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Representation of integers as sums of fractional powers of primes and powers of 2

by

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1. Introduction. Linnik [11] established that each large even integer N is a sum of two primes and a bounded number of powers of 2,

$$N = p_1 + p_2 + 2^{\nu_1} + \cdots + 2^{\nu_k},$$

where p_i and ν_j denote prime numbers and positive integers respectively. Gallagher [1] established a stronger result by a different method. Liu, Liu and Wang [13] first established that $k = 54000$ is admissible. Heath-Brown and Puchta [4] showed that $k = 13$ is admissible. Liu and Lü [16] showed that $k = 12$ is admissible.

Motivated by the works of Linnik and Gallagher, Liu, Liu and Zhan [14] proved that every large even integer N can be written as a sum of four squares of primes and powers of 2,

$$N = p_1^2 + p_2^2 + p_3^2 + p_4^2 + 2^{\nu_1} + \cdots + 2^{\nu_k}.$$

And in [12], it is showed that $k = 8330$ is admissible. The value of k was improved by Liu and Lü [15] and Li [10]. Recently Zhao [19] showed that $k = 46$ is admissible.

In this paper we consider the representation of integers in the form

$$N = [p_1^c] + [p_2^c] + 2^{\nu_1} + \cdots + 2^{\nu_k}. \tag{1.1}$$

We establish the following theorem.

Theorem 1.1. *For $1 < c < 29/28$, there exists an integer k such that each large integer N can be represented in the form* (1.1).

Notation. Let $[x]$ denote the integer part of x, $\{x\}$ denote the fractional part of x, $e(x)$ denote $e^{2\pi ix}$ and $\rho(x)$ denote $1/2 - \{x\}$. Let $\gamma = 1/c$ and

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$L = \log_2 N$. Let ε denote a sufficiently small positive real number which may not be the same everywhere it occurs.

2. The set up. Let

$$T(x) = \sum_{p^c \le N} (\log p) e(x[p^c]), \qquad G(x) = \sum_{2^\nu \le N} e(2^\nu x) = \sum_{\nu \le L} e(2^\nu x)$$

and

$$r_k(N) = \sum_{N=[p_1^c]+[p_2^c]+2^{\nu_1}+\cdots+2^{\nu_k}} (\log p_1)(\log p_2).$$

Let $\omega = N^{-1/3-\eta}$ and η be a sufficiently small positive real number. We divide $[-\omega, 1-\omega)$ into the major arc and minor arc as in [9]. We have

$$\begin{aligned} r_k(N) &= \int_0^1 T(x)^2 G(x)^k e(-xN)\, dx \\ &= \int_{-\omega}^{1-\omega} T(x)^2 G(x)^k e(-xN)\, dx \\ &= \Big\{ \int_{-\omega}^{\omega} + \int_{\omega}^{1-\omega} \Big\} T(x)^2 G(x)^k e(-xN)\, dx. \end{aligned}$$

We set $\mathfrak{M} = [-\omega, \omega]$ and $\mathfrak{m} = (\omega, 1-\omega)$. In the rest of the paper, we treat them separately.

3. Major arc $\mathfrak{M}$. In this section we consider the integral over the major arc $\mathfrak{M}$. For $|x| \le \omega$, we have

$$T(x) = \sum_{p^c \le N} (\log p) e(xp^c) + O(\omega N^\gamma).$$

Let

$$\begin{aligned} S(x,N) &= \sum_{p^c \le N} (\log p) e(xp^c), & U(x,N) &= \sum_{m^c \le N} e(xm^c), \\ \tilde{S}(x,X) &= \sum_{X < p^c \le 2X} (\log p) e(xp^c), & \tilde{U}(x,X) &= \sum_{X < m^c \le 2X} e(xm^c). \end{aligned}$$

In order to control the error term, we need the following L^2 estimate.

LEMMA 3.1 ([8, Theorem 3.1]). *For $0 < Y \le 1/2$, we have*

$$\int_{-Y}^{Y} |\tilde{S}(x,X) - \tilde{U}(x,X)|^2\, dx \ll \frac{X^{2/c-2} \log^2 X}{Y} + Y^2 X + Y^2 J_c\Big(X, \frac{1}{2Y}\Big),$$

where $J_c(X,h)=\int_X^{2X}(\theta((x+h)^{1/c})-\theta(x^{1/c})-((x+h)^{1/c}-x^{1/c}))^2\,dx$ *and* $\theta(x)=\sum_{p\le x}\log p$.

In order to state the following result, we introduce a hypothesis on the density of the zeros of the Riemann zeta-function. With classical notation, we assume that there exist constants $B\ge 0$ and $C\ge 2$ such that for $\sigma\in[1/2,1]$ and $T\ge 2$, we have

$$N(\sigma,T)\ll T^{C(1-\sigma)}(\log T)^B. \tag{3.1}$$

Huxley [6] proved that (3.1) holds with $C=12/5$ and some $B\ge 0$.

LEMMA 3.2 ([8, Theorem 3.2]). *Let $c>0$ be a real number and ε be an arbitrarily small positive constant. Assuming that* (3.1) *holds, there exists a positive constant $c_1=c_1(\varepsilon)$, which does not depend on c, such that*

$$J_c(X,h)\ll_c h^2X^{2/c-1}\exp\biggl(-c_1\biggl(\frac{\log X}{\log\log X}\biggr)^{1/3}\biggr)$$

uniformly for $X^{1-2/(Cc)+\varepsilon}\le h\le X$.

The following two lemmas are classical results on exponential sums and exponential integrals.

LEMMA 3.3 ([7, Lemma 8.8]). *Let $f(t)$ be a real function with $|f'(t)|\le 1-\theta$ and $f''(t)\ne 0$ on $[a,b]$. Then*

$$\sum_{a<m<b}g(m)e(f(m))=\int_a^b g(t)e(f(t))\,dt+O(G\theta^{-1}),$$

where

$$G=|g(b)|+\int_a^b|g'(y)|\,dy.$$

LEMMA 3.4 ([7, Lemma 8.10]). *Let $f(t)$ be a real function with $|f'(x)|\ge h$ on $[a,b]$. Then*

$$\int_a^b e(f(t))\,dt\ll 1/h.$$

We also need the following integral estimate.

LEMMA 3.5 ([18, Lemma 7]).

$$\int_{-\omega}^{\omega}|T(x)|^2\,dx\ll N^{2\gamma-1}\log^2 N.$$

The lemma below is the key result in this section.

LEMMA 3.6. *For any* $2 \le n \le N$,

$$\int_{\mathfrak{M}} T(x)^2 e(-xn)\,dx = \frac{\Gamma^2(1+\gamma)}{\Gamma(2\gamma)} n^{2\gamma-1} + O\Big(N^{2\gamma-1}\exp\Big(-c_2\Big(\frac{\log N}{\log\log N}\Big)^{1/3}\Big)\Big)$$

for some constant c_2 *smaller than* c_1 *in Lemma* 3.2.

Proof. By the Cauchy–Schwarz inequality and Lemma 3.5, we have

$$\int_{-\omega}^{\omega} |T(x)^2 - U(x,N)^2|\,dx \le 2\Big(\int_{-\omega}^{\omega} (T(x)^2 + U(x,N)^2)\,dx\Big)^{1/2}\Big(\int_{-\omega}^{\omega} |T(x) - U(x,N)|^2\,dx\Big)^{1/2} \ll (N^{2\gamma-1}\log^2 N)^{1/2}\Big(\int_{-\omega}^{\omega} |S(x,N) - U(x,N)|^2\,dx + \omega^3 N^{2\gamma}\Big)^{1/2}.$$

Let

$$\tilde{S}(x) = \sum_{N^{1-\gamma/3} < p^c \le N} (\log p) e(xp^c), \qquad \tilde{U}(x) = \sum_{N^{1-\gamma/3} < m^c \le N} e(xm^c).$$

We have

$$S(x,N) = S(x,N^{1-\gamma/3}) + \tilde{S}(x), \qquad U(x,N) = U(x,N^{1-\gamma/3}) + \tilde{U}(x).$$

Thus

$$\int_{-\omega}^{\omega} |S(x,N) - U(x,N)|^2\,dx \ll \int_{-\omega}^{\omega} |S(x,N^{1-\gamma/3}) - U(x,N^{1-\gamma/3})|^2\,dx + \int_{-\omega}^{\omega} |\tilde{S}(x) - \tilde{U}(x)|^2\,dx.$$

By the prime number theorem,

$$\int_{-\omega}^{\omega} |S(x,N^{1-\gamma/3}) - U(x,N^{1-\gamma/3})|^2\,dx \ll N^{2\gamma-1-\eta}.$$

Let $Y = \omega$. By the dyadic method and Lemmas 3.1–3.2, we get

$$\int_{-\omega}^{\omega} |\tilde{S}(x) - \tilde{U}(x)|^2\,dx \ll N^{2\gamma-1}\exp\Big(-c_2\Big(\frac{\log N}{\log\log N}\Big)^{1/3}\Big)$$

for some constant c_2 smaller than the c_1 of Lemma 3.2.

Hence

$$\int_{-\omega}^{\omega} |S(x,N)-U(x,N)|^2\,dx \ll N^{2\gamma-1}\exp\biggl(-c_2\biggl(\frac{\log N}{\log\log N}\biggr)^{1/3}\biggr).$$

Since $\omega^3 N^{2\gamma} \le N^{2\gamma-1-3\eta}$, we have

$$\begin{aligned}(3.2)\qquad &\int_{-\omega}^{\omega} T(x)^2 e(-xn)\,dx\\ &\quad= \int_{-\omega}^{\omega} U(x)^2 e(-xn)\,dx + O\biggl(N^{2\gamma-1}\exp\biggl(-c_2\biggl(\frac{\log N}{\log\log N}\biggr)^{1/3}\biggr)\biggr).\end{aligned}$$

Now we analyze $\int_{-\omega}^{\omega} U(x)^2 e(-xn)\,dx$. Lemma 3.3 with $a=1$, $b=N^\gamma$, $g(t)=1$ and $f(t)=t^c x$ yields

$$U(x) = \int_{1}^{N^\gamma} e(t^c x)\,dt + O(1). \tag{3.3}$$

Lemma 3.3 with $a=1$, $b=N$, $g(t)=\gamma t^{\gamma-1}$ and $f(t)=tx$ yields

$$\int_{1}^{N^\gamma} e(t^c x)\,dt = \gamma\int_{1}^{N} t^{\gamma-1} e(tx)\,dt = \gamma\sum_{1\le m\le N} m^{\gamma-1} e(xm) + O(1).$$

Thus

$$U(x) = \gamma\sum_{1\le m\le N} m^{\gamma-1} e(xm) + O(1). \tag{3.4}$$

Let $V(x)=\gamma\sum_{1\le m\le N} m^{\gamma-1}e(xm)$. We get

$$\begin{aligned}\int_{-\omega}^{\omega} (U(x)^2 - V(x)^2)e(-xn)\,dx &\ll N^\gamma \int_{-\omega}^{\omega} |U(x)-V(x)|\,dx \ll \omega N^\gamma\\ &\ll N^{2\gamma-1-\eta}.\end{aligned}$$

Hence

$$\int_{-\omega}^{\omega} U(x)^2 e(-xn)\,dx = \int_{-\omega}^{\omega} V(x)^2 e(-xn)\,dx + O(N^{2\gamma-1-\eta}).$$

By Lemma 3.4, (3.3) and (3.4), we have

$$V(x) = \int_{1}^{N^\gamma} e(t^c x)\,dt + O(1) \ll 1 + 1/x.$$

Thus

$$\int_{-\omega}^{\omega} V(x)^2 e(-xn)\,dx - \int_{-\omega}^{1-\omega} V(x)^2 e(-xn)\,dx \ll \int_{\omega}^{1} x^{-2}\,dx \ll \omega^{-1} \ll N^{2\gamma-1-\eta}.$$

Therefore,

$$\int_{-\omega}^{\omega} U(x)^2 e(-xn)\,dx = \int_{-\omega}^{1-\omega} V(x)^2 e(-xn)\,dx + O(N^{2\gamma-1-\eta}).$$

Since

$$\int_{-\omega}^{1-\omega} V(x)^2 e(-xn)\,dx = \gamma^2 \sum_{m_1+m_2=n} m_1^{\gamma-1} m_2^{\gamma-1},$$

it follows that

$$\int_{-\omega}^{\omega} U(x)^2 e(-xn)\,dx = \gamma^2 \sum_{m_1+m_2=n} m_1^{\gamma-1} m_2^{\gamma-1} + O(N^{2\gamma-1-\eta}).$$

By [5, Lemma 7.17], we get

$$\sum_{m_1+m_2=n} m_1^{\gamma-1} m_2^{\gamma-1} = \frac{\Gamma(\gamma)^2}{\Gamma(2\gamma)} n^{2\gamma-1} + O(N^{\gamma-1}).$$

Thus

$$\int_{-\omega}^{\omega} U(x)^2 e(-xn)\,dx = \frac{\Gamma(1+\gamma)^2}{\Gamma(2\gamma)} n^{2\gamma-1} + O(N^{2\gamma-1-\eta}).$$

By (3.2), we have

$$\int_{-\omega}^{\omega} T(x)^2 e(-xn)\,dx = \frac{\Gamma^2(1+\gamma)}{\Gamma(2\gamma)} n^{2\gamma-1} + O\Bigg(N^{2\gamma-1} \exp\bigg(-c_2\Big(\frac{\log N}{\log\log N}\Big)^{1/3}\bigg)\Bigg). \quad \blacksquare$$

In order to get a lower bound of the integral over $\mathfrak{M}$, we also need the following simple lemma.

LEMMA 3.7. *Let* $\Xi(N,k) = \{n \ge 2 : n = N - 2^{\nu_1} - \cdots - 2^{\nu_k}\}$ *with* $k \ge 2$. *Then*

$$\sum_{n\in\Xi(N,k)} n^{2\gamma-1} \ge (1-\varepsilon) N^{2\gamma-1} L^k.$$

Proof. We follow [12, Lemma 6.1], but the present case is easier. Let $\nu_1, \ldots, \nu_k$ satisfy the stronger conditions $((\nu))$:

$$1 \le \nu_1, \ldots, \nu_k \le \log_2(N/(kL)).$$

Then

$$\sum_{n \in \Xi(N,k)} n^{2\gamma-1} \ge \sum_{((\nu))} (N - 2^{\nu_1} - \cdots - 2^{\nu_k})^{2\gamma-1} \ge \left(N - \frac{N}{L}\right)^{2\gamma-1} \sum_{((\nu))} 1$$
$$\ge (1-\varepsilon) N^{2\gamma-1} L^k. \ \blacksquare$$

Thus we have the following lemma for $\mathfrak{M}$.

Lemma 3.8. *If $k \ge 2$, then*

$$\int_{\mathfrak{M}} T(x)^2 G(x)^k e(-xN)\,dx \ge \frac{\Gamma^2(1+\gamma)}{\Gamma(2\gamma)} (1-\varepsilon) N^{2\gamma-1} L^k$$

for N large enough.

Proof. This follows easily from Lemmas 3.1 and 3.7. $\blacksquare$

4. Minor arc $\mathfrak{m}$. In this section we estimate $\max_{x \in \mathfrak{m}} |T(x)|$ from above. We have

$$T(x) = \sum_{n \le N^\gamma} \Lambda(n) e(xn^c) e(-x\{n^c\}) + O(N^{\gamma/2}).$$

By Fourier expansion (see [9]),

$$e(-x\{y\}) = \sum_{|m| \le M} c_m(x) e(my) + O\left(\min\left(1, \frac{1}{M\|y\|}\right)\right),$$

where

$$c_m(x) = \frac{1 - e(-x)}{2\pi i (x+m)}.$$

Hence

$$T(x) = \sum_{|m| \le M} c_m(x) \sum_{n \le N^\gamma} \Lambda(n) e((x+m)n^c)$$
$$+ O\left(\log N \sum_{n \le N^\gamma} \min\left(1, \frac{1}{M\|n^c\|}\right)\right).$$

Thus

$$\max_{x \in \mathfrak{m}} |T(x)| \ll \log N \left(\sum\nolimits_1 + \sum\nolimits_2\right),$$

where

$$\sum\nolimits_1 = \max_{\omega \le x \le M+1} \Big| \sum_{n \le N^\gamma} \Lambda(n) e(xn^c) \Big|, \qquad \sum\nolimits_2 = \sum_{n \le N^\gamma} \min\left(1, \frac{1}{M\|n^c\|}\right).$$

We first estimate $\sum_2$. We have the following lemma.

Lemma 4.1 ([3, p. 246]). *We have*

$$\sum\nolimits_2 \ll \big(N^\gamma M^{-1}\log N + (MN)^{1/2}\big)\log N.$$

Let $M = N^{\gamma/2-1/6+\eta}$. We get

$$\sum\nolimits_2 \ll N^{\gamma/4+5/12+\eta}. \tag{4.1}$$

Lemma 4.2. *We have*

$$\sum\nolimits_1 \ll N^{5\gamma/6+5/36+\eta/6}. \tag{4.2}$$

Proof. The proof is similar to that of [18, Lemma 10] which used Vaughan identities; we omit the details. ■

Lemma 4.3. *We have*

$$\max_{x\in\mathfrak{m}} |T(x)| \ll N^{5\gamma/6+5/36+\eta/2}.$$

Proof. Choosing η sufficiently small, by (4.1) and Lemma 4.2 we get the lemma. ■

5. A mean square estimate. Let $r(h)$ denote the number of representations of h in the form

$$[p_1^c] - [p_2^c] = h, \quad p_1^c, p_2^c \le N,\ |h| \le N. \tag{5.1}$$

In the following, we use the sieve method to give an upper bound of $r(h)$.

Let $z = N^\alpha$ and $P(z) = \prod_{p<z} p$. Set

$$R(h) = \sum_{\substack{m,p\le N^\gamma\\ [m^c]-[p^c]=h\\ (m,P(z))=1}} 1.$$

It is not difficult to see that $r(h) \le R(h) + 3z$. Let $D = N^\delta$ and $\lambda(d)$ the upper bound Rosser's weight of level D (see [2]). Then

$$\sum_{d\mid k}\mu(d) \le \sum_{d\mid k}\lambda(d)$$

for every $k \in \mathbb{N}$. Furthermore, we know that

$$|\lambda(d)| \le 1, \quad \lambda(d) = 0 \quad \text{if } d > D \text{ or } \mu(d) = 0.$$

We also have

$$\sum_{d\mid P(z)} \frac{\lambda(d)}{d} \le \prod_{p<z}\left(1 - \frac{1}{p}\right)\big(F(s) + O(\log D)^{-1/3}\big),$$

where

$$s = \frac{\log D}{\log z} = \frac{\delta}{\alpha}$$

and $F(s)$ is the upper function of the linear sieve which satisfies

$$F(s) = 2e^{\gamma_0} s^{-1}, \quad 1 \le s \le 3$$

(here γ_0 is the Euler constant).

Let

$$A(N) = \sum_{p \le N^{\gamma}} \big(([p^c] + h + 1)^{\gamma} - ([p^c] + h)^{\gamma}\big).$$

Then

$$A(N) = \gamma \sum_{p \le N^{\gamma}} ([p^c] + h)^{\gamma - 1} + O(N^{\gamma - 1})$$

via [17]. Also by [17], it yields

$$R(h) \le R_0 + \Sigma_0 + \Sigma_1,$$

where

$$R_0 = A(N) \sum_{d \mid P(z)} \frac{\lambda(d)}{d},$$

$$\Sigma_j = \bigg| \sum_{d \mid P(z)} \lambda(d) \sum_{p \le N} \rho\bigg(-\frac{1}{d}([p^c] + h + j)^{\gamma}\bigg)\bigg|, \quad j = 0, 1.$$

Lemma 5.1 (Mertens' theorem).

$$\prod_{p < z} \bigg(1 - \frac{1}{p}\bigg) = \frac{e^{-\gamma_0}}{\log z} + O\bigg(\frac{1}{\log^2 z}\bigg).$$

Proof. This lemma can be found in many books; we omit the proof. ∎

Now we can give an upper bound of $r(h)$.

Lemma 5.2. *When* $1 < c < 29/28$, *we have*

$$r(h) \le (K(c) + \varepsilon) \frac{N^{2\gamma - 1}}{\log^2 N}$$

for any $\varepsilon > 0$ *and* $K(c) = 52c/(29 - 28c)$.

Proof. From [17], we know that

$$\Sigma_j \ll \frac{N^{2\gamma - 1}}{\log^3 N}, \quad j = 0, 1.$$

By the prime number theorem, it is easy to see that

$$A(N) \le (1 + \varepsilon) \frac{N^{2\gamma - 1}}{\log N}.$$

By Lemma 5.1, we have

$$\sum_{d \mid P(z)} \frac{\lambda(d)}{d} \le \frac{2(1 + \varepsilon)}{\delta \log N}.$$

Thus

$$R_0 \le \frac{2(1+\varepsilon)}{\delta}\,\frac{N^{2\gamma-1}}{\log^2 N}.$$

Since $\delta < (29\gamma - 28)/26$, we have

$$R(h) \le \frac{52(1+\varepsilon)}{29\gamma-28}\,\frac{N^{2\gamma-1}}{\log^2 N}.$$

Since $\alpha < (29\gamma - 28)/52$, we get

$$r(h) \le \frac{52(1+\varepsilon)}{29\gamma-28}\,\frac{N^{2\gamma-1}}{\log^2 N}.$$

This completes the proof of the lemma. ■

Now we can give a mean square estimate of $T(x)G(x)$.

Lemma 5.3. *We have*

$$\int_0^1 |T(x)G(x)|^2\,dx \le (K(c)+\varepsilon)N^{2\gamma-1}L^2.$$

Proof. This is obvious by Lemma 5.2. ■

6. Proof of Theorem 1.1. In this section we combine the estimates of the integral over the major arc $\mathfrak{M}$ and the minor arc $\mathfrak{m}$ to prove Theorem 1.1.

To control the integral over the minor arc $\mathfrak{m}$, we set $\mathcal{E}_\lambda = \{\alpha \in (\omega, 1-\omega] : |G(\alpha)| \ge \lambda L\}$. We have the following lemma.

Lemma 6.1 ([4, pp. 561–564]). *Let*

$$G_h(\alpha) = \sum_{0\le n\le h-1} e(\alpha 2^n), \qquad F(\xi,h) = \frac{1}{2^h}\sum_{r=0}^{2^h-1} \exp\Big\{\xi\,\mathrm{Re}\Big(G_h\Big(\frac{r}{2^h}\Big)\Big)\Big\}.$$

Then

$$\mathrm{meas}(\mathcal{E}_\lambda) \le N^{-E(\lambda)}, \quad \textit{where} \quad E(\lambda) = \frac{\xi\lambda}{\log 2} - \frac{\log F(\xi,h)}{h\log 2} - \frac{\varepsilon}{\log 2}$$

for any $h \in N$ and any $\xi, \varepsilon > 0$.

Now we divide the minor arc $\mathfrak{m}$ into $\mathfrak{m}\cap\mathcal{E}_\lambda$ and $\mathfrak{m}\setminus\mathcal{E}_\lambda$.

By (4.3) and Lemma 6.1, we have

$$\int_{\mathfrak{m}\cap\mathcal{E}_\lambda} T(x)^2G(x)^k e(-xN)\,dx \ll N^{-E(\lambda)}N^{5\gamma/3+5/18+2\eta}L^k$$
$$= N^{5\gamma/3+5/18+2\eta-E(\lambda)}L^k \ll N^{2\gamma-1},$$

provided that $E(\lambda) > 23/18 - \gamma/3$. Using Mathematica 9.0 on a PC with $\xi = 1.91$ and $h = 22$ in Lemma 6.1, we get $\lambda = 0.986175$ (see [15] for details).

By Lemma 5.2, we have

$$\int_{\mathfrak{m}\setminus\mathcal{E}_\lambda} T(x)^2G(x)^k e(-xN)\,dx \le (\lambda L)^{k-2}\int_0^1 |T(x)G(x)|^2\,dx$$
$$\le \lambda^{k-2}(K(c)+\varepsilon)N^{2\gamma-1}L^k.$$

By Lemma 3.8, we get

$$r_k(N) \ge (1-\varepsilon)^2\biggl(\frac{\Gamma^2(1+\gamma)}{\Gamma(2\gamma)} - \lambda^{k-2}K(c)\biggr)N^{2\gamma-1}L^k.$$

Solving the inequality

$$\lambda^{k-2}K(c) < \frac{\Gamma^2(1+\gamma)}{\Gamma(2\gamma)}, \tag{6.1}$$

we get

$$k > 2 + \frac{\log K(c) + \log \Gamma(2\gamma) - 2\log\Gamma(1+\gamma)}{|\log\lambda|}. \tag{6.2}$$

Thus $k = 3 + \bigl[\frac{\log K(c)+\log\Gamma(2\gamma)-2\log\Gamma(1+\gamma)}{|\log\lambda|}\bigr]$ is admissible.

COROLLARY 6.2. *If c is very close to 1, then $k = 286$ is admissible.*

Proof. If c is very close to 1, by (6.1) we have

$$k > 2 + \frac{\log 52}{|\log 0.986175|} \ge 285.824.$$

Thus we can take $k = 286$. ∎

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