi-dimensional domain of the type $(0,\Lambda_1)\times\dots\times(0,\Lambda_n)$ (cf. Remark 3.11). It is easy to introduce a regular mesh and perform numerical computations in this case. However, we cannot give a theoretical estimation of the error for $n>1$, because of possible singularities of the derivatives of PDE solutions. There are theorems on the regularity of PDE solutions and their derivatives in the case of Ω with a smooth boundary, but descriptions and implementations of numerical schemes become difficult. A construction of a scheme on a cylindrical domain is proposed in [18], whose authors extend a convex nonrectangular domain to a sufficiently large cube. Then they construct a difference scheme using reflection with respect to the boundary.

REMARK 3.11. To formulate MOL for (1.1), (1.2) in the case of an n-dimensional interval $\Omega=(0,\Lambda)\subset\mathbb{R}^n$, where $\Lambda=(\Lambda_1,\dots,\Lambda_n)$, it is enough to define $x_k=(k_1h_1,\dots,k_nh_n)$ with $k=(k_1,\dots,k_n)$, $h_j=\Lambda_j/N_j$ $(1\le j\le n)$. Then we define the following sets of indices:

$$J=\{k=(k_1,\dots,k_n):0\le k_j\le N_j \text{ for } 1\le j\le n\},$$
$$J_0=\{k=(k_1,\dots,k_n):0< k_j< N_j \text{ for } 1\le j\le n\}.$$

The space l^N, $N = (N_1, \ldots, N_n)$, is the set of all $p = \{p_k\}_{k\in J}$ with $p_k \in \mathbb{R}$. If l_0^N is the set of $p \in l^N$ with $p_k = 0$ for $k \in J \setminus J_0$, then the discrete Laplace operator $\Delta_h : l_0^N \to l_0^N$ is as follows:

$$(\Delta_h\, p)_k = \sum_{j=1}^{n} \frac{p_{k+e_j} - 2p_k + p_{k-e_j}}{h_j^2}, \qquad k \in J_0,$$

where e_j is the jth vector of the standard basis in $\mathbb{R}^n$, and $(\Delta_h\, p)_k = 0$ for $k \in J \setminus J_0$. Then we consider the method like (3.2)–(3.4) with the above sets of indices and the operator Δ_h.

4. Numerical examples. We present examples of PDE problems (1.1), (1.2) with rapidly increasing nonlinearities. We give theoretical estimates of blow-up times in the continuous case (see Proposition 3.10 for the discrete case). Our aim is to show that the results of numerical experiments for these examples are consistent with our theory. The numerical computations were performed using GNU Octave.

EXAMPLE 4.1. We apply the method of lines for the PDE problem

$$\begin{aligned} &\frac{\partial}{\partial t}(u - \Delta u) = e^{3u} \quad \text{on } (0,1) \times [0,T], \\ &u|_{t=0} = 0, \quad u|_{x=0} = u|_{x=1} = 0. \end{aligned} \tag{4.1}$$

In view of the theory (see Example 2.8) we have

$$\hat{T} = T_\beta < \frac{1}{3c_\varphi c_\gamma} \simeq 2.945, \qquad T_\varrho < T_\varrho^* = \frac{\lambda}{3e^{3\varrho_0}} \simeq 3.623$$

($c_\varphi = 1$, $c_\gamma \simeq 0.1139$, $\lambda = \pi^2 + 1$, $\varrho_0 = 0$). Figure 1 shows the approximate solution for some t-values. We observe the blow-up at $T^* \simeq 3.52$.

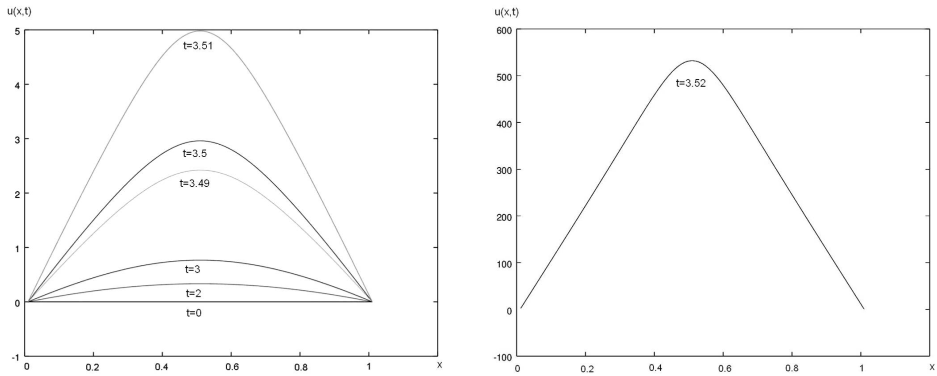


Fig. 1. The numerical solution of the problem (4.1)

Example 4.2. For the PDE problem

$$\frac{\partial}{\partial t}(u - \Delta u) = e^{4u} \quad \text{on } (0,1)\times[0,T], \tag{4.2}$$
$$u|_{t=0} = 20(x-x^2)(x-0.5)^2, \quad u|_{x=0} = u|_{x=1} = 0,$$

we have

$$\hat{T} = T_\beta < \frac{1}{4c_\varphi c_\gamma} \simeq 0.6292, \quad T_\varrho < T_\varrho^* = \frac{\lambda}{4e^{4\varrho_0}} \simeq 1.5608$$

($c_\varphi \simeq 3.4886$, $c_\gamma \simeq 0.1139$, $\lambda = \pi^2+1$, $\varrho_0 = 0.1386$). The solution to MOL for this problem is presented in Figure 2. It blows up at $T^* \simeq 1.54$.

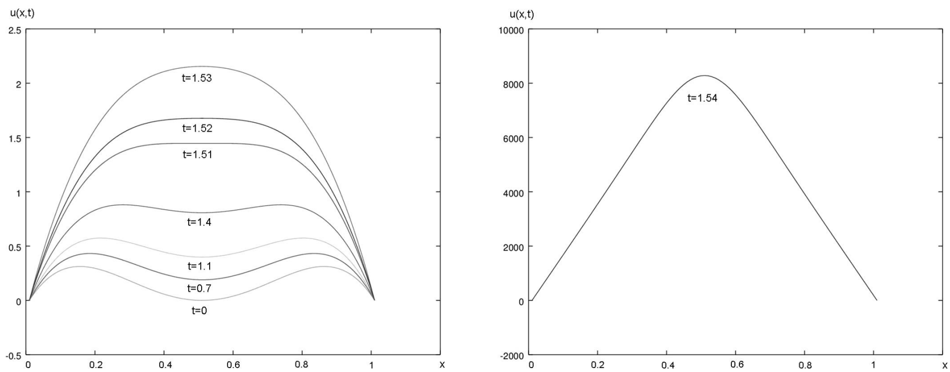


Fig. 2. The numerical solution of the problem (4.2)

Remark 4.3. A similar example, with e^u on the right-hand side and with a different initial function, is given in [1]. The authors analyse blow-up times experimentally with so called effective accuracy order.

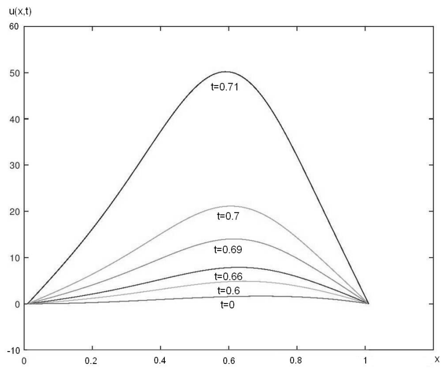
Example 4.4. For the PDE problem

$$\frac{\partial}{\partial t}\left(u - \frac{1}{\pi^2}\Delta u\right) = u^3 e^{-t/2} \quad \text{on } (0,1)\times[0,T], \tag{4.3}$$
$$u|_{t=0} = \frac{3\pi^2}{4}x^2\cos\frac{\pi}{2}x, \quad u|_{x=0} = u|_{x=1} = 0,$$

we have the blow-up condition $\frac{1}{2\varrho_0^2} < 1$ (see Example 2.10) and

$$\hat{T} = T_\beta < \frac{1}{\delta|\varphi|_\infty^\delta c_\gamma} \simeq 0.3056, \quad T_\varrho < T_\varrho^* = -2\ln\left(1 - \frac{1}{2\varrho_0^2}\right) \simeq 1.1261$$

($\delta = 2$, $|\varphi|_\infty \simeq 1.6493$, $c_\gamma \simeq 0.6015$, $\lambda = 2$, $\varrho_0 \simeq 1.0777$). The numerical solution of this problem is presented in Figure 3. It blows up for $T^* \simeq 0.73$.



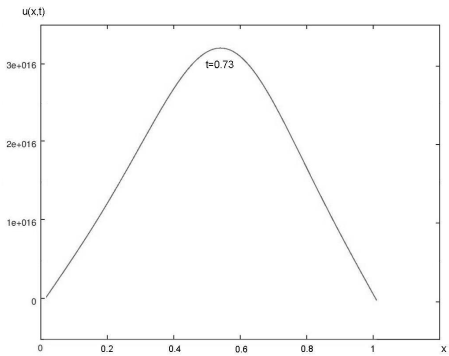


Fig. 3. The numerical solution of the problem (4.3)

Example 4.5. Consider the problem (4.3). We compare approximate solutions for two values of the space step h with that obtained for $h = 0.0001$ (it may be treated as the exact solution). The errors are given in Table 1.

Table 1. The errors of MOL for the problem (4.3).

	$t = 0.6$		$t = 0.66$		$t = 0.69$		$t = 0.7$		$t = 0.71$	
x	$h = .01$	$h = .005$	$h = .01$	$h = .005$	$h = .01$	$h = .005$	$h = .01$	$h = .005$	$h = .01$	$h = .005$
0.0	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000
0.1	.00011	.00003	.00069	.00017	.00390	.00097	.01116	.00278	.07336	.01822
0.2	.00023	.00006	.00146	.00036	.00820	.00205	.02341	.00583	.15351	.03812
0.3	.00041	.00010	.00241	.00060	.01327	.00331	.03758	.00936	.24453	.06072
0.4	.00070	.00018	.00367	.00092	.01918	.00478	.05343	.01331	.34158	.08482
0.5	.00114	.00028	.00514	.00128	.02495	.00622	.06771	.01687	.42066	.10446
0.6	.00153	.00038	.00612	.00153	.02764	.00689	.07292	.01817	.43865	.10893
0.7	.00149	.00037	.00559	.00140	.02427	.00605	.06299	.01570	.37111	.09216
0.8	.00095	.00024	.00366	.00091	.01620	.00404	.04227	.01053	.24927	.06191
0.9	.00034	.00008	.00157	.00039	.00754	.00188	.02009	.00501	.12031	.02988
1.0	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000	.00000

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