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SOME RESULTS ON COMBINED MATRICES

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Abstract. A combined matrix of a nonsingular matrix is the Hadamard (entrywise) product of the matrix and its transposed inverse. A characterization of the diagonal entries of the combined matrix of a positive definite matrix was known. In the paper one investigates behaviour of the diagonal entries of combined matrices for some other classes of matrices. In particular, one proves that the diagonal entries of the combined matrix of an oscillatory Hessenberg CBM-matrix are characterized by some four conditions. It leads to the formulation of a conjecture that the first three from these four conditions characterize the diagonal entries of the combined matrix of a square totally positive matrix of arbitrary order. The conjecture is proven for the order equal to 3.

1. Introduction. In [2], it was proved that a necessary and sufficient condition for the diagonal entries a_{ii} of a positive definite matrix A and the diagonal entries α_{ii} of the inverse matrix A^{-1} is:

$$a_{ii}\alpha_{ii} \geq 1 \quad \text{for all } i;$$
$$2\max_i(\sqrt{a_{ii}\alpha_{ii}} - 1) \leq \sum_i(\sqrt{a_{ii}\alpha_{ii}} - 1).$$

Starting from another point, we studied in this connection the matrices—interesting by themselves—of the form $A \circ (A^{-1})^T$, where $\circ$ means the Hadamard (entrywise) product. We called such matrix the *combined matrix* of the nonsingular matrix A. It has been known for a long time that multiplication from either side of A by a nonsingular diagonal matrix does not change the combined matrix, also the row and column sums of the combined matrix are for any nonsingular matrix A always equal to one. For positive definite A, the behaviour of the diagonal entries of the combined matrix was completely

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described above. It was then natural to ask the question about such behaviour of other classes, e.g. for totally positive matrices.

The class of totally positive matrices, shortly TP-matrices, i.e. matrices, every square submatrix of which has a positive determinant, is on one side rather narrow, on the other side enough large to contain important subclasses of matrices, such as (somehow restricted) Cauchy matrices, the Hilbert matrix and its submatrices, etc. In the recent years, many new discoveries about them were made. For totally positive matrices, we shall need the following important fact [4]. There, a submatrix of a matrix is called *relevant* if it is square, has consecutive rows and consecutive columns, and either its first row or its first column (or both) is in the first row or in the first column of A.

THEOREM 1.1. *Let* $A = [a_{ik}]$ *be a real* $m \times n$ *matrix. Then a necessary and sufficient condition that* A *is totally positive is*:

All the determinants of all relevant submatrices of A *are positive.*

We also recall a well known factorization theorem on totally positive matrices from [6] (see also [5]):

THEOREM 1.2. *An* $n \times n$ *real matrix* A *is totally positive if and only if it can be expressed as*

$$A = B_1 B_2 \cdots B_{n-1} D C_{n-1} C_{n-2} \cdots C_1, \tag{1}$$

where for $i = 1, \dots, n-1$, B_i *is the lower bidiagonal matrix*

$$B_i = \begin{bmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ & \ddots & \ddots & & & & \\ & & 0 & 1 & & & \\ & & & \alpha_{n-i+1,1} & 1 & & \\ & & & & \ddots & \ddots & \\ & & & & & \alpha_{ni} & 1 \end{bmatrix}, \tag{2}$$

and C_i *is the upper bidiagonal matrix*

$$C_i = \begin{bmatrix} 1 & 0 & & & & & \\ & \ddots & \ddots & & & & \\ & & 1 & 0 & & & \\ & & & 1 & \beta_{1,n-i+1} & & \\ & & & & \ddots & \ddots & \\ & & & & & 1 & \beta_{in} \\ & & & & & & 1 \end{bmatrix}, \tag{3}$$

D *is a diagonal matrix, and all the numbers* α_{ik}, $i > k$, β_{ik}, $i < k$, *and all the diagonal entries of* D *are positive.*

In the following well known theorem, we say that a square matrix is strongly nonsingular if all matrices in the nested sequence of the upper-left corner matrices are nonsingular. Also, a totally nonnegative matrix, some power of which is totally positive, is called *oscillatory.*

THEOREM 1.3. *Let A be a totally nonnegative matrix. Then A is oscillatory if and only if the matrix A is strongly nonsingular and both matrices A_1 and A_2 have positive diagonal entries, where A_1, resp. A_2 is the matrix obtained from A by deleting the first column and the last row, resp. the first row and the last column.*

2. Results.

We start with a lemma.

LEMMA 2.1. *Let $A = [a_{ik}]$, $i = 1, \ldots, n-1$, $k = 1, \ldots, n$, be a totally positive matrix. If $n \geq 3$ and $a_{11} = a_{12}$, then*

$$\det \begin{bmatrix} a_{11} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{23} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n-1,1} & a_{n-1,3} & \ldots & a_{n-1,n} \end{bmatrix} > \det \begin{bmatrix} a_{12} & a_{13} & \ldots & a_{1n} \\ a_{22} & a_{23} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n-1,2} & a_{n-1,3} & \ldots & a_{n-1,n} \end{bmatrix}. \quad (4)$$

Proof. We use induction on n. If $n = 3$, the result is true since

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} > 0$$

(and $a_{11} = a_{12}$) implies $a_{22} > a_{21}$ and indeed

$$\det \begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix} > \det \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}.$$

Suppose now that $n > 3$ and that for $n-1$ the result holds. Let B be the matrix which coincides with A in all entries except for $a_{n-1,n}$, and such that its entry $\hat{a}_{n-1,n}$ satisfies

$$\det \begin{bmatrix} a_{12} & a_{13} & \ldots & a_{1n} \\ a_{22} & a_{23} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n-1,2} & a_{n-1,3} & \ldots & \hat{a}_{n-1,n} \end{bmatrix} = 0. \quad (5)$$

For this B, the inequality (4) holds by Theorem 1.1, possibly with equality, since for any $\varepsilon > 0$ the matrix $\hat{B}$ with $\hat{a}_{n-1,n} + \varepsilon$ instead of $\hat{a}_{n-1,n}$ is totally positive since $\hat{B}$ and A have the same system of relevant submatrices except that on the left-hand side of (5). Now, by the induction hypothesis, the cofactor C_1 of $a_{n-1,n}$ on the left-hand side of (4) is greater than the corresponding cofactor C_2 of $a_{n-1,n}$ on the right-hand side of (4). The left-hand side of (4) can be written as

$$C_1(a_{n-1,n} - \hat{a}_{n-1,n}) + \det \begin{bmatrix} a_{11} & a_{13} & \ldots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{n-1,1} & a_{n-1,3} & \ldots & \hat{a}_{n-1,n} \end{bmatrix},$$

the right-hand side as

$$C_2(a_{n-1,n} - \hat{a}_{n-1,n}) + \det \begin{bmatrix} a_{12} & a_{13} & \ldots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{n-1,2} & a_{n-1,3} & \ldots & \hat{a}_{n-1,n} \end{bmatrix}.$$

Since the last determinant is by (5) equal to zero whereas the first is nonnegative, the inequalities $C_1 > C_2$ and $a_{n-1,n} - \hat{a}_{n-1,n} > 0$ imply (4). ■

Theorem 2.2. *Let $A = [a_{ik}]$ be an $n \times n$ totally positive matrix, let $A^{-1} = [\alpha_{ik}]$. Then, for $n = 2$, $a_{11}\alpha_{11} = a_{22}\alpha_{22}$, and if $n \geq 3$,*

$$a_{11}\alpha_{11} < a_{22}\alpha_{22}, \tag{6}$$

as well as

$$a_{n-1,n-1}\alpha_{n-1,n-1} > a_{nn}\alpha_{nn}. \tag{7}$$

Proof. By the mentioned property of totally positive matrices, multiplication of a row or a column by a positive number does not change the combined matrix. We can thus assume that $a_{11} = 1$, $a_{12} = 1$, $a_{22} = 1$, and we can also use the adjoint matrix instead of the inverse. Our problem is then to show that in the partitioning of A as

$$A = \begin{bmatrix} 1 & 1 & A_{13} \\ a_{21} & 1 & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix},$$

$$\det \begin{bmatrix} 1 & A_{23} \\ A_{32} & A_{33} \end{bmatrix} < \det \begin{bmatrix} 1 & A_{13} \\ A_{31} & A_{33} \end{bmatrix}. \tag{8}$$

By Lemma 2.1, removing the second row from A,

$$\det \begin{bmatrix} 1 & A_{13} \\ A_{32} & A_{33} \end{bmatrix} < \det \begin{bmatrix} 1 & A_{13} \\ A_{31} & A_{33} \end{bmatrix}. \tag{9}$$

Using Lemma 2.1 for columns after removing from A the first column, we obtain

$$\det \begin{bmatrix} 1 & A_{13} \\ A_{32} & A_{33} \end{bmatrix} > \det \begin{bmatrix} 1 & A_{23} \\ A_{32} & A_{33} \end{bmatrix}. \tag{10}$$

From (9) and (10), the required inequality (8), and thus (6), follows. The inequality (7) follows from the fact that the transformation $A \to JAJ$, where J is the skew identity, does not change the property of A to be totally positive. The rest is obvious. ■

Corollary 2.3. *Let $A = [a_{ik}]$ be an $n \times n$ totally nonnegative nonsingular matrix, let $A^{-1} = [\alpha_{ik}]$. Then, if $n \geq 3$*

$$a_{11}\alpha_{11} \leq a_{22}\alpha_{22},$$

as well as

$$a_{n-1,n-1}\alpha_{n-1,n-1} \geq a_{nn}\alpha_{nn}.$$

Remark 2.4. Theorem 1.3 implies that for oscillatory matrices also strict inequality in the inequalities above holds.

Remark 2.5. Theorem 2.2 is a step towards the solution of the problem to characterize the n-tuples of diagonal entries of combined matrices of square totally positive matrices. By the Hadamard inequalities, all the diagonal entries should be greater than one. For the case of symmetric Cauchy matrices, and thus also for all principal minors of the Hilbert matrix, the author proved in [1] that the sequence $\{\sqrt{a_{ii}\alpha_{ii}} - 1\}$ for $n \times n$ matrices has the remarkable property that the sum of its odd terms is equal to the sum of the even terms. Another close property is in the next theorem.

THEOREM 2.6. *Let $A = [a_{ik}]$ be an arbitrary (even complex) nonsingular tridiagonal matrix, $A^{-1} = [\alpha_{ik}]$. Then the sequence $\{a_{ii}\alpha_{ii} - 1\}$ has the property that the sum of its odd terms is equal to the sum of the even terms.*

Proof. Let us form the combined matrix $C = A \circ (A^T)^{-1}$. Since all its row and column sums are equal to one, the matrix $C - I$ has all such sums equal to zero. It is also tridiagonal. Let now r_o, r_e, respectively, be the sum of all entries in the odd, respectively even rows of $C - I$, and analogously c_o, c_e the sums of the columns of $C - I$. It is evident that the sum $r_o - r_e + c_o - c_e$, which is zero, is at the same time twice the alternate sum of the diagonal entries of $C - I$. ■

In [3], the author introduced the so called *complementary basic matrices* (CBM-matrices) as matrices, if of order n, of the form $G_{i_1}G_{i_2}\cdots G_{i_{n-1}}$, where $(i_1, i_2, \dots, i_{n-1})$ is some permutation of $(1, 2, \dots, n-1)$ and the matrices G_k, $k = 1, \dots, n-1$, have the form

$$G_k = \begin{bmatrix} I_{k-1} & & \\ & C_k & \\ & & I_{n-k-1} \end{bmatrix} \tag{11}$$

for 2×2 matrices C_k. Because of its shape, we call matrices of the form $G_1G_2\cdots G_{n-1}$ *Hessenberg CBM-matrices.* It is immediate that if all the C_k's are totally positive, then the Hessenberg CBM-matrices are oscillatory since all the subdiagonal and all the super-diagonal entries are positive (and the sequence of the nested principal minors is positive).

We can now prove the following:

THEOREM 2.7. *Let A be the matrix $A = G_1G_2\cdots G_{n-1}$, where G_i have the form* (11) *with nonsingular C_i's. Then the combined matrix of A has the property that the products of the diagonal entries in even positions and in odd positions coincide.*

Proof. For simplicity, we will suppose that the matrix C_i has the form

$$C_i = \begin{bmatrix} 1 & 1 \\ a_i & b_i \end{bmatrix},$$

$a_i \neq b_i$. Then, denoting $\frac{1}{b_i - a_i}$ as δ_i, one can prove easily by induction that the diagonal entries of $A = G_1G_2\cdots G_{n-1}$ are 1, b_1, b_2, $\dots$, b_{n-1}, the diagonal entries of A^{-1} then $b_1\delta_1$, $b_2\delta_1\delta_2$, $\dots$, $b_{n-1}\delta_{n-2}\delta_{n-1}$, δ_{n-1}. The result thus follows since both products in the combined matrix are equal to $\prod_{i=1}^{n-1} b_i\delta_i$. ■

THEOREM 2.8. *If A is an oscillatory Hessenberg CBM-matrix, then the diagonal entries $d_1, d_2, \dots, d_n$ of the combined matrix $C(A)$ of A are characterized by the following four facts*:

1. $d_i > 1$ *for all* i;
2. $d_1 < d_2$ *and* $d_{n-1} > d_n$;
3. $d_k < d_{k-1}d_{k+1}$ *for* $k = 2, \dots, n-1$;
4. $d_1d_3\cdots = d_2d_4\cdots$.

Proof. Necessity follows from Theorems 2.2 and 2.7 and Remark 2.5. Sufficiency follows easily from solvability of the conditions in the proof of Theorem 2.7 if 3. is satisfied. ■

Let us formulate a fact the proof of which is immediate.

Theorem 2.9. *All products $A_P = \prod_P G_i$ have the same sequence of the diagonal entries. If all the matrices C_i are nonsingular, the sequence of the diagonal entries of $C(A_P)$ has the form*

$$\{c_1, c_1c_2, c_2c_3, \dots, c_{n-2}c_{n-1}, c_{n-1}\}$$

for some $c_1, c_2, \dots, c_{n-1}$.

Corollary 2.10. *If A has the form $\prod_P G_i$ where the G_is are defined as in* (11) *and all the C_is are totally positive, then the diagonal entries $d_1, \dots, d_n$ of $C(A)$ have all the properties* 1.–4. *in Theorem* 2.8. *Conversely, if $d_1, d_2, \dots, d_n$ satisfy these conditions, then there exists a totally nonnegative nonsingular CBM-matrix A whose combined matrix $C(A)$ has $d_1, d_2, \dots, d_n$ as the sequence of its diagonal entries.*

We conjecture the following.

Conjecture. *If A is an $n \times n$ totally positive matrix, then the sequence $\{d_1, d_2, \dots, d_n\}$ of the diagonal entries of $C(A)$ is characterized by the conditions* 1.–3. *in Theorem* 2.8.

We shall prove now the conjecture in the case $n = 3$.

Theorem 2.11. *A necessary and sufficient condition that the sequence $\{u_1, u_2, u_3\}$ be the sequence of diagonal entries of the combined matrix of a totally positive 3×3 matrix is*:

1. $u_i > 1$ *for* $i = 1, 2, 3$.
2. $u_2 > u_1 + u_3 - 1$.
3. $u_2 < u_1u_3$.

Proof. Let A be a totally positive 3×3 matrix. By Theorem 1.2, it can be written in the form (1)–(3) for $n = 3$, with positive numbers α_{21}, α_{31}, α_{32}, β_{12}, β_{13}, β_{23}, d_1, d_2 and d_3, where d_1, d_2, d_3 are the diagonal entries of the matrix D. Then, the inverse $(A^T)^{-1}$ is equal to

$$\begin{bmatrix} 1 & -\alpha_{21} & \alpha_{21}\alpha_{32} \\ 0 & 1 & -(\alpha_{31}+\alpha_{32}) \\ 0 & 0 & 1 \end{bmatrix} D^{-1} \begin{bmatrix} 1 & 0 & 0 \\ -\beta_{12} & 1 & 0 \\ \beta_{12}\beta_{23} & -(\beta_{13}+\beta_{23}) & 1 \end{bmatrix}.$$

It follows that

$$u_1 = d_1\Big(\frac{1}{d_1} + \frac{1}{d_2}\alpha_{21}\beta_{12} + \frac{1}{d_3}\alpha_{21}\alpha_{32}\beta_{12}\beta_{23}\Big), \tag{12}$$

$$u_2 = (\alpha_{21}\beta_{12}d_1 + d_2)\Big[\frac{1}{d_2} + \frac{1}{d_3}(\alpha_{31}+\alpha_{32})(\beta_{13}+\beta_{23})\Big], \tag{13}$$

$$u_3 = \frac{1}{d_3}\big[\alpha_{21}\alpha_{31}\beta_{12}\beta_{13}d_1 + (\alpha_{31}+\alpha_{32})(\beta_{13}+\beta_{23})d_2 + d_3\big]. \tag{14}$$

The condition 1. is clearly fulfilled. The condition 2. follows from the expression $u_1 + u_3 - u_2 - 1$ using (12)–(14) since only a sum of negative terms remains. To prove 3., observe that in the expression $u_1u_3 - u_2$ after substitution in (12)–(14) all negative terms cancel and at least one positive term remains.

Let conversely u_1, u_2 and u_3 fulfil all three conditions 1.–3. Observe that the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ u_1 - 1 & u_1 & u_1 \\ x & u_3 - 1 & u_3 \end{bmatrix}, \tag{15}$$

where $x = u_3 - \frac{u_2}{u_1}$, is totally nonnegative with all subdeterminants positive except that in the first two rows and last two columns. The matrix $(A^T)^{-1}$ is then the matrix

$$\begin{bmatrix} u_1 & u_3(1-u_1) + u_1 x & (1-u_1)(1-u_3) - u_1 x \\ -1 & u_3 - x & 1 - u_3 + x \\ 0 & -1 & 1 \end{bmatrix}.$$

The combined matrix $A \circ (A^T)^{-1}$ has thus diagonal entries u_1, u_2 and u_3. The only problem is the singularity of that 2×2 submatrix of A. To get rid of that gap, observe the following. The mapping Φ which assigns to every totally positive 3×3 matrix A the ordered triple $D[C(A)]$ of the diagonal entries of $C(A)$ is a continuous (even differentiable) mapping from an open set S_1 into a set S_2 contained by the previous part of the proof in the open set $\tilde{S}_2$ given by the conditions 1.–3. The matrix A in (15) is on the boundary of S_1 and $D[C(A)]$ is in $\tilde{S}_2$. By the implicit function theorem, there exists in S_1 some matrix $\hat{A}$ such that $D[C(\hat{A})] = D[C(A)]$. Thus $\Phi(S_1) = \tilde{S}_2$. ∎

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